4. Other applications. We mention two other applications of the results. If there are s individuals with possibly different loss functions, $W_{ijk}(x)$ can denote the loss suffered by individual k when d_j is made and F_i is true and x is observed. Or different true situations may lead to the same distribution of the observable chance variable, so that $W_{ijk}(x)$ is the loss incurred under the kth true situation leading to the distribution F_i. The range of k may depend upon i, and all the results hold.

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CORRECTION OF A PROOF*

By J. Kiefer

Cornell University

In the proof of Theorem 3 of "On Wald's Complete Class Theorems" (*Ann. Math. Stat.*, Vol. 24 (1953), pp. 70–75), the inequality appearing in the definition of $r_{2,m}(\xi)$ should be altered to read $r(\xi, \delta^m) \geqq r(\xi, \delta_2) - \epsilon/2$; the remainder of the proof is then easily altered to give the desired result. Without the $\epsilon/2$, one would still have to prove that the space $\mathfrak{D}$ is large enough to give $\lim_{m\to\infty} r_{2,m}(\xi) < \infty$. The author is indebted to Mr. Jerome Sacks for pointing out this fact.

* Received 7/11/53.

ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Stanford meeting of the Institute, June 19–20, 1953*)

1. **On the Probability Function of the Quotient of Sample Ranges from a Rectangular Distribution.** Leo A. Aroian, Hughes Aircraft and Development Laboratories, Culver City.

In a recent paper Paul R. Rider (*J. Amer. Stat. Assn.*, Vol. 46 (1951), pp. 502–507) has derived the probability functi n of $u = R_1/R_2$, the quotient of the sample ranges of two independent random samples from $f(x) = 1/x_0$ for $0 \leqq x \leqq x_0$, $f(x) = 0$ elsewhere, where R_1 is the sample range in a sample of m and R_2 is the sample range in a sample of n from $f(x)$. The power function of the test is derived, the tables are extended for the 5 per cent, $2\frac{1}{2}$ per cent, 1 per cent, and $\frac{1}{2}$ per cent levels of significance. In case m and n large a Cornish-Fisher expansion for the levels of significance is derived. The transformation $w = \frac{1}{2} \log_e u$ is found convenient and use is made of the moment generating function of w to find the cumulants of w, which are needed in the Cornish-Fisher expansion. Limiting distributions are given for m large, $n \to \infty$ and vice versa.

2. Actuarial Validity of the Binomial Distribution for Large Numbers of Lives with Small Mortality Probabilities. JOHN E. WALSH, U. S. Naval Ordnance Test Station, China Lake.

In actuarial work, one of the most widely used probability tools is the binomial distribution. Sufficient conditions for the validity of the binomial distribution for a group of lives observed during some time or age interval are: (a) The probability of death within the interval considered is the same for each person of the group. (b) The lives are statistically independent with respect to mortality. Then the deaths occurring in this group during the observation period have a binomial probability distribution. For practical situations, these conditions are never exactly satisfied. Condition (a) is undoubtedly violated appreciably for many groups of lives. The close association between friends, relatives, and neighbors indicates that condition (b) also may be noticeably violated. Thus use of the binomial distribution in actuarial work might seem a very questionable procedure. This note investigates the applicability of the binomial distribution for a situation which appears to be of common actuarial occurrence and where both (a) and (b) can be noticeably violated. The binomial distribution is found to yield reasonably accurate probabilities for the case of a large number of lives if the average death probability is used in place of the common mortality value specified by (a).

3. On the Distribution of the Likelihood Ratio. HERMAN CHERNOFF, Stanford University.

A classical result on the distribution of the likelihood ratio λ is the following. Under suitable regularity conditions, if the hypothesis that a parameter θ lies on an r-dimensional hyperplane of k-dimensional space is true, the distribution of $-2 \log \lambda$ is asymptotically that of chi square with $k - r$ degrees of freedom. On the basis of n independent observations, let λ be computed for a test of the hypothesis that θ lies in ω, against the alternative that θ lies in τ where ω and τ are disjoint subsets of k-dimensional space. Let the origin be a limit point of both ω and τ, and let J represent the information matrix per observation at $\theta = 0$. If ω and τ may be suitably approximated near the origin by positively homogeneous sets, the asymptotic distribution of $-2 \log \lambda$ when $\theta = 0$ is the same as for the problem where the observations have a joint normal distribution with mean θ and covariance matrix J^{-1}, and ω and τ are replaced by their approximations.

4. Testing the Approximate Validity of Statistical Hypotheses. J. L. HODGES, JR. AND ERICH L. LEHMANN, University of California, Berkeley.

A statistical hypothesis H, in the customary formulation, is frequently known a priori not to be exactly true. A much discussed example of this situation is the problem of testing for normality. The large-sample paradoxes inherent in such a formulation (see for example Berkson, *J. Amer. Stat. Assn.*, Vol. 33 (1938), p. 526) may be avoided by testing instead the hypothesis H' that H is approximately valid. The chi square test of goodness of fit is modified to provide a large-sample test of H'. A number of related small-sample parametric problems are also treated. For example, a strictly unbiased test is found for the hypothesis that the mean ξ of a normal population of unknown variance, differs from its hypothetical value ξ_0 by not more than a stated amount δ.

5. Distribution of Correlated Means. D. S. VILLARS, U. S. Naval Ordnance Test Station, China Lake.

The maximum likelihood estimate of a population mean as computed from a set of sample means correlated by a known amount, turns out to be a weighted average with definite weighting coefficients. These coefficients have been worked out for sets from four to twelve and correlations corresponding to degrees of overlap of one third, one half, two thirds and three quarters. The weighted sum of squares and products of deviations of correlated means $n \sum_{i,j}^{p} A^{ij}(x_i - \bar{x})(x_j - \bar{x})$ is shown to be distributed as chi square with degrees of freedom equal to $p - 1$. These results are applicable to the distribution of moving averages on the null hypothesis of zero trend with time.

6. **On the Detection of Sure Signals in Noise.** R. C. DAVIS, U. S. Naval Ordnance Test Station, Pasadena Annex.

The purpose of this paper is to simplify and extend known results on the detection of sure signals in the presence of noise. For a sure signal; that is, of completely specified form and nonrandom, upon which a background noise is superimposed linearly, several criteria for an optimum pre-detection filter are compared. The background noise may be any continuous stochastic process possessing mean value zero and a known continuous covariance function. Specifically, it is shown that when the input noise to a predetection filter is Gaussian, the filter which maximizes—for a fixed false alarm probability—the probability of detecting the signal when present is identical with the linear filter which maximizes the output signal-to-noise ratio. An explicit expression is obtained for the probability of detection. The stability of the optimum filter is discussed in some detail. Finally it is shown how an optimum signal shape of given energy content can be chosen.

7. **A Statistic Associated with the Joint Distribution of n Successive Amplitudes. Preliminary Report.** WILLIAM C. HOFFMAN, U. S. Navy Electronics Laboratory, San Diego.

In previous work the joint distribution of the n random variables $R(t_j) = \{X^2(t_j) + Y^2(t_j)\}^{\frac{1}{2}}$, $(j = 1, 2, \cdots, n)$, was found for $X(t_j)$, $Y(t_j)$ from a stationary Gaussian process. A statistic $q = 2n^{-1} \sum_1^n r_j^2$ is now defined, and its characteristic function and small sample distribution determined. The statistic q, which is an estimate of the parameter σ^2, is shown to be unbiased, consistent, and in the case of simple Markov dependence, asymptotically efficient in the strict sense. It is also shown that q is asymptotically normal. Expressions for maximum likelihood estimates are found for the case of simple Markov dependence, and $\hat{\sigma}^2$ is shown to be asymptotically equivalent to q in probability. A test of independence versus simple Markov dependence is given.

8. **Some Two-Sample Tests Based on a Particular Measure of Discrepancy.** LOUIS H. WEGNER, University of Oregon.

Let F and G be continuous cumulative probability functions. The quantity $\theta(F, G) = \int_{-\infty}^{\infty} (F - G)^2 \, dG$ is a measure of discrepancy between F and G and is such that $\theta(F, G) = 0$ if and only if $F = G$. E. Lehmann has proposed a distribution-free statistic which is the minimum variance unbiased estimate of the functional $\phi(F, G) = \frac{1}{3} + \theta(F, G) + \theta(G, F)$. Three other distribution-free statistics based on $\theta(F, G)$ are $\theta(F^*, G^*)$, $\theta(F^*, G^*) + \theta(G^*, F^*)$, and $\phi(F^*, G^*)$, where F^* and G^* are the corresponding sample cumulative probability functions. The above four statistics, minus their expected values and multiplied by suitable functions of the sample sizes, are shown to have the same asymptotic distribution. Under $H_0 : F = G$, the asymptotic distribution is the same as that of the von Mises statistic. Under $H_1 : F \neq G$, the asymptotic distribution is normal provided F and G are restricted slightly. The tests based on large values of these statistics are shown to be consistent tests of H_0 against H_1. An example showing that these tests are not in general unbiased is given. The variance of Lehmann's statistic is found in terms of F and G and the power of the corresponding test is investigated for the alternatives $G = F^2$ and $G = F^3$.

9. Confidence Intervals for a Proportion. Preliminary Report. E. L. Crow, U. S. Naval Ordnance Test Station, China Lake.

Certain direct modifications of the Clopper and Pearson confidence intervals for a binomial proportion p are discussed. The sample information consists of the number r of items with a stated characteristic in a random sample of size n. To obtain confidence intervals with confidence coefficient at least $1 - \alpha$, Clopper and Pearson determined, for each given p, an interval of values of r by excluding at *each* end of the distribution values of r with probability mass totaling not more than $\alpha/2$. If the excluded probability mass α is not so restricted, then generally shorter intervals for r, and correspondingly for p, are obtained. One modification consists of following Clopper and Pearson except that all the probability mass α is removed at one end of the distribution of r when none can be removed at the other. Another modification, proposed by Theodore E. Sterne, consists of excluding from the interval of r for each given p those values of r having the smallest probabilities. For $1 - \alpha = 0.90, 0.95$ and 0.99 and $n = 1, 2, \cdots, 20$, confidence intervals for p based on these two modifications are tabulated and compared.

10. On Estimating Both Mean and Standard Deviation of a Normal Population from the Lowest r out of n Observations. John V. Breakwell, North American Aviation Company, Los Angeles.

Maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of both mean and standard deviation are obtainable from the solution of a transcendental equation involving the ratio r/n and the ratio D/S, where D is the difference between the mean of and the highest of the lowest r observations, while s is the standard deviation of these r observations. The asymptotically bivariate normal distribution of $\sqrt{n}(\hat{\mu} - \mu)$ and $\sqrt{n}(\hat{\sigma} - \sigma)$ is investigated, the elements of the covariance matrix being decreasing functions of the ratio r/n. The biases in both $\hat{\mu}$ and $\hat{\sigma}$ are negative, of order $1/n$, and are certain numerically decreasing functions of the ratio r/n.

11. Strong Consistency of Stochastic Approximation Methods. Julius R. Blum, University of California, Berkeley.

Robbins and Monro have constructed a stochastic approximation scheme which estimates consistently the root of a regression equation. Similarly Kiefer and Wolfowitz have proposed a consistent sequence of estimates for the point where an unknown regression function achieves its maximum. It is shown that both of these schemes have the property of strong consistency, under somewhat weaker restrictions. A semimartingale theorem due to Doob is generalized and applied to prove strong convergence of a certain sequence of random vectors. This is applied to problems of solving k regression equations in k unknowns and of estimating the point where a regression function in k variables achieves its maximum.

12. Some Probability Results for Mortality Rates Based on Insurance Data. John E. Walsh, U. S. Naval Ordnance Test Station, China Lake.

This paper considers the probability distribution of an observed mortality rate based on a large amount of insurance data. A computationally feasible method of obtaining significance tests and confidence intervals for the "true" mortality rate estimated by the observed rate is presented. It is only necessary to know the value of the observed rate, the number of units (policies, amounts, etc.) exposed to risk, and the number of units associated with each person who died during his observation period. In the derivations the lives are assumed to be statistically independent but need not have the same mortality rate nor be observed during the same period. The tests and confidence intervals obtained are nearly 100 per cent efficient. A generalization of the basic technique is used to derive the probability distribution of actuarial cost functions and other quantities of interest. A method whereby past data may be used as a help in estimating the future probability distribution of these actuarial functions is outlined.

13. Extensions of the U-Test to Three Populations. Louis H. Wegner, University of Oregon.

Let F, G, and H be continuous cumulative probability functions. Denote the classes of triples (F, G, H) such that (1) $F = G = H$, (2) $F < G < H$, and (3) $F \leqq G \leqq H$ (where at least one of the inequality signs holds) by C_1, C_2, and C_3 respectively. Two extensions of the Wilcoxen-Mann-Whitney U-test are proposed and are shown to be consistent and unbiased tests of C_1 against C_3. The test statistics are shown to be asymptotically normal under C_1 and also under C_3 provided that for one of the statistics C_3 is slightly restricted. Certain moments are found in terms of F, G, and H. Finally, it is shown that a test of C_1 against C_2 proposed by D. R. Whitney is unbiased against C_3, consistent against C_2, but not consistent against C_3.

14. Normal Regression Theory and Some Classical Statistics in Multivariate Analysis. Junjiro Ogawa, Osaka University.

The purpose of this paper is to give a new method of derivation of the sampling distributions of the multiple correlation coefficient and the Hotelling's squared generalized Student's ratio T^2. Although the results here presented are well known and now are classical, there seems to be some interest from the methodological point of view. The fundamental idea of this paper was suggested by Prof. G. Elfving's 1947 paper (G. Elfving, "A simple method of deducing certain distributions connected with multivariate sampling," *Skand. Aktuarietids.*, Vol. 29–30 (1947), pp. 56–74). In this paper Prof. G. Elfving had attempted the systematic derivation of the sampling distributions of the classical statistics in multivariate analysis utilizing the geometrical interpretations of the results of the normal regression theory, but with repect to the two statistics mentioned above, he succeeded in deriving their sampling distributions only in the null cases. Here we shall show that our method gives their sampling distributions in general cases. In this connection, we had to describe the main results of the normal regression theory somewhat more precisely than those which are seen in the literatures, at least as far as the writer knows.

15. The Use of Maximum Likelihood Estimates in Chi Square Tests of Goodness of Fit. Herman Chernoff and Erich L. Lehmann, Stanford University and University of California, Berkeley.

Consider the problem of testing that a sample comes from a distribution of given form. The test is performed by counting the number of observations falling into specified cells and applying the χ^2 test to these frequencies. In estimating the parameters for this test one may use the maximum likelihood (or asymptotically equivalent) estimates based (1) on the cell frequencies or (2) on the original observations. It is pointed out that in (2) (unlike the well known result for (1)) the test statistic does not have a limiting χ^2-distribution, but that it is stochastically larger than would be expected under the χ^2 theory. The limiting distribution is obtained and some examples are computed. These indicate that the error is not serious in the case of fitting a Poisson distribution, but may be so for the fitting of a normal.

16. On the Treatment of Ties in Nonparametric Tests. Joseph Putter, University of California, Berkeley.

In applying rank tests to tied observations, two alternative procedures are customarily used: either the tied observations are "randomized," that is, ordered in a way depending on the outcome of an additional random experiment, or the definition of the test statistic is appropriately extended to cover the tied case. For (1) the Wilcoxon two-sample test and (2) the sign test, the asymptotic distributions of the statistics concerned are derived. The performances of the two alternative procedures are then compared, using Pitman's concept of asymptotic relative efficiency (cf. Noether, *Ann. Math. Stat.* Vol 21 (1950), p. 241). In both cases, the "randomized" test is proved to be less efficient than the "nonrandomized" test given by the modified test statistic. In (1), the modified test statistic is essentially the one suggested by Kruskal and Wallis (*Ann. Math. Stat.*, Vol 23 (1952), p. 538); the asymptotic relative efficiency of the randomized test with respect to the nonrandomized test is $1 - \Sigma p_k^3$, where p_k are the jumps of the relevant underlying distribution at its discontinuities. In (2), the nonrandomized test consists essentially of ignoring the zero differences $x_i - y_i$; the analogous asymptotic relative efficiency is $1 - p_0$, where $p_0 = P(X_i = Y_i)$. (Research sponsored by the Bureau of Naval Research.)

17. Asymptotic Relative Efficiency of Some Rank Tests for Analysis of Variance Problems. F. C. Andrews, Stanford University.

Let $\{X_{ij}; i = 1, 2, \cdots, c; j = 1, 2, \cdots, n_i\}$ be independent random variables with $F_i(x)$ the continuous distribution function of X_{ij}, $n_i = s_i n$. Several nonparametric tests for the hypothesis $F_1 = F_2 = \cdots = F_c$ have been proposed. To study the asymptotic behavior of two of these tests against translation differences, the sequence of alternative hypotheses $F_i(x) = F(x + \theta_i/\sqrt{n})$, $i = 1, 2, \cdots, c$, $\Sigma_{i=1}^c (\theta_i - \bar{\theta})^2 > 0$ is assumed. With mild assumptions on F, for this type of alternative hypothesis, the limiting distribution as $n \to \infty$ of the Wallis-Kruskal H statistic is shown to be $\chi_{c-1}^2(\lambda^H)$, (noncentral χ^2 with $c - 1$ degrees of freedom and noncentral parameter λ^H), $\lambda^H = 12[\int_{-\infty}^{+\infty} (F'(x))^2\, dx]^2 \cdot \Sigma_{i=1}^c s_i(\theta_i - \bar{\theta})^2$; for that of the Mood-Brown median statistic $\chi_{c-1}^2(\lambda^M)$ with $\lambda^M = 4(F'(a))^2 \cdot \Sigma_{i=1}^c s_i(\theta_i - \bar{\theta})^2$, $F(a) = \frac{1}{2}$. These results are used to determine the asymptotic relative efficiency (a.r.e.) of the median test with respect to the H-test which is $\frac{1}{3}[F'(a)/\int_{-\infty}^{+\infty} [F'(x)]^2\, dx]^2$ and the a.r.e. of the H-test with respect to the classical F test which is $12[\sigma_F \int_{-\infty}^{+\infty} (F'(x))^2\, dx]^2$, σ_F^2 is the variance of the d.f.F. These last a.r.e. results are independent of the power, level of significance, and c, and so agree with the known results in the two sample case.

18. Application of the Studentized Maximum Chi-Distribution. Preliminary Report. T. A. Jeeves, University of California, Berkeley.

If U is the maximum of $X_1, X_2, \cdots X_s$, where X_i has a chi-distribution with m degrees of freedom, and $n^{\frac{1}{2}}Y$ has a chi-distribution with n degrees of freedom, then the distribution of $Z = U/Y$ is termed the studentized maximum chi-distribution. This statistic can be applied to obtain confidence bands in problems of multiple regression and analysis of covariance. For the comparison of many regression lines (hyperplanes), the bands so obtained are the Neyman-narrowest bands about each line (hyperplane). With a slight modification the variance need not be assumed the same about each line. This statistic has also been applied to obtain bands which are Neyman-shortest about each coefficient of the regression hyperplane and to develop families of bands with flat boundaries. These later can be used (i) in their own right (ii) as an approximation to the Neyman-narrowest bands, or (iii) to facilitate construction of the later bands. Asymptotic expressions for the distribution have been obtained and short tables prepared.

(Abstracts of papers presented at the Kingston meeting of the Institute, August 31–September 4, 1953)

19. Sequential Probability Ratio Confidence Sets. (Preliminary Report.) Allan Birnbaum, Columbia University.

Let $x = (x_1, x_2, \cdots)$ denote a sequence of observed values of a random variable distributed according to $f(x, \theta)$. Let $T(\theta')$ denote the sequential probability ratio test of the hypothesis $H(\theta'): \theta = \theta'$ against the hypothesis $H(\theta' + \Delta): \theta = \theta' + \Delta$, at size α and power $1 - \beta$, $\Delta > 0$, with operating characteristic denoted by $L_{\theta'}(\theta)$, for θ', $\theta' + \Delta$ and θ in Ω, with Ω a (possibly infinite) interval. Let $\theta''(x) = \inf \{\theta' \mid T(\theta') \text{ accepts } H(\theta') \text{ when } x \text{ is observed}\}$. Under general conditions, $L_{\theta'}(\theta)$ is monotone for all θ' and $\theta''(x)$ is convex for all x. Let $m(x, \theta')$ be the number of observations required for $T(\theta')$ to terminate when the sequence x is observed. Let $n(x) = \sup \{m(x, \theta') \mid \theta' \varepsilon \Omega\}$; $n(x)$ and $\theta''(x)$ are functions of the first $n(x)$ components of x only. $n(X)$ is finite with probability one. If θ is the true parameter value, then $Pr\{\theta''(X) \leqq \theta\} = 1 - \alpha$ and $Pr\{\theta''(X) \leqq \theta - \Delta\} = \beta$. Hence the assertion $I(X)$: "$\theta''(X) \leqq \theta < \theta''(X) + \Delta$" will be true with probability $1 - \alpha - \beta$. The method has the advantages: (a) that it constructs confidence intervals of prescribed length Δ and confidence coefficient $1 - \alpha - \beta$ without need to consider the distribution of any statistics, and (b) that it is applicable to certain problems for which there seem to be no alternative methods available. Application of the method to sequential tests of composite hypotheses is being studied.

20. Optimum Sample Size for Choosing the Population Having the Smaller Variance. Paul N. Somerville, University of North Carolina.

Assume we have $k + 1$ populations, normally distributed and with variances $1 \leqq \theta_1 \leqq \cdots \leqq \theta_k$. Let it be required to select N individuals from one of the populations, where those individuals that differ by more than an amount d from the mean are rejected, and where the loss involved in rejecting an individual is r. Suppose we take a preliminary sample of size $n + 1$ from each population, and select the population having the smallest sample variance for the selection of the N individuals. Let the cost of the preliminary sample be $c_1 n + c_0$. If we use the sample size which minimizes the maximum expected loss, then the "optimum" sample size is an increasing function of N_r/c_1 provided N_r/c_1 is sufficient large with respect to c_0 that it is profitable to sample. Tables giving "optimum" sample sizes for $d = 1, 2, 3$ for $k = 1$, $k = 2$, for various values of N_r/c_1 are given. (This research was supported in part by the United States Air Force under Contract AF 18(600)-83.)

21. The Generation of Pseudo-Random Numbers on a Decimal Calculator. Jack Moshman, Oak Ridge National Laboratory.

Let $\rho = 7^{4K+1}$ and define $\rho_0 = 1$. The digits of the sequence $\rho_{i+1} \equiv \rho_i \cdot \rho \pmod{10^s}$, under certain conditions, fulfill specified tests for randomness and provide $5 \cdot 10^{s-3} (s > 4)$ decimal numbers before repetition of the basic cycle. It is found that the five last significant digits should be omitted from the number before application and there is an uncertainty about the sixth. Relevant theorems from number theory are cited and experimental values of χ^2 for various tests are displayed for 10,000 generated numbers.

22. The Integral Solution of Pearson's Random Walk Problem and Related Matters. David Durand and J. Arthur Greenwood, National Bureau of Economic Research and Manhattan Life Insurance Company.

Von Mises has shown that the vector mean of a sample of n from the population $dp = (2 J_0(ik))^{-1} \cdot \exp(k \cos x) \cdot dx$ furnishes a significance test for the hypothesis $k = 0$. This vector mean, multiplied by n, has the distribution found by Kluyver in 1906 as a solution to Pearson's random walk problem. Kluyver's distribution is computed by quadratures and tabled, for $n = 6(1)24$. Two series expansions of the distribution are considered; first, an expansion in Laguerre functions essentially due to Pearson, and, second, an expansion in descending powers of n. For $n = 7, 14, 21$, the goodness of approximation of these series is compared. For use in significance tests, the 5 per cent and 1 per cent tail-area points of the distribution are tabled. The expansion in descending powers of n is inverted for use in extending the table of percentage points.

23. On Optimal Systems. David Blackwell, Howard University.

For any sequence $x_1, x_2, \cdots$ of chance variables satisfying $| x_n | \leqq 1$ and $E(x_n | x_1, \cdots, x_{n-1}) \leqq$ -umax $(| x_n | | x_1, \cdots, x_{n-1})$, where u is a fixed constant, $0 < u < 1$, Pr $\{x_1 + \cdots + x_n \geqq t \text{ for some } n \geqq 0\} \leqq [(1 - u)/(1 + u)]^t$ for all $t \geqq 0$, with equality, for integral t, when the x_n are independent and Pr $\{x_n = \pm 1\} = (1 \mp u)/2$. This result has a simple interpretation in terms of gambling systems; a corollary is that for any chance variables $x_1, x_2, \cdots$ satisfying $| x_n | \leqq 1$ and $E(x_n | x_1, \cdots, x_{n-1}, = 0$, Pr $\{| n^{-1}(x_1 + \cdots + x_n) | \geqq \epsilon \text{ for some } n \geqq N\} \leqq (1 + \epsilon)^{-\epsilon N/(2+\epsilon)}$, yielding Lévy's result that $n^{-1}(x_1 + \cdots + x_n) \to 0$ with probability one.

24. Maximum Likelihood Regression Equations. H. Leon Harter, Wright-Patterson Air Force Base.

Consider the application of the principle of maximum likelihood to the problem of determining the regression equation of one variable on p others. For a normal distribution of residuals, the maximum likelihood solution is the familiar least squares solution, found by minimizing the sum of squares of the residuals. For a Laplace distribution of residuals, the maximum likelihood solution is found by minimizing the sum of the absolute values of the residuals. For distributions of residuals with finite limits, only certain solutions are admissible. For the truncated normal distribution, the maximum likelihood solution is found by minimizing the sum of squares of residuals for the set of admissible solutions. For the truncated Laplace distribution, the maximum likelihood solution is found by minimizing the sum of absolute values of the residuals for the set of admissible solutions. For a rectangular distribution of residuals, the likelihood function is a constant, and there is no unique maximum likelihood solution, one admissible solution being just as likely as another.

25. Spherical Distributions. (Preliminary Report.) G. E. P. Box, Imperial Chemical Industries, Blackley, Manchester, England and North Carolina State College.

A wide class of test criteria, including the t test, analysis of variance test, Bartlett test, and tests of normality are independent of scale. Their null distributions are usually derived on the assumptions of independence, normality and equality of variance. Such test criteria follow the same null distribution and consequently the tests are equally valid under the less stringent conditions that the observations $y_1, y_2, \cdots, y_n$ follow what may be called a "spherical" distribution, that is the contours of the joint density functions are spheres

$$p(y_1, y_2, \cdots, y_n) = kf(\Sigma y^2) \qquad 0 < \Sigma y^2 < L \tag{1}$$

where L may be infinite and k is chosen so that the integral taken over the whole space is unity. The condition for validity is necessary as well as sufficient. The y's would not be independent except with the normal density function, (J. Clerk Maxwell, *Philos. Mag.*, Vol. 19 (1860), p. 19; M. S. Bartlett, "The vector representation of a sample," *Proc. Cambridge Philos. Soc.*, Vol. 30 (1934), pp. 327–340), nevertheless spherical distributions are important because (i) The spherical distribution, but not necessarily the normal distribution, is an approximation to the distribution generated by standard randomization procedures, (ii) The spherical distribution is generated exactly by the process of angular randomization, which can be used with certain multi-factor designs. (G. E. P. Box, "Multi-factor designs of first order," *Biometrika*, Vol. 39 (1952), pp. 49–57.) (iii) Using a parent spherical distribution certain distribution problems may be attacked from a novel and useful angle. When the null-hypothesis is not true, the power of the test criterion is not independent of the function f chosen. However tests which are U.M.P. on the usual assumptions are also U.M.P. for any spherical distribution in which f is a decreasing function.

26. On the Monotonic Character of the Power of a Certain Test in Multivariate Analysis of Variance. S. N. Roy, University of North Carolina.

A test of the hypothesis H_0 of equality of means for k p-variate normal populations (assumed to have the same dispersion matrix Σ) has been put forward, (S. N. Roy, "On a heuristic method of test construction and its use in multivariate analysis", *Ann. Math. Stat.*, June, 1953) having the critical region: $\theta_q \geqq c$, where θ_q is the largest (necessarily positive) characteristic root of the matrix $S^* S^{-1}$ and S^* is the sample "between" covariance matrix, everywhere at least p.s.d. of rank $q = \min(p, k - 1)$ and S is the sample "within" covariance matrix, everywhere p.d., and where c is given by: $P(\theta_q \geqq c \mid H_0) = \alpha$ (say). If we denote by H, the usual nonnull hypothesis and by Σ^*, the usual weighted "between" covariance matrix of the k populations, it is well known and also has been shown, (see above reference), that the power of the critical region, that is, $P(\theta_q \geqq c \mid H)$ is a function of just the characteristic roots (all nonnegative) of the matrix $\Sigma^* \Sigma^{-1}$. It is shown in the present paper that, for a given c, that is, α, this power is a monotonically increasing function of each of the population characteristic roots, which incidentally proves that the proposed test is unbiased.

27. Some Large-Sample Results on Estimation and Power for a Method of Paired Comparisons. (Preliminary Report.) Ralph A. Bradley, Virginia Polytechnic Institute.

Certain large-sample results are obtained for a method of paired comparisons developed by Terry and the author (*Biometrika* Vol. 39 (1952), p. 324). In that paper a parameter Π_i is postulated for each of t items or treatments with $\Sigma_i \Pi_i = 1$ and each $\Pi_i \geqq 0$. It was further postulated that in a comparison of item i with item j the probability that item i obtain rank 1 be $\Pi_i/(\Pi_i + \Pi_j)$. A null hypothesis, $\Pi_i = 1/t$ for all i, was tested against a class of alternatives using maximum likelihood estimates p_i of Π_i and likelihood ratio tests. In the present paper formulas are developed for the variances of the estimators for large samples and confidence limits placed on Π_i, $(\Pi_i - \Pi_j)$, and $(\log \Pi_i - \log \Pi_j)$. It is further shown that, if $\theta_i = \sqrt{n}\,(\Pi_i - 1/t)$ and if R is the likelihood ratio statistic for homogeneity of treatment ratings, then $-2 \log R$ has for large samples the distribution of a noncentral chi square with $(t - 1)$ degrees of freedom and parameter $\lambda = t^3 \Sigma_i \theta_i^2/4$. The test is shown to be asymptotically more powerful than a multi-binomial test formulated and to have a relative efficiency, when compared with the analysis of variance, of $t/\{(t - 1)\Pi\}$. Illustrative examples in taste testing are given. (Research sponsored by the Bureau of Agricultural Economics, United States Department of Agriculture.)

28. Nonparametric Estimation of Survivorship. Paul Meier, Johns Hopkins University.

A standard problem of life testing in general and medical follow-up in particular is the estimation of the proportion of individuals surviving to time T, for example, the proportion of newly diagnosed cancer cases who survive 5 years. For the case of follow-up with unbiased losses it is shown that the simple limiting form of the usual "actuarial" estimate (taking the limit as the interval size goes to zero) is unbiased with variance well approximated by a formula proposed by Greenwood. The estimate is also derived as the maximum likelihood solution of the nonparametric estimation problem and the procedure is extended to the case of competing risks. Various methods of estimating survivorship are compared, with special reference to their sensitivity to biases in the data.

29. Comparison of Two Rank Order Tests for the Two-Sample Problem. GOTTFRIED E. NOETHER, Boston University.

Recently, two rank order tests have been suggested for testing the hypothesis H_0 that two samples of sizes m and n come from the same continuous population, the alternative being that the two samples come from normal populations with different means, but common variance. Let r stand for the ranks of the n observations in the second sample in the over-all ranking of all $m + n = N$ observations. Then Terry's test (*Ann. Math. Stat.*, Vol. 23 (1952), pp. 346–366) is based on the statistic $c_1(R) = \Sigma_r E(Z_{Nr})$ where Z_{Nr} is the *rth* order statistic in a sample of size N from a standard normal population. Van der Waerden's test (*Nederl. Akad. Wetensch. Proc. Ser. A*, Vol. 55 (1952), pp. 453–458) is based on the statistic $X = \Sigma_r \psi(r/N + 1)$ where $\psi(p)$ is the p-quantile of the standard normal distribution. On the basis of examples, it is easily shown that the two tests do not always lead to the same decision. However, when H_0 is true, the correlation coefficient between $c_1(R)$ and X tends to 1 as N increases, and the two tests are asymptotically equivalent.

30. The Poisson Distribution as a Limit of Dependent Binomial Distributions with Unequal Probabilities. JOHN E. WALSH, U. S. Naval Ordnance Test Station, Inyokern.

It is well known that the Poisson probability distribution approximates the binomial probability distribution for situations where the sample size n is large and the probability of "success" small. This result was extended to the case of n independent binomial events with possibly different probabilities for "success" by B. O. Koopman ("Necessary and sufficient conditions for Poisson's distribution," *Proc. Amer. Math. Soc.*, Vol. 1 (1950), pp. 813–823). This paper presents a further extension which appears to be of practical interest and where an event is not required to be statistically independent of all the $n - 1$ other events. Roughly stated, the limiting conditions assumed are: First, each event is statistically independent of at least $n - m - 1$ of the other events and $m/n \to 0$ as $n \to \infty$. Second, although the conditional probability of "success" for an event can be greatly changed in ratio by knowledge of the outcomes for other events, m times this probability tends to zero as $n \to \infty$. Third, the sum over all events of the unconditional probabilities of "success" converges to a finite value as $n \to \infty$. An approximate form of these conditions for large but finite n is presented along with an outline of a general method of intuitive verification for practical applications.

31. An Estimate of the Number of States in a Discrete Markov Chain. A. T. REID, University of Chicago.

In this note we point out a way of obtaining an estimate of the number of states in a discrete Markov chain with two absorbing states. Let us call the states $E_0, E_1, \cdots, E_a$; and define transition probabilities $r_{i,i+1} = i/a$, $r_{i,i-1} = 1 - i/a$ $(i = 1, \cdots, a - 1)$, and $r_{ii} = 1 (i = 0, a)$. The above chain (with E_0 and E_a representing recovery and death of an irradiated organism) has been used as a model in radiobiology (*Bull. Math. Biophysics*, Vol. 13 (1951), pp. 153–163). It is of interest to obtain an estimate of a from the experimentally observed times required for the system to enter either E_0 or E_a. Suppose we observe the system on n occasions, m of which it ends up in E_0. Call $t_{j0} (j = 1, \cdots, m)$ the time required for the system to enter E_0 if initially it was in E_1. Assuming α steps or transitions per unit time let $k_{j0}(= \alpha t_{j0})$ represent the number of steps required for the system to pass from E_1 to E_0. Now $r_{10}^{k_{j0}}$ is the probability of going from E_1 to E_0 in k_{j0} steps. The likelihood function is $P(k_{10}, \cdots, k_{m0}; a) = \Pi_{j=1}^{m} r_{10}^{k_{j0}}$. Put $\xi = \Sigma_{j=1}^{m} k_{j0}$, then $L = ln\, P = ln\, r_{10}^{\xi}$, where r_{10}^{ξ} is the element in the first column and second row of the matrix R raised to the ξ-th power. Since R is decomposable we have $r_{10}^{\xi} = r_{11}^{\xi-1} r_{10}$. The probability $r_{11}^{\xi-1}$ can be calculated using algebraic methods. Differentiating L with respect to a, and proceeding in the usual manner we can obtain a maximum likelihood estimate of the number of states.

32. On a Test of the Rank of a Matrix of Means for k p-variate Normal Populations. S. N. Roy, University of North Carolina.

Suppose we have random samples of sizes n_r $(r = 1, 2, \cdots, k)$ from k p-variate normal populations with a common dispersion matrix Σ. Let Σ^* be the weighted *raw* "between" covariance matrix of the k populations, that is, the covariance matrix of means (without reducing to the grand means) and S^* be the *raw* "between" and S the "within" covariance matrices of the k samples. Almost everywhere, S^* is at least p.s.d. of rank $q = \min(p, k)$ and S is p.d. Also Σ is p.d. and Σ^* is at least p.s.d. of rank, say, $r \leqq \min(p, k)$. To test the hypothesis that Σ^* is of a specific rank r, that is, the $p \times k$ matrix of means is of rank r, the technique of an earlier paper, (S. N. Roy, "On a heuristic method of test construction and its use in multivariate analysis", *Ann. Math. Stat.*, June, 1953.), is used, leading to a critical region in terms of the characteristic roots of the matrix S^*S^{-1}, which are all nonnegative and of which q roots are, almost everywhere, positive. Some properties of the test are also discussed.

33. On the Monotonic Character of the Power of a Test of Independence in Multivariate Analysis. S. N. Roy, University of North Carolina.

A test has been offered (S. N. Roy, "On a heuristic method of test construction and its use in multivariate analysis", *Ann. Math. Stat.*, June, 1953.) for the hypothesis H_0 of independence of two sets of variates, p and q in number (with a joint $(p + q)$-variate normal distribution), the critical region of the test being: $\theta_p \geqq c$, where $p \leqq q$, and θ_p is the largest characteristic root of the sample matrix $S_{11}^{-1} S_{12} S_{22}^{-1} S_{12}'$, and S_{11} and S_{22} are the sample covariance matrices of the p-set and the q-set and S_{12} is the sample covariance matrix between the p-set and the q-set, and where $P(\theta_p \geqq c \mid H_0) = \alpha$ (say). Almost everywhere, the $(p + q)$th order sample covariance matrix is p.d. and S_{12} is of rank p and thus all the p characteristic roots are positive. If we denote by Σ_{11}, Σ_{22} and Σ_{12} the corresponding population roots, then assuming that the $(p + q)$th population covariance matrix is p.d., all the p characteristic roots of the population matrix $\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}'$ are nonnegative. It is well known and has also been shown (see above reference) that the power of the test, that is, $P(\theta_p \geqq c \mid H)$, is a function of just the characteristic roots of the population matrix $\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}'$. It is shown in the present paper that it is a monotonically increasing function of each of these characteristic roots, which incidentally proves that the test is an unbiased one.

34. The Asymptotic Variance of Estimates of the Mean Life of a Radioactive Source (Preliminary Report.) Richard F. Link, Princeton University.

Suppose the number of particles which disintegrate in K nonoverlapping time intervals of equal length t is recorded. Suppose further that these particles come from a source with mean life τ or from a background of constant intensity α. The intensity of the background may or may not be known. Let τ be estimated by the method of maximum likelihood. The asymptotic variance of this estimate is calculated for several values of: K, the number of time intervals; Kt/τ, the number of mean lives the source is observed; and α, the intensity of the background.

35. Testing the Equality of Means of Rectangular Populations. ROBERT V. HOGG, State University of Iowa.

Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ be ordered samples from two rectangular populations having equal ranges but possibly different means. Using the likelihood ratio criterion we find that the statistic $t = \max(x_n - x_1, y_m - y_1)/[\max(x_n, y_m) - \min(x_1, y_1)]$ is used to test the hypothesis that the two means are equal. To find the distribution of this ratio under the null hypothesis we proceed as follows. First, show that the ratio and its denominator are stochastically independent using an extension of a theorem of Neyman concerning the independence of sufficient statistics and statistics whose distributions do not involve the parameters. Second, calculate the moments of the numerator and of the denominator and thus, by dividing, those of the ratio. Third, observe from these moments the interesting fact that the distribution function of the ratio is a combination of the discrete and continuous types; namely $0, t < 0; 2nmt^{n+m-2}/(n + m)(n + m - 1), 0 \leqq t < 1; 1, 1 \leqq t$. This problem is extended to more than two populations.

36. Structure of the Sample Space for Group Organization Theory. LEO KATZ AND JAMES H. POWELL, Michigan State College.

An organization of n individuals is connected by t directional bonds between pairs. Any particular configuration of the bonds produces a directed graph. In the social psychological applications, certain functions of graphs are employed. These are random variables over the graphs, considered as points in an appropriate sample space. The context of the particular psychological investigation may induce a conditioning of the sample space. In the null case, each directed graph of t joins on n nodes is equally likely. These are partitioned disjointly and exhaustively, first, by point-wise restrictions on the outgoing lines and, second, by further point-wise restrictions on incoming lines. An unpublished result of Katz and Powell on graph theory gives the number of points in a second-order subspace. The second-order subspaces in a first-order space are obtained by standard combinatorial methods. In special cases, many second-order spaces are isomorphic in sets; consequently, calculations may be materially abridged. Applications are given to classes of unsolved problems of group organization theory. (Work sponsored by Office of Naval Research).

37. A Family of Cumulative Frequency Functions for J-shaped Frequency Functions. C. W. TOPP AND F. C. LEONE, Case Institute of Technology.

A three-parameter family of cumulative frequency functions is presented. These cover a wide variety of J-shaped frequency functions. Upon testing a number of J-shaped empirical distributions it was found that many of these had third and fourth moments within the range covered by this family of curves. These empirical distributions are composed primarily of life testing and failure data. A graph of α_3^2 and δ has been prepared. This includes some members of the family of frequency functions for different values of the parameters. After a proper choice of a specific function from the graph, the cumulative frequency function can be used directly for the chi square test of goodness of fit.

38. Multilayer Significance Procedures. (Preliminary Report.) JOHN W. TUKEY, Princeton University.

Even when satisfactory confidence procedures are available for multiple comparisons, there is some real need, and more supposed need, for significance procedures. The framework of multilayer significance procedures includes the procedures proposed by Fisher, Newman, Duncan, Tukey (2 versions, one with minor modification), Keuls, Nandi, and Cornfield and co-workers (at least in part) among others, (It is easy to imagine procedures which do not fit into its framework.) and is patterned after Duncan's recent discussion. Conceptually, at least, it involves testing every subgroup of the k determinations for *apparent significance*. A subgroup which, *together with all* subgroups which include it, is apparently significant, is adjudged *significant*. Various kinds of levels of significance are defined. As John Mandel pointed out in connection with the author's first multiple comparison procedure, the overall null hypotheses most likely to cause error are those in which the determinands ("true" values of the determinations) are equal in pairs, these common values being widely separated. If WSD_j is the allowance for j determinations based on the studentized range, then $\frac{1}{2}(WSD_k + WSD_h)$ may be used to test the range of subgroups of h from a group of k without exceeding the nominal error rate. A gap procedure may be added.

39. Estimation in Truncated Multivariate Normal Distributions. A. C. COHEN, JR., University of Georgia.

This paper represents an extension of results which the author presented before a joint meeting of I.M.S. and the Biometric Society in Washington, D. C., April 30, 1953, concerning bivariate normal distributions. Maximum likelihood estimators are derived for parameters of a multivariate normal population which are functions of a random sample in which one of the variates has been subjected to a truncation at known terminals. Single and double truncations both for known and unknown numbers of eliminated observations are considered. The estimators are reduced to simple algebraic forms for easy application to practical problems. Asymptotic variances and covariances of the estimates are obtained from the likelihood information matrices.

40. The Extrema of Certain Functionals of Distribution Functions. (Preliminary Report.) WASSILY HOEFFDING, University of North Carolina.

Two types of problems are considered. *Problem I.* Let $\varphi(F) = \int \cdots \int K(x_1, \cdots, x_n)$ $dF(x_1) \cdots dF(x_n)$, where F is a cumulative distribution function (cdf) on the real line and K a given function. Let D be the class of all cdf's $\underline{F}(x)$ with $\int x^i \, dF(x) = c_i$, $i = 1, \cdots, r$, where $c_1, \cdots, c_r$ are given numbers. To determine $\sup_{F \varepsilon D} \varphi(F)$. *Problem II.* Let $\underline{F} = (F_1, \cdots, F_n)$ be a vector of n cdf's on the real line, $\varphi(\underline{F}) = \int \cdots \int K(x_1, \cdots, x_n)$ $dF_1(x_1) \cdots dF_n(x_n)$, and let V be the class of vectors $\underline{F}$ with $\int x^i \, dF_j(x) = c_{ij}$, $i = 1, \cdots,$ r; $j = 1, \cdots, n$, where the c_{ij} are given. To determine $\sup_{\underline{F} \varepsilon V} \varphi(\underline{F})$. For Problem II it is found that if $\varphi(\underline{F})$ is continuous with respect to the metric $d(\underline{F}, G) = \max_j \sup_x | F_j(x) - G_j(x) |$, then $\sup_{\underline{F} \varepsilon V} \varphi(\underline{F}) = \sup_{\underline{F} \varepsilon V_{r+1}} \varphi(\underline{F})$, where V_{r+1} is the subclass of V where the components of $\underline{F}$ are step functions with at most $r + 1$ steps. For Problem I a similar reduction to discrete distributions is possible only under more restrictive assumptions. Preliminary results for certain cases where K takes on the values 0 or 1 have been obtained. The results permit a sharpening of inequalities of the Tchebycheff type when the chance variable involved is assumed to be a sum of independent chance variables.

41. Probability Distributions Related to Random Transformations of a Finite Set. (Preliminary Report.) HERMAN RUBIN AND ROSEDITH SITGREAVES, Stanford University.

Let X be a set of n elements, and let $\mathfrak{J}$ be a set of transformations of X into X. For a given $x \varepsilon X$ and $T \varepsilon \mathfrak{J}$, the smallest set of elements $y \varepsilon X$ closed under T and including x, is called the structure in T containing x.

Let $\underline{m}$ be the number of structures in T, $\underline{c}$ be the number of elements in the structure containing x, $\underline{s}$ be the number of successors of x (including x), and $\underline{p}$ the number of predecessors of x (including x). We assume that elements are selected at random from $X \times \mathfrak{I}$, each pair (x, T) having probability $1/nt$ of being chosen where t is the number of transformations in $\mathfrak{I}$. For each of these functions, exact probability distributions, together with asymptotic expressions for these probabilities as n becomes large, have been found when $\mathfrak{I}$ is the set of all transformations of X into X, and when $\mathfrak{I}$ is restricted to transformations for which an element $x \varepsilon X$ has either zero or k immediate predecessors. Asymptotic expressions for several of these probability distributions have been obtained when $\mathfrak{I}$ is the set of transformations for which an element x has no more than k immediate predecessors.

42. **Characterization of Tolerance Regions.** D. A. S. FRASER, University of Toronto.

A distribution-free or nonparametric tolerance region for a class of distributions $\{P_x^\theta/\theta \varepsilon \Omega\}$ over $\mathfrak{X}(\mathfrak{A})$ is defined as a mapping $S(x_1, \cdots, x_n)$ from $\mathfrak{X}^n$ into $\mathfrak{A}$ for which the distribution of $P_x^\theta(S(X_1, \cdots, x_n))$ induced by the distribution P_x^θ for each x_i is independent of $\theta \varepsilon \Omega$. If $\varphi_y(x_1, \cdots, x_n)$ is the characteristic function of the set $S(x_1, \cdots, x_n)$ then a necessary and sufficient condition that $S(x_1, \cdots, x_n)$ be distribution-free is that there be a sequence $\alpha_1, \alpha_2, \cdots$ such that $\varphi_{x_{n+1}}(x_1, \cdots, x_n) - \alpha_1, \varphi_{x_{n+1}}(x_1, \cdots, x_n)\varphi_{x_{n+2}}(x_1, \cdots, x_n) - \alpha_2, \cdots$ are unbiased estimates of zero over $\mathfrak{X}^{n+1}, \mathfrak{X}^{n+2}, \cdots$.

43. **A Nonparametric Model for the Linear Hypothesis.** D. A. S. FRASER, University of Toronto.

If the errors of a linear hypothesis design are assumed to have a spherically symmetric joint distribution, then an orthogonal rotation putting the problem in canonical form permits the use of standard methods: rank tests, most powerful tests for specific alternatives, or tests based on substitution of order statistics from an independent normal sample.

44. **On the Analysis of Diurnal Fluctuations in Physiological States and Performance. (Preliminary Report.)** E. CHRISTINE KRIS, Illinois Institute of Technology and University of Chicago.

In studies dealing with the quantitative relationship between physiological states and work performed on various tasks over a perod of time, the problem of locating optimal periods for testing arises. Previous studies have utilized either the method of constant interval or random time sampling. We know, however, that certain physiological states exhibit periodic variations adapted to some cycle; for example, body-temperature fluctuations coincide with the day-and-night as well as with the menstrual cycle. Now if variations in work-output and performance on tests can be related to these known metabolic cycles, prediction of low and high points on a person's curve should be possible in terms of expected changes which can be calculated from the beginning of an iterative series to later portions of it. In this vein a time series analysis of diurnal variations in body-temperature, heart-rate and a work-output test based on five measures per day was undertaken upon data gathered on a female subject over a period of three months. For this a punch-card technique was used. Significant auto- and cross correlations are indicated. In addition a two-way analysis of variance showed the following: the within day variation is five times greater than the day to day variation in all three variables, although both are significant at the .01 level. The greater part of the variation is contributed by the consistently lower measures obtained in the early morning and late night hours. Since the peak periods differ over the middle of the day in the three variables, no one best testing time for all can be established. It is however apparent that time-portions of established cycles can be used as indices of known conditions of variation.