In this case the characteristic function $\psi(t, \theta, \sigma) = e^{i\theta t - \sigma|t|}$, so that $h(t\sigma) = e^{-\sigma|t|} = e^{-|\sigma t|}$. It is immediately verified that $h(t\sigma_1)\, h(t\sigma_2) = h(t\sigma_3)$, where $\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$. Therefore, by Theorem 4.1, $\bar{x}$ has a Cauchy distribution. In fact, since in this case $E(e^{it\bar{x}}) = [h(t\sigma/n)]^n = h(t\sigma)$, $\bar{x}$ has exactly the same distribution as does x itself, namely $p(x, \theta, \sigma)$. It follows from Theorem 4.2 that $\bar{x}$ is a density unbiased point estimate of the location parameter θ.

It is readily seen that Theorems 4.1 and 4.2 are valid also when $\bar{x}$ is replaced by any linear homogeneous estimate $\sum_{i=1}^{n} a_i x_i$, where $\sum_{i=1}^{n} a_i = 1$.

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SOME PROPERTIES OF BETA AND GAMMA DISTRIBUTIONS

By M. V. JAMBUNATHAN

University of Mysore

1. Summary and Introduction. The object of this paper is to present certain important properties of the Gamma distribution and the two kinds of Beta distributions, and to indicate certain useful applications of these two to sampling problems. The distribution of the Studentised D^2-statistic under the null hypothesis is obtained in two different ways.

2. The Gamma distribution. If a random variable x has probability density

$$\frac{1}{\Gamma(a)} e^{-x} x^{a-1}, \qquad 0 \leqq x < \infty, \tag{2.1}$$

then x is said to have a *Gamma distribution*; furthermore, x is called a *Gamma variate* with parameter a, and is symbolically written $\gamma(a)$. The Gamma distribution is known to possess, among others, the mean conserving property (m.c.p.), provided the variates are independent.[1] Symbolically

$$\gamma(a) + \gamma(b) = \gamma(a + b).$$

If x is a Gamma variate with parameter a, then $2x$ is distributed as χ^2 with $2a$ degrees of freedom.

3. Beta distribution of the first kind. A random variable having the probability density

$$\frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1}, \qquad 0 \leqq x \leqq 1, \tag{3.1}$$

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[1] This necessary proviso which was left out in the original paper has since been inserted at the suggestion of a referee to whom the author is thankful for several other suggestions besides.

is said to have a *Beta distribution of the first kind*, and x is called a Beta variate of the first kind with parameters a and b, written symbolically $\beta_1(a, b)$.

Under certain mild conditions, the product of two independent Beta variates of the first kind also follows the same distribution, as stated in the following theorem.

THEOREM 1. *If x and y are two independent Beta variates of the first kind with parameters (a, b) and $(a + b, c)$, then the product xy is a $\beta_1(a, b + c)$ variate.*

The theorem is proved by transforming variables, the transformation being $u = xy, v = (y - u)/(1 - u)$. When this is done, u and v are independent variables and u is a $\beta_1(a, b + c)$ variate, v is a $\beta_1(b, c)$ variate.

The result of the theorem can be generalized, as shown by Rao [2]. We then get

THEOREM 2. *If $x_1, x_2, \cdots, x_p$ be p independent Beta variates of the first kind with parameters (a_i, b_i) for $i = 1, 2, \cdots, p$, and if $a_{i+1} = a_i + b_i$ for $i = 1, 2, \cdots, (p - 1)$, then the product $x_1x_2 \cdots x_p$ is a β_1 variate with parameters a_1 and $b = \sum_{i=1}^p b_i$.*

This theorem is proved by the repeated application of Theorem 1. The properties of Gamma variates in conjunction with the results of Theorems 1 and 2 lead to the following

THEOREM 3. *If x_i, y_i for $i = 1, 2, \cdots, p$ be independent Gamma variates with parameters a_i, b_i which are connected by the relation $a_{i+1} = a_i + b_i$, for $i = 1, 2, \cdots, (p - 1)$, then the product $\prod_{i=1}^p x_i/(x_i + y_i)$ is a β_1 variate with parameters (a_1, b) where $b = \sum_{i=1}^p b_i$.*

This may be expressed symbolically as

$$\frac{\gamma(a_1)}{\gamma(a_1 + b_1)} \cdot \frac{\gamma(a_1 + b_1)}{\gamma(a_1 + b_1 + b_2)} \cdots \frac{\gamma(a_1 + b_1 + \cdots + b_{p-1})}{\gamma(a_1 + b_1 + \cdots + b_p)}$$

$$= \beta_1(a_1, \sum_{i=1}^p b_i) = \frac{\gamma(a_1)}{\gamma(a_1 + b_1 + \cdots + b_p)}.$$

Since the variates are independent, and have Gamma distributions, we note that $x_i/(x_i + y_i)$ is a $\beta_1(a_i, b_i)$ variate. Application of Theorem 2 establishes the result.

4. Beta distribution of the second kind. A random variable x is said to have a *Beta distribution of the second kind* with parameters a and b if x has the probability density

$$\frac{x^{a-1}}{B(a, b)(1 + x)^{a+b}}, \qquad 0 \leqq x < \infty,$$

and is symbolically written $\beta_2(a, b)$.

The two kinds of Beta distributions are connected by a transformation. If x is a $\beta_1(a, b)$ variate, then $u = (1 - x)/x$ is a $\beta_2(b, a)$ variate. Conversely, if u is a $\beta_2(b, a)$ variate, then $x = 1/(1 + u)$ is a $\beta_1(a, b)$ variate. This mutual relationship enables us to deduce a new result connecting the two distributions.

THEOREM 4. *If $u = (1 + y)/(1 + x)$, and if u is a $\beta_1(b - d, d)$ variate while y*

is a $\beta_2(a, b)$ variate, then x is a $\beta_2(a + d, b - d)$ variate, provided that u and y are independent.

Since y is a $\beta_2(a, b)$ variate, $1/(1 + y)$ is a $\beta_1(b, a)$ variate. Therefore $1/(1 + x) = u[1/(1 + y)]$ is the product of two β_1 variates with parameters $(b - d, d)$ and (b, a), and is, therefore, by Theorem 1, equal to a $\beta_1(b - d, a + d)$ variate. Hence x is a $\beta_2(a + d, b - d)$ variate.

5. Distribution of Studentised D^2 under the null hypothesis. The results deduced in the preceding sections enable us to obtain easily the derivation of the distribution of the Studentised D^2-statistic. Consider two independent random samples of size n_1 and n_2 drawn from two multivariate normal populations. It is assumed in what follows that the two populations possess the same covariance matrix $\Lambda = \| \sigma_{ij} \|$ for $i, j = 1, 2, \cdots, p$.

Let $n = n_1 + n_2$ and $1/c = 1/n_1 + 1/n_2$. Also let $\bar{x}_i^{(1)}$ and $\bar{x}_i^{(2)}$ be the mean values of the ith character for the first and the second samples respectively.

Let S_{ij} be the pooled corrected sum of products within the two samples for the variates x_i and x_j; that is, $S_{ij} = S_{ij}^{(1)} + S_{ij}^{(2)}$, where

$$S_{ij}^{(1)} = \sum_{r=1}^{n_1} (x_{ir}^{(1)} - \bar{x}_i^{(1)})(x_{jr}^{(1)} - \bar{x}_j^{(1)}), \tag{5.1}$$

the upper suffix (1) indicating the first sample; $S_{ij}^{(2)}$ is defined similarly. The statistic $s_{ij} = S_{ij}/(n - 2)$ is an estimate of σ_{ij}, the covariance between x_i and x_j in the two populations. If (S^{ij}) be the matrix inverse to (S_{ij}), and (s^{ij}) the matrix inverse to (s_{ij}), we might call (s^{ij}) the estimate matrix of the covariances. The statistic appropriate for testing the significance of the difference between the means of the several characters is given by

$$D_p^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} s^{ij} d_i d_j, \tag{5.2}$$

where d_i is the difference between the means of the ith character in the two samples, and d_j the difference between the means of the jth character, see [1] and [3]. This is called the Studentised D^2-statistic. Two ways of obtaining the derivation of the distribution of this statistic are given below.

First Method. Let

$$R_k = | S_{ij} |_k / | S_{ij} |_{k-1}, \qquad k = 1, 2, \cdots, p, \tag{5.3}$$

where, following Rao [2], the symbol $| S_{ij} |_k$ is used to denote the determinant of order k of the matrix $(S_{ij})_k$ obtained by giving i, j the values $1, 2, \cdots, k$, $(k \leqq p)$. The symbol $| S_{ij} |_0$ may, without causing inconsistency, be defined to be equal to unity. Also let

$$R'_k = | S'_{ij} |_k / | S'_{ij} |_{k-1}, \qquad k = 1, 2, \cdots, p, \tag{5.4}$$

where S'_{ij} is the corrected sum of products, that is,

$$S'_{ij} = \sum (x_i - \bar{\bar{x}}_i)(x_j - \bar{\bar{x}}_j)$$

summed over all the $(n_1 + n_2)$ sample elements, $\bar{x}_i$ being the pooled means of both the samples. It is easy to show that $S'_{ij} = S_{ij} + cd_i d_j$, whence

$$R'_k = \frac{| S_{ij} + cd_i d_j |_k}{| S_{ij} + cd_i d_j |_{k-1}}. \tag{5.5}$$

But

$$| S_{ij} + cd_i d_j |_k = | S_{ij} |_k \left(1 + c \sum_1^k \sum_1^k d_i d_j S_k^{ij}\right) = | S_{ij} |_k (1 + CD_k^2), \tag{5.6}$$

where (S_k^{ij}) is the matrix inverse to $(S_{ij})_k$, and $C = (n - 2)c$. The quantities R_k and $(R'_k - R_k)$ when divided by $2\sigma_k^2$ are *independently* distributed as Gamma variates with parameters $\frac{1}{2}(n - k - 1)$ and $\frac{1}{2}$ respectively for $k = 1, 2, \cdots, p$, if the true means of the characters of the populations are equal. Hence R_k/R'_k is a $\beta_1(\frac{1}{2}(n - k - 1), \frac{1}{2})$ variate. But by (5.3) and (5.5) in combination with (5.6),

$$R_k/R'_k = (1 + CD_{k-1}^2)/(1 + CD_k^2), \qquad k = 1, 2, \cdots, p. \tag{5.7}$$

It can be shown that $R_p, R_{p-1}, \cdots, R_2, R_1$ are all independently distributed, as are also

$$\frac{1 + CD_{p-1}^2}{1 + CD_p^2}, \frac{1 + CD_{p-2}^2}{1 + CD_{p-1}^2}, \cdots, \frac{1 + CD_1^2}{1 + CD_2^2}, \frac{1}{1 + CD_1^2}.$$

Now putting $a = \frac{1}{2}(p - 1)$, $b = \frac{1}{2}(n - p)$, $d = \frac{1}{2}$ in Theorem 4, we find that, *assuming* CD_{p-1}^2 is a β_2-variate with parameters $\frac{1}{2}(p - 1)$, $\frac{1}{2}(n - p)$, the quantity CD_p^2 is a β_2-variate with parameters $\frac{1}{2}p$, $\frac{1}{2}(n - p - 1)$. It is well known that CD_1^2 is a β_2-variate with parameters $\frac{1}{2}$, $\frac{1}{2}(n - 2)$. Hence, by the principle of *finite induction*, CD_p^2 is a β_2-variate with parameters $\frac{1}{2}p$, $\frac{1}{2}(n - p - 1)$, and therefore its distribution is

$$\frac{1}{B(\frac{1}{2}p, \frac{1}{2}(\text{n} - p - 1))} \cdot \frac{(CD_p^2)^{(p-2)/2} d(CD_p^2)}{(1 + CD_p^2)^{(n-1)/2}},$$

which simplifies to

$$\frac{1}{B(\frac{1}{2}p, \frac{1}{2}(n - p - 1))} \cdot \frac{C^{p/2}(D_p^2)^{(p-2)/2} dD_p^2}{(1 + CD_p^2)^{(n-1)/2}}. \tag{5.8}$$

SECOND METHOD. Another method which is capable of yielding some additional results is the following. From (5.3) it easily follows that

$$S = | S_{ij} |_p = R_p \cdot R_{p-1} \cdots R_1. \tag{5.9}$$

Similarly

$$S' = | S_{ij} + cd_i d_j |_p = R'_p \cdot R'_{p-1} \cdots R'_1. \tag{5.10}$$

Therefore

$$\frac{S}{S'} = \prod_{i=p}^{1} \frac{R_i}{R'_i} = \prod_{i=p}^{1} \beta_1\left(\frac{n - i - 1}{2}, \frac{1}{2}\right), \tag{5.11}$$

employing the symbolic notation of Section 3. By Theorem 2, the last member of the above equation is seen to reduce to a β_1-variate with parameters $\frac{1}{2}(n - p - 1)$, $\frac{1}{2}p$, so that (S/S') follows the Beta distribution of the first kind with parameters $\frac{1}{2}(n - p - 1)$ and $\frac{1}{2}p$. This result has been obtained by Wilks [4], by deriving expressions for the moments of the distribution of (S/S'). The above is a simple and direct method of establishing the distribution of the statistic (S/S').

From (5.6) it readily follows that (S/S') is equal to $1/(1 + CD_p^2)$, so that the latter is distributed as a β_1-variate with parameters $\frac{1}{2}(n - p - 1)$ and $\frac{1}{2}p$, whence CD_p^2 is a β_2-variate with parameters $\frac{1}{2}p$ and $\frac{1}{2}(n - p - 1)$, leading to the distribution as shown in (5.8).

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ON SOME FUNCTIONS INVOLVING MILL'S RATIO[1]

By D. F. Barrow and A. C. Cohen, Jr.

University of Georgia

1. Introduction and Summary. In this note, we prove that, for all (finite) values of h,

$$\psi(h) = \frac{m_2}{m_1^2} = \frac{1 - h(Z - h)}{(Z - h)^2}, \tag{1}$$

is monotonic increasing[2], that

$$2m_1^2 - m_2 > 0, \tag{2}$$

and that

$$1 < \psi(h) < 2, \tag{3}$$

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[2] While this paper was being considered by the referees, the authors learned of a proof by Des Raj [5] which establishes the monotonic property of $\psi(h)$ for negative values of h.