employing the symbolic notation of Section 3. By Theorem 2, the last member of the above equation is seen to reduce to a β_1-variate with parameters $\frac{1}{2}(n - p - 1)$, $\frac{1}{2}p$, so that (S/S') follows the Beta distribution of the first kind with parameters $\frac{1}{2}(n - p - 1)$ and $\frac{1}{2}p$. This result has been obtained by Wilks [4], by deriving expressions for the moments of the distribution of (S/S'). The above is a simple and direct method of establishing the distribution of the statistic (S/S').

From (5.6) it readily follows that (S/S') is equal to $1/(1 + CD_p^2)$, so that the latter is distributed as a β_1-variate with parameters $\frac{1}{2}(n - p - 1)$ and $\frac{1}{2}p$, whence CD_p^2 is a β_2-variate with parameters $\frac{1}{2}p$ and $\frac{1}{2}(n - p - 1)$, leading to the distribution as shown in (5.8).

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ON SOME FUNCTIONS INVOLVING MILL'S RATIO[1]

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1. Introduction and Summary. In this note, we prove that, for all (finite) values of h,

$$\psi(h) = \frac{m_2}{m_1^2} = \frac{1 - h(Z - h)}{(Z - h)^2}, \tag{1}$$

is monotonic increasing[2], that

$$2m_1^2 - m_2 > 0, \tag{2}$$

and that

$$1 < \psi(h) < 2, \tag{3}$$

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[2] While this paper was being considered by the referees, the authors learned of a proof by Des Raj [5] which establishes the monotonic property of $\psi(h)$ for negative values of h.

where Z is the reciprocal of Mill's ratio,

$$Z(h) = e^{-h^2/2} \Big/ \int_h^\infty e^{-t^2/2}\, dt, \tag{4}$$

and where m_1 and m_2 are respectively the first and second moments of a singly truncated normal distribution about the point of truncation.

The function $\psi(h)$ arises in connection with maximum likelihood estimation of population parameters from singly truncated normal samples (cf. for example [1] and references cited therein). The inequality (2) arises in connection with three-moment estimates based on samples of the same type (cf. [2] and [3]).

2. Some preliminary results. To prove that $\psi(h)$ is monotonic increasing, it is sufficient to establish that $\psi'(h) > 0$. Differentiating (4) gives

$$Z' = Z(Z - h). \tag{5}$$

Using this result and differentiating (1), we obtain

$$\psi'(h) = [hZ(Z - h)^2 - 3Z(Z - h) + 2]/(Z - h)^3. \tag{6}$$

For subsequent use, it can be shown (cf. for example Sampford [4]) that

$$(7)\quad 0 < Z' < 1, \qquad \lim_{h\to\infty} Z' = 1, \qquad \lim_{h\to-\infty} Z' = 0,$$

$$(8)\quad (Z - h) > 0, \qquad \lim_{h\to\infty} (Z - h) = 0, \qquad \lim_{h\to-\infty} (Z - h) = \infty,$$

$$(9)\quad h(Z - h) < 1, \qquad \lim_{h\to\infty} h(Z - h) = 1, \qquad \lim_{h\to-\infty} h(Z - h) = -\infty.$$

3. Proof that $\psi'(h) > 0$. Since from (8), $(Z - h) > 0$, a sufficient condition that $\psi'(h) > 0$ is that

$$\theta(h) = [hZ(Z - h)^2 - 3Z(Z - h) + 2] > 0.$$

To prove this latter inequality, we first write $\theta(h)$ in the form

$$\theta(h) = -Z(Z - h)^3 + Z^2(Z - h)^2 - 3Z(Z - h) + 2. \tag{10}$$

Using (7), (8), and (9), it can be shown that $\lim_{h\to\infty}\theta(h) = 0$. Therefore to prove $\theta(h) > 0$, it would be sufficient though not necessary to show that $\theta'(h) < 0$. Using (5), we find

$$\theta'(h) = -Z(Z - h)^4 - Z^2(Z - h)^3 + 2Z^3(Z - h)^2 - 5Z^2(Z - h) + 3Z. \tag{11}$$

Proof that $\theta'(h) < 0$ does not follow readily, so we introduce the auxiliary function

$$g(h) = e^{\omega(h)}\theta(h), \tag{12}$$

where

$$\omega(h) = -\int_0^h Z(x)\, dx, \tag{13}$$

and thus

$$\omega'(h) = -Z(h). \tag{14}$$

Since $e^{\omega(h)} > 0$, a necessary and sufficient condition that $\theta(h) > 0$ is that $g(h) > 0$. It can be shown that $\lim_{h\to\infty} g(h) = 0$, and consequently to prove $\theta(h) > 0$, it is sufficient to show that $g'(h) < 0$.

On differentiating (12), we obtain

$$g'(h) = e^{\omega(h)}[\theta'(h) - Z\theta(h)]. \tag{15}$$

Again using the fact that $e^{\omega(h)} > 0$, it follows that $g'(h) < 0$ if and only if

$$\theta'(h) - Z\theta(h) < 0. \tag{16}$$

From (10) and (11), we have

$$\begin{aligned}\theta'(h) - Z\theta(h) &= -Z(Z-h)^4 + Z^3(Z-h)^2 - 2Z^2(Z-h) + Z \\ &= Z\{[Z(Z-h)-1]^2 - (Z-h)^4\} \\ &= Z\{Z(Z-h) - 1 - (Z-h)^2\}\{Z(Z-h) - 1 + (Z-h)^2\}, \\ &= Z\{h(Z-h) - 1\}\{(2Z-h)(Z-h) - 1\}.\end{aligned} \tag{17}$$

Sampford (loc. cit.) proved[3]

$$(2Z - h)(Z - h) - 1 > 0, \qquad \text{for all finite } h. \tag{18}$$

From (4), $Z > 0$, and from (9), $h(Z - h) - 1 < 0$. Therefore $\theta'(h) - Z\theta(h) < 0$, and accordingly $\psi'(h) > 0$ for all finite h. With this result, the proof that $\psi(h)$ is monotonic increasing, for all finite h, is complete.

4. Proof that $2m_1^2 - m_2 > 0$. As shown in [1], m_1 and m_2 may be expressed as

$$m_1 = \sigma[Z - h], \qquad m_2 = \sigma^2[1 - h(Z - h)], \tag{19}$$

and it follows that

$$2m_1^2 - m_2 = \sigma^2[2(Z-h)^2 + h(Z-h) - 1]. \tag{20}$$

Since $\sigma^2 > 0$, it is sufficient to demonstrate that the expression within brackets on the right side, above, is positive. After certain simplifications, we obtain

$$\begin{aligned}[2(Z-h)^2 + h(Z-h) - 1] &= [(Z-h)(2Z - 2h + h) - 1] \\ &= [(2Z-h)(Z-h) - 1] > 0,\end{aligned}$$

which is Sampford's inequality (18), and the proof is complete.

5. Proof that $1 < \psi(h) < 2$. From (19), (5), and (7), it follows that

$$m_2 - m_1^2 = \sigma^2[1 - Z(Z-h)] = \sigma^2(1 - Z') > 0.$$

[3] This inequality can also be established by employing the multiplier $e^{\omega(h)}$ in a role similar to that in which it appears above.

Using this result and inequality (2), which was established in Section 4, we have $m_1^2 < m_2 < 2m_1^2$, and the required result follows immediately on dividing by m_1^2. We also note that $\lim_{h\to-\infty}\psi(h) = 1$, and $\lim_{h\to\infty}\psi(h) = 2$. Thus no narrower limits can be found. To obtain these limits, we use the result, $\lim_{h\to-\infty} Z/h = 0$, which follows from $\lim_{h\to-\infty} Ze^{h^2/2} = \left[\int_{-\infty}^{\infty} e^{-t^2/2}\,dt\right]^{-1} = (\sqrt{2\pi})^{-1}$. Thereby we have

$$\lim_{h\to-\infty} \psi(h) = \lim_{h\to-\infty} \frac{1/h^2 - Z/h + 1}{(Z/h - 1)^2} = \frac{0 - 0 + 1}{(0 - 1)^2} = 1,$$

and

$$\lim_{h\to\infty} \psi(h) = \lim_{h\to\infty} \frac{e^{\omega(h)}[1 - h(Z - h)]}{e^{\omega(h)}(Z - h)^2}$$

which is indeterminate of the form 0/0 as given. Using L'Hospital's rule and making certain obvious simplifications, we obtain

$$\lim_{h\to\infty} \psi(h) = \lim_{h\to\infty} \frac{-2}{Z(Z - h) - 2} = \frac{-2}{1 - 2} = 2.$$

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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Ithaca meeting of the Institute, March 18–20, 1954)

1. Confidence Region Procedures Based on the Logarithm of the Likelihood.

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Let $f(x, \theta_0)$ be a probability function where θ_0 is one of a set of permissible parameter points $\theta = (\theta_1, \cdots, \theta_h)$ contained in some subspace of R_h. A sample $(x_1, \cdots, x_n)$ of size n is observed and a set of h functions, $\varphi_j = (1/\sqrt{n}) \sum_{i=1}^{j} k_{ij}L_i$, $j = 1, \cdots, h < n$, computed, where $L_i = \partial \log f/\partial\theta_i$, $f = \Pi_{i=1}^{n} f(x_i, \theta)$, and the k_{ij} are chosen so that $E(\varphi_j) = 0$, $E(\varphi_i\varphi_j) = \delta_{ij}$. For a given sample, the φ_j are functions of θ, and $(\varphi_1(\theta), \cdots, \varphi_h(\theta))$ is a point in the pivotal space $\Phi \subseteq R_h$. If a region W can be constructed in Φ so that $\Pr\{(\varphi_1, \cdots, \varphi_h) \varepsilon W\} = \alpha$ independently of θ_0, the corresponding region in the parameter