ON THE REDUCED MOMENT PROBLEM

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1. Summary. For a special class of cumulative distribution functions which are solutions of a given reduced moment problem (cf. paragraph 3, pages 27 and 28, of [4]), the well known expression for the least upper bound of the absolute difference between *any* two solutions of the same reduced moment problem is improved upon by the introduction of a constant nonnegative multiplier which is smaller than unity in the case of the special class of solutions. Useful properties of the determinantal form of the classical expression for the least upper bound are derived. The numerical value of the constant multiplier is computed in the case of a well known class of cumulative distribution functions.

In addition, a simple method is given for constructing, over a finite range, an infinite set of continuous and differentiable cumulative distribution functions which are solutions of the same reduced moment problem when one such solution is known. The new expression for the least upper bound, when applied to members of the constructed class of continuous solutions, may be helpful in deriving *general*, but crude, inequalities among orthogonal polynomials over a *finite* interval.

2. Introduction. Let $\Phi(x)$ be a cumulative distribution function (cdf) (by this we mean a nonnegative, nondecreasing function, which need however not be normalized) defined on the interval $a \leqq x \leqq b$, where either or both of a and b may be infinite. If either (or both) of a and b is (are) finite, we may speak of the range as being infinite provided we define

$$\Phi(x) = \begin{cases} 0 & \text{for} \quad x < a \\ \mu_0 & \text{for} \quad x \geqq b. \end{cases} \tag{1}$$

We assume further that

(i) $\Phi(x)$ has at least $n + 1$ points of increase in the interval $[a, b]$.

(ii) The moments of $\Phi(x)$, defined by the Stieltjes integrals

$$\mu_r = \int_a^b x^r \, d\Phi(x)$$
$$= \int_{-\infty}^{\infty} x^r \, d\Phi(x)$$

exist for $r = 0, 1, 2, \cdots, 2n$.

Let $\Psi(x)$ be another cdf defined for $a' \leqq x \leqq b'$ and having all the properties of $\Phi(x)$ mentioned above (with obvious modifications.) If the corresponding

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moments of $\Phi(x)$ and $\Psi(x)$, for $r = 0, 1, \cdots, 2n$, are equal respectively, then we have the inequality

$$(2) \quad |\Phi(x) - \Psi(x)| \leqq \{Q'_{n+1}(x)Q_n(x) - Q_{n+1}(x)Q'_n(x)\}^{-1} = \rho_n(x),$$

say, where $Q_r(x)$, $(r = 0, 1, 2, \cdots, n)$, is a polynomial of exact degree r given by the denominator of the rth convergent of the continued fraction associated with the integral $\int_{-\infty}^{\infty} [d\phi(x)/(x - t)]$ or $\int_{-\infty}^{\infty} [d\Psi(x)/(x - t)]$ and possessing the following three properties:

$$(3) \qquad \int_{-\infty}^{\infty} Q_r(x)Q_s(x)\, d\phi(x) = \int_{-\infty}^{\infty} Q_r(x)Q_s(x)\, d\psi(x) = \begin{cases} 0 & \text{for } r \neq s \\ c_r & \text{for } r = s \end{cases}$$

where c_r is a positive constant,

and $$Q'_{r+1}(x)Q_r(x) - Q_{r+1}(x)Q'_r(x) > 0,$$

$$(4) \qquad Q_r(x) \equiv (\alpha_r x + \beta_r)Q_{r-1}(x) - Q_{r-2}(x), \quad r = 1, 2, 3, \cdots n,$$

where α_r and β_r are determinable coefficients independent of x and where $Q_{-1}(x) \equiv 0$, $Q_0(x) \equiv 1$ and $\alpha_1 = 1/\mu_0$.

Inequality (2) is easily deducible from the well known Tchebycheff inequalities [1]. A proof of (2), based on a method due to Stieltjes [8], is given by Uspensky in appendix II of [10].

The function $\rho_n(x)$ appearing on the right-hand side of (2) has also been expressed in forms (a) and (b) below [4] (cf. [4], pp. 42–44 and p. 72 for derivation and equivalence of forms (a) and (b)).

(a)

$$(5) \qquad \rho_n(x) = \{\textstyle\sum_{r=0}^{n} \omega_r^2(x)\}^{-1}$$

where $\omega_r(x)$, $(r = 0, 1, 2, \cdots n)$, is the orthonormal polynomial of exact degree r associated with $d\Phi(x)$ or $d\Psi(x)$, that is, with the moment sequence $\{\mu_r\}$, $r = 0, 1, 2, \cdots, 2n$.

(b)

$$(6) \qquad \rho_n(x) = -\Delta_n/D_n(x),$$

where

$$(7) \qquad \Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} = |\mu_{i+j}|_{i,j=0,1,2,\cdots,n}$$

and

$$(8) \qquad D_n(x) = \begin{vmatrix} 0 & 1 & x & \cdots & x^n \\ 1 & \mu_0 & \mu_1 & \cdots & \mu_n \\ x & \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} = \begin{vmatrix} 0 & x^j \\ x^i & \mu_{i+j} \end{vmatrix}_{i,j=0,1,2,\cdots,n} .$$

The equivalence of (2) and (5) may be established by the properties (3) and (4) of $Q_p(x)$ and the well known properties of the orthonormal set $\omega_r(x)$ (cf. pp. 41–42 of [9], in particular, equations 3.2.1 and 3.2.4).

3. Some properties of the determinants Δ_n and $D_n(x)$. Two properties of Δ_n and $D_n(x)$, believed to be new, are derived in this section. These properties, especially useful in the numerical evaluation of the two determinants, are given as two theorems.

THEOREM 1. *The determinant Δ_n is an arithmetical invariant under a transformation of the origin of moments, that is, if*

$$\mu_k(a) = \int_{-\infty}^{\infty} (x - a)^k \, d\Phi(x)$$

then

$$\Delta_n = | \mu_{i+j} | = | \mu_{i+j}(a) |_{i,j=0,1,2,\ldots,n} ,$$

for any arbitrary real number, a.[1]

PROOF. We apply to the determinant Δ_n in (7) the following two difference operations.

(a) For each element μ_{k+j} in the $(k + 1)$st row of Δ_n substitute

$$(9) \qquad \nu_{k+j}(a) = \int_{-\infty}^{\infty} x^j (x - a)^k \, d\phi(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^r \mu_{k-r+j} .$$

Obviously such a substitution leaves Δ_n unchanged in value as it merely adds to the $(k + 1)$st row a linear sum of multiples of the preceding rows.

If this difference operation is applied first to the $(n + 1)$st row, then to the nth row and so on, the determinant Δ_n will retain the same value. Thus we have

$$(10) \qquad \Delta_n = | \mu_{i+j} | = | \nu_{i+j}(a) |_{i,j=0,1,2,\ldots,n} .$$

[1] J. Geronimus, in his paper, "On some persymmetric determinants formed by the polynomials of M. Appell," *J. London Math. Soc.*, Vol. 6 (1951) pp. 55–59, obtained indirectly and as by-products of a solution to an extremal problem results equivalent to Theorems 1 and 2 given in the present paper. Geronimus assumes an absolutely continuous cumulative distribution function but his proofs apply equally well to the general case treated above. The author is indebted to Dr. H. P. Mulholland of the University of Birmingham who in a letter to the author, dated 25 October 1953, outlined the relevant results of Geronimus.

(b) For each element $\nu_{i+k}(a)$ in the $(k+1)$st column of (10) substitute

$$\xi_{i+k}(a) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^r \nu_{i+k-r}(a) \tag{11}$$

and apply this substitution first to the $(n+1)$st column, then to the nth column, and so on. The resulting determinant is again equal to Δ_n as each column is replaced by a linear sum of multiples of the preceding columns.

The element $\xi_{i+j}(a)$ in the resulting determinant is, in virtue of (9), (10) and (11) equal to $\mu_{i+j}(a)$. Hence,

$$\Delta_n = | \mu_{i+j}(a) |_{i,j=0,1,2,\dots,n}, \tag{12}$$

as required.

THEOREM 2. *The determinant $D_n(x)$ may also be expressed in the form* $-| \mu_{i+j}(x) |_{i,j=1,2,3,\dots,n}$. *That is, the following relation holds identically for all x,*

$$\begin{vmatrix} 0 & 1 & x & \cdots & x^n \\ 1 & \mu_0 & \mu_1 & \cdots & \mu_n \\ x & \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} = - \begin{vmatrix} \mu_2(x) & \mu_3(x) & \cdots & \mu_{n+1}(x) \\ \mu_3(x) & \mu_4(x) & \cdots & \mu_{n+2}(x) \\ \vdots & \vdots & & \vdots \\ \mu_{n+1}(x) & \mu_{n+2}(x) & \cdots & \mu_{2n}(x) \end{vmatrix} \tag{13}$$

where $\mu_i = \mu_i(0)$ for $i = 0, 1, 2, \cdots 2n$.

PROOF. Applying two differencing operations to the left-hand side of (13) similar to the differencing operations used in the proof of Theorem 1, replacing a by x throughout (with obvious modifications due to the difference between the orders of the determinants $D_n(x)$ and $\Delta_n(x)$) relation (13) may be easily established.

In view of Theorems 1 and 2 one obtains a new expression for $\rho_n(x)$ given by

$$\rho_n(x) = | \mu_{i+j}(a) |_{i,j=0,1,2,\dots,n} / | \mu_{i+j}(x) |_{i,j=1,2,3,\dots,n} ; \tag{14}$$

where a is any real number and where again

$$\mu_r(x) = \int_{-\infty}^{\infty} (t - x)\, d\phi(t).$$

The inequality

$$| \Phi(x) - \Psi(x) | \leqq \rho_n(x) \tag{15}$$

where $\rho_n(x)$ is given by any of the expressions specified above, gives a gauge of the error involved in approximating to an unknown cdf by another known cdf whose moments, up to order $2n$, are equal to the corresponding moments of the unknown function. However, the gauge given by (15) is usually much larger than the actual error as is well known in practical problems of this type. We shall show below that under certain conditions inequality (15) can be improved upon.

4. Improvements upon inequality (15). Let $\Phi(x)$ and $\Psi(x)$ be defined as in Section 2. We need only consider the case when $\Phi(x)$ and $\Psi(x)$ are not identically equal, as the special case is trivial. Consider the two functions

$$\Phi(x) - \Psi(x) + A\Psi(x) \tag{16}$$

and

$$A\Psi(x) \tag{17}$$

where A is a positive constant chosen so that (16) is a cdf with at least $n + 1$ points of increase. As both $\Phi(x)$ and $\Psi(x)$ are never decreasing functions, A need not exceed unity. Thus we may write the condition

$$0 < A \leqq 1. \tag{18}$$

Since $\Phi(x)$ and $\Psi(x)$ have identical moments up to order $2n$, the functions (16) and (17) have also identical moments up to order $2n$. Therefore, the functions

$$[\phi(x) - (1 - A)\Psi(x)]/A \equiv \Phi_1(x), \qquad \text{say} \tag{19}$$

and

$$\Psi(x) \tag{20}$$

have equal moments up to order $2n$, which in turn are identical with the corresponding moments of $\Phi(x)$. Further, the functions (19) and (20) are nonnegative, never-decreasing, and have at least $n + 1$ points of increase each. Therefore, by (15), we have

$$|\Phi_1(x) - \Psi(x)| \leqq \rho_n(x)$$

which reduces to

$$|\Phi(x) - \Psi(x)| \leqq A\rho_n(x) \tag{21}$$

where $\rho_n(x)$ is given by any of the expressions (2), (5), (6) and (14) of Section 2.

Similarly, by considering the functions

$$\frac{1}{B}[\psi(x) - \Phi(x) + B\phi(x)]$$

and $\Phi(x)$ we obtain the inequality

$$|\Phi(x) - \Psi(x)| \leqq B\rho_n(x) \tag{22}$$

where $0 < B \leqq 1$.

When A and B are each equal to unity, inequalities (21) and (22) will be identical with (15). However, when it is possible to choose either or both of A and B less than unity, then we have an improvement upon inequality (15) (see Section 5 below). When both of A and B may be chosen less than unity, the inequality involving the smaller of the two constants naturally would lead to a better improvement.

When the functions $\Phi(x)$ and $\Psi(x)$ are assumed to be continuous and differentiable for all x in the ranges $[a, b]$ and $[a', b']$ respectively, (at a, a' and b, b' continuity and differentiability only on the right and on the left respectively are assumed) (in which case each of the functions $\phi(x)$ and $\Psi(x)$ possess an infinite number of points of increase), one may obtain the smallest positive number that could be assigned to the constant A as follows.

Since A must be chosen so that $\Phi_1(x)$ becomes a nonnegative, never-decreasing function with at least $n + 1$ points of increase, we must choose A so that

$$f'(x) \equiv \Phi'(x) - \Psi'(x) + A\Psi'(x) \geqq 0$$

for all x, and such that $f(x)$ has at least $n + 1$ points of increase. The minimum value of A which makes $f(x)$ a nonnegative, and never-decreasing function is

$$A_0 = 1 + \text{l.u.b.}(-\Phi'(x)|\,\Psi'(x)) \tag{23}$$

where the least upper bound is taken over min $(a, a') \leqq x \leqq$ max (b, b') provided $-\Phi'(x)|\,\Psi'(x)$ is bounded above.

If we choose $A = A_0 + \epsilon$ where ϵ is an arbitrary positive number, then $f(x)$ will have more than $n + 1$ points of increase. Hence

$$|\,\Phi(x) - \Psi(x)| \leqq (A_0 + \epsilon)\rho_n(x)$$

and since ϵ is arbitrary, we have,

$$|\,\Phi(x) - \Psi(x)| \leqq A_0\rho_n(x) \tag{24}$$

where A is given by (23), provided $-\Phi'(x)|\,\Psi'(x)$ is bounded above. That $f(x)$ has an infinite number of points of increase when $0 < A = A_0 \leqq 1$ is, in fact, obvious because $\Phi(x)$ and $\Psi(x)$ are distinct continuous and differentiable cumulative distribution functions possessing equal moments up to order $2n$ [10].

Similarly we may choose the minimum value of B given by

$$B_0 = 1 + \underset{c \leqq x \leqq d}{\text{l.u.b.}}(-\Psi'(x)/\Phi'(x)) \tag{25}$$

provided $-\Psi'(x)/\Phi'(x)$ is bounded above.

Therefore, we may write instead of (15) the inequality

$$|\,\Phi(x) - \Psi(x)| \leqq \min\,(A_0\,, B_0)\rho_n(x)) \tag{26}$$

where A_0 and B_0 are given by (23) and (25).

The above results may be summarized by the following theorem.[2]

THEOREM 3. *If $\Phi(x)$, $a \leqq x \leqq b$ and $\Psi(x)$, $a' \leqq x \leqq b'$ are two nonidentical continuous and differentiable cumulative distribution functions whose corresponding moments up to order $2n$ exist and are equal and if at least one of the ratios*

[2] Theorem 3 and its proof were first given, in a slightly modified form, in an appendix to a Ph.D. thesis submitted by the author to the University of London in May, 1950.

$-\Phi'(x)/\Psi'(x)$ *and* $-\Psi'(x)/\Phi'(x)$ *is bounded above for* $c \leqq x \leqq d$, *where* $c = \min(a, a')$ *and* $d = \max(b, b')$, *then*

$$|\Phi(x) - \Psi(x)| \leqq C\rho_n(x)$$

where

$$0 < C = 1 + \min\left\{\underset{c \leqq x \leqq d}{\text{l.u.b.}}(-\Phi'(x)/\Psi'(x)), \underset{c \leqq x \leqq}{\text{l.u.b.}}(-\Psi'(x)/\Phi'(x))\right\}$$

and where

$$\rho_n(x) = |\mu_{i+j}|_{i,j=0,1,2,\ldots,n} / |\mu_{i+j}(x)|_{i,j=1,2,\ldots,n}$$

and μ_r *and* $\mu_r(t)$ *are the* rth *moments about the origin and* t *respectively.*

A second theorem, which leads to an improvement of inequality (26) when each of A_0 and B_0 exists and is less than unity, may be stated as follows.

THEOREM 4. *If* $\Phi(x)$ *and* $\Psi(x)$ *satisfy the conditions of Theorem* 3, *and if both of* A_0 *and* B_0 *of equations* (23) *and* (25) *exist, then*

$$|\Phi(x) - \Psi(x)| \leqq K\rho_n(x) \tag{27}$$

where

$$0 < K = A_0B_0/(A_0 + B_0 - A_0B_0) \leqq \min(A_0, B_0) \leqq 1. \tag{28}$$

PROOF OF THEOREM 4. The two functions

$$\Phi_1(x) = \{\Phi(x) - \Psi(x) + A_0\Psi(x)\}/A_0$$

and

$$\Psi_1(x) = \{\Psi(x) - \Phi(x) + B_0\Phi(x)\}/B_0$$

satisfy all the conditions for the application of inequality (15) and possess the same moments up to order $2n$ as the original functions $\Phi(x)$ and $\Psi(x)$. Hence

$$|\Phi_1(x) - \Psi_1(x)| = (A_0 + B_0 - A_0B_0)|\Phi(x) - \Psi(x)|/(A_0B_0) \leqq \rho_n(x).$$

Since $0 < A_0 \leqq 1$ and $0 < B_0 \leqq 1$, then $(A_0 + B_0 - A_0B_0)/A_0B_0 > 0$ and therefore

$$|\Phi(x) - \Psi(x)| \leqq \{A_0B_0/(A_0 + B_0 - A_0B_0)\}\rho_n(x) = K\rho_n(x)$$

as required.

If either A_0 or B_0 is equal to unity, inequality (28) reduces to inequality (26). However, if each of A_0 and B_0 is less than unity, inequality (28) is an improvement upon inequality (26) since, in this case,

$$0 < K = A_0B_0/(A_0 + B_0 - A_0B_0) < \min(A_0, B_0).$$

Examples are given in the following two sections which show the existence of classes of cdf's for which inequality (27) is an improvement upon inequality (15). At most (even when the least upper bound of the ratio $-\Phi'(x)/\Psi'(x)$ and that of its reciprocal do not exist) the maximum value that both A_0 and B_0 can take is unity.

5. An application of inequality (27). As an illustration of a case when all the constants A_0, B_0 and K exist and each is less than unity let us apply inequality (27) to the class of cumulative distribution functions given on page 106 of [2]. In particular consider any two cdf's of this class defined by

$$\Phi(x) = \int_0^x ke^{-\alpha t^\lambda}(1 + \epsilon_1 \sin (\beta t^\lambda \tan \lambda\pi))\, dt$$

and

$$\Psi(x) = \int_0^x ke^{-\alpha t^\lambda}(1 + \epsilon_2 \sin (\beta t^\lambda \tan \lambda\pi))\, dt,$$

where $0 \leqq x \leqq \infty$; $k > 0$; $\alpha > 0$; $0 < \lambda < \frac{1}{2}$; $\beta = \alpha \tan \lambda\pi$; and $-1 < \epsilon_2 < \epsilon_1 < 1$.

The two cdf's $\Phi(x)$ and $\Psi(x)$ are distinct and have equal moments of all orders irrespective of the distinct values of ϵ_1 and ϵ_2 in the open interval $(-1, 1)$. Applying inequality (27) one obtains the inequality

$$|\Phi(x) - \Psi(x)| \leqq K/\textstyle\sum_{r=0}^{\infty} \omega_r^2(x)$$

with $A_0 = (\epsilon_1 - \epsilon_2)/(1 + \epsilon_1)$, $B_0 = (\epsilon_1 - \epsilon_2)/(1 - \epsilon_2)$, and $K = \frac{1}{2}(\epsilon_1 - \epsilon_2)$, and where $\omega_r(x)$ is the orthonormal polynomial of degree r associated with the given distribution functions. The series $\sum_{r=0}^{n} \omega_r^2(x)$ converges as $n \to \infty$ because the two distributions are distinct. Obviously inequality (27) applies in this case and gives improved limits for $|\Phi(x) - \Psi(x)|$, in comparison with inequality (15). In this case both A_0 and B_0, and therefore K, exist. Other applications are given in the following section.

6. Class of cumulative distribution functions possessing moments equal up to a specified order to those of a given cdf. We consider the case of a cdf which is continuous and differentiable and has a finite range. The extension to a cdf with a finite number of points of increase is simple. However, I have not succeeded yet in extending the following results to the case of a cdf with an infinite range.

Let $F(x) = \int_a^x f(t)\, dt$ be any continuous and differentiable cdf with moments μ_r, $r = 0, 1, \cdots$, and $a \leqq x \leqq b$, where both a and b are finite. Let $p_r(x)$, $r = 0, 1, \cdots$, be the set of orthogonal polynomial over the range $[a, b]$ associated with $f(x)$ as a weight function. Then we have the following theorem.

THEOREM 5. *For all ϵ such that $|\epsilon| \leqq 1$, the class of cumulative distribution functions*

$$F_n(x, \epsilon) = \int_a^x f(t)(+ \epsilon p_{n+i}(t)/L_{n+i})\, dt, \qquad a \leqq x \leqq b \tag{29}$$

where

$$L_r = \underset{a \leqq x \leqq b}{\text{l.u.b.}} |p_r(x)| \quad \text{and} \quad i = 1, 2, \cdots,$$

possess the same moments up to order n, provided the required moments exist.

The proof is obvious since $| \epsilon p_{n+i}(x) |/L_{n+i} \leqq 1$ and because of the orthogonality property of the polynomials $p_r(x)$. Over an infinite range the polynomials $p_r(x)$ are not bounded and therefore the theorem does not hold.

It is to be noted above that $F_n(x, 0) = F(x)$ and therefore the set $F_n(x, \epsilon)$ possesses the same moments as $F(x)$ up to order n. Thus given any cdf over a finite interval which possesses moments up to order $2n + i$ with $i \geqq 1$, one may construct by Theorem 5 an infinite number of cumulative distribution functions which possess the same moments up to order n as the given cdf. The existence of the moments up to order $2n + 1$ implies the existence of the associated orthogonal polynomial $p_{n+1}(x)$.

The class of cumulative distribution functions $F_n(x, \epsilon)$ provides a suitable set for illustrating the improvement obtained by introducing the constant K in inequality (27). As an illustration consider the two functions $F_{2n}(x, 0)$ and $F_{2n}(x, \epsilon_1)$ where $\epsilon_1 \neq 0$. Applying inequality (27) to these two functions one gets $A_0 = | \epsilon_1 |/(1 + | \epsilon_1 |)$, $B_0 = | \epsilon_1 |$, and $K = | \epsilon_1 |/2$,
and therefore

$$(30) \qquad | F_{2n}(x, \epsilon_1) - F_{2n}(x, 0)| \leqq | \epsilon_1 |/(2\textstyle\sum_{r=0}^{n} \omega_r^2(x))$$

where

$$\omega_r^2(x) = p_r^2(x) \Big/ \int_a^b p_r^2(x)f(x)\, dx.$$

This represents a reduction in the bound for the absolute difference which is at least equal to half the bound given by inequality (15).

Incidentally inequality (30), upon substitution from (29) reduces to a *general* inequality among the orthogonal polynomials for all sets of orthogonal polynomials over a finite interval, say $[a, b]$. In this particular case the inequality is given by

$$(31) \qquad \left| \int_a^x (p_{2n+1}(t)/L_{2n+1})f(t)\, dt \right| \leqq \tfrac{1}{2} \cdot \frac{1}{\sum_{r=0}^{n} \omega_r^2(x)} .$$

Applying inequalities (27) to other pairs of the class $F_{2n}(x, \epsilon)$ one can obtain other general inequalities among orthogonal polynomials. Of course, because of the generality of the class $F_{2n}(x, \epsilon)$, these may be expected to be rather crude, in the sense that the right-hand side of (31) is in *special* cases (e.g. Legendre polynomials) much larger than the left-hand side in (31) (cf. [9], chapter 7, p. 154).

In the particular case when $p_r(x)$ is the Legendre polynomial of order r defined over the interval $[-1, 1]$ by

$$\int_{-1}^{1} p_r(x)p_s(x)\, dx = \begin{matrix} 2/(2r + 1) & \text{if } r = s \\ 0 & \text{if } r \neq s, \end{matrix}$$

inequality (31) becomes

$$| p_{2n+2}(x) - p_{2n}(x)| \leqq (4n + 3)/\textstyle\sum_{r=0}^{n} (2r + 1)p_r^2(x),$$

which is not as strong an inequality as those known in the case of Legendre polynomials.

Theorem 5 is particularly of interest in respect of the prevailing practice of fitting a Pearsonian cdf to an unknown cdf when the ranges of the Pearsonian cdf is finite. If the fitted cdf is denoted by $F(x)$ and if say the first four moments have been fitted, then any member of the class $F_{3+i}(x, \epsilon)$, with $| \epsilon | \leqq 1$, $i = 1, 2, \cdots$, has the same first four moments as the unknown cdf. There is no indication, however, that the fitted Pearson cdf gives a better approximation than other members of the class $F_{3+i}(x, \epsilon)$. In other words, for a finite range, Theorem 5 leads to a method of fitting which is more general than that provided by the usual methods. It may be possible in particular cases to choose a value of ϵ which gives a better fit than $F(x)$.

The class $F_n(x, \epsilon)$ of Theorem 5 may be extended further into the class $F_n(x, \epsilon_1, \epsilon_2, \cdots, \epsilon_m)$ with $| \epsilon_1 | + | \epsilon_2 | + \cdots + | \epsilon_m | \leqq 1$ where

$$F_n(x, \epsilon_1, \epsilon_2, \cdots, \epsilon_m) = \int_a^x \left(1 + \sum_{i=1}^{m} (\epsilon_i p_{n+i}(t)/L_{n+i})\right) f(t)\, dt$$

provided the required number of moments, necessary for the existence of $p_{n+m}(x)$ exist.

Finally, Theorem 5 proves the existence of an infinite class of continuous cumulative distribution functions which are solutions of a given reduced moment problem *over a finite range* provided that there is at least one continuous cdf which is a solution to the given moment problem.

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