SEQUENTIAL PROCEDURES IN CERTAIN COMPONENT OF VARIANCE PROBLEMS[1]

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1. Introduction and Summary. The primary purpose of this paper is to discuss certain sequential procedures for discriminating between two values for the ratio of variance components in a simple one-way classification. In order to clarify the presentation of the three procedures discussed, they will first be applied (in Section 2) to the problem of discrimination between two values for the ratio of variances in two distinct normal populations. The arguments in Section 3 closely parallel those in Section 2, and the form of the two sections has been made as similar as possible, to emphasize this parallelism. Section 4 contains some formulae used in calculating approximate average sampling numbers presented in the earlier sections.

2. Comparison of two variances. Let Π_1, Π_2 be two normal populations with variances σ_1^2, σ_2^2 respectively. It is desired to discriminate between the hypotheses $H'(\sigma_1^2/\sigma_2^2 = \theta')$ and $H''(\sigma_1^2/\sigma_2^2 = \theta'')$. It has been shown [1] that the following sequential procedure (I) will give approximate probabilities $\alpha'(\alpha'')$ of choice of $H''(H')$ when $H'(H'')$ is valid.

Procedure (I). "Start by taking samples of two (x_{11}, x_{12}; x_{21}, x_{22}) from Π_1, Π_2 respectively. At each subsequent stage (if required) one further individual is taken from each population.

"At the $(n - 1)$st stage calculate

$$g_n = \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)^2 \Big/ \sum_{i=1}^{n} (x_{2i} - \bar{x}_2)^2. \tag{1}$$

Accept H' if $\qquad p(g_n \mid \theta'')/p(g_n \mid \theta') < A.$

Accept H'' if $\qquad p(g_n \mid \theta'')/p(g_n \mid \theta') > B.$

Otherwise proceed to the nth stage."

In the above statement $A = \alpha''/(1 - \alpha')$, $B = (1 - \alpha'')/\alpha'$; and

$$p(g_n \mid \theta) = \frac{\theta^{\frac{1}{2}(n-1)}}{B(\frac{1}{2}(n-1), \frac{1}{2}(n-1))} \cdot \frac{g_n^{\frac{1}{2}(n-3)}}{(\theta + g_n)^{n-1}} \tag{2}$$

so that

$$p(g_n \mid \theta'')/p(g_n \mid \theta') = (\theta''/\theta')^{\frac{1}{2}(n-1)}((\theta' + g_n)/(\theta'' + g_n))^{n-1}.$$

The validity of procedure (I) can be demonstrated by showing that it is equivalent to a sequential likelihood ratio test based on the sequence $\{g_n\}$.

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These variables, however, are not independent and do not have a common distribution so that the approximate formulae for the ASN (average sample number) given by Wald [2] cannot be applied. Two alternative procedures, (II) and (III), will now be described. It is possible to use the approximate ASN formulae for these procedures.

PROCEDURE (II). "At each stage take a sample of r individuals from each of Π_1 and Π_2. Calculate g_r by (1) for these $2r$ values and call it $g_{r,n}$ if based on the sample available at the nth stage.

Accept H' if $$\prod_{j=1}^{n} [p(g_{r,j} \mid \theta'')/p(g_{r,j} \mid \theta')] < A.$$

Accept H'' if $$\prod_{j=1}^{n} [p(g_{r,j} \mid \theta'')/p(g_{r,j} \mid \theta')] > B.$$

Otherwise proceed to the $(n + 1)$st stage."

In the above statement $p(g_{r,j} \mid \theta)$ is given by (2) with g_n replaced by $g_{r,j}$ and n by r. Hence

$$\prod_{j=1}^{n} [p(g_{r,j} \mid \theta'')/p(g_{r,j} \mid \theta')] = (\theta''/\theta')^{\frac{1}{2}n(r-1)} \prod_{j=1}^{n} ((\theta' + g_{r,j})/\theta'' + g_{r,j}))^{r-1}.$$

PROCEDURE (III). Here we use the same sampling scheme as in (I), but a different decision rule.

"Start by taking samples of two (x_{11}, x_{12}; x_{21}, x_{22}) from Π_1, Π_2 respectively. At each subsequent stage (if required) one further individual is taken from each population.

"At the $(n - 1)$st stage calculate

$$g'_n = (x_{11} + x_{12} + \cdots + x_{1,(n-1)} - (n - 1)x_{1n})^2/ (x_{21} + x_{22} + \cdots + x_{2,(n-1)} - (n - 1)x_{2n})^2. \tag{3}$$

Accept H' if $$\prod_{j=2}^{n} [p(g'_j \mid \theta'')/p(g'_j \mid \theta')] < A.$$

Accept H'' if $$\prod_{j=2}^{n} [p(g'_j \mid \theta'')/p(g'_j \mid \theta')] < B.$$

Otherwise proceed to the nth stage."

In the above statement

$$p(g'_j \mid \theta) = p(g_2 \mid \theta)_{g_2 = g'_j} \qquad \text{for all } j \tag{4}$$

where $p(g_2 \mid \theta)$ is defined as in (2), so that

$$\prod_{j=2}^{n} [p(g'_j \mid \theta'')/p(g'_j \mid \theta')] = \left(\frac{\theta''}{\theta'}\right)^{\frac{1}{2}(n-1)} \prod_{j=2}^{n} [(\theta' + g'_j)/(\theta'' + g'_j)].$$

In Procedure (II) the $g_{r,j}$'s are evidently mutually independent and have a common distribution. In Procedure (III) the quantities

$$y_{t,j} = (x_{t1} + x_{t2} + \cdots + x_{t,(j-1)} - (j - 1)x_{tj})/\sqrt{j(j - 1)}$$

are mutually independent normal variables [3] each with expected value zero and variance $\sigma_t^2 (t = 1, 2)$. Hence the g_j''s are mutually independent and have the common distribution (4).

In both (II) and (III) therefore we can use Wald's approximate formulae for the ASN. The results of applying this formulae are given below in (6) to (9). In these formulae

$$a = \alpha' \log B + (1 - \alpha') \log A; \qquad b = \alpha'' \log A + (1 - \alpha'') \log B$$

and

$$E(\nu_1, \nu_2; \theta) = \frac{\theta^{\frac{1}{2}\nu_2}}{B(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \int_0^\infty \frac{g^{\frac{1}{2}\nu_1 - 1}}{(\theta + g)^{\frac{1}{2}(\nu_1+\nu_2)}} \log \left[\left(\frac{\theta''}{\theta'}\right)^{\frac{1}{2}\nu_2} \left(\frac{\theta' + g}{\theta'' + g}\right)^{\frac{1}{2}(\nu_1+\nu_2)} \right] dg. \tag{5}$$

For Procedure (II), the approximate ASN are

$$2ra/E(r - 1, r - 1; \theta'), \quad \text{when } H' \text{ is true,} \tag{6}$$

$$2rb/E(r - 1, r - 1; \theta''), \quad \text{when } H'' \text{ is true.} \tag{7}$$

For Procedure (III), the approximate ASN are

$$2 + 2a/E(1, 1; \theta'), \quad \text{when } H' \text{ is true,} \tag{8}$$

$$2 + 2b/E(1, 1; \theta''), \quad \text{when } H'' \text{ is true.} \tag{9}$$

Table I shows the ASN calculated by these formulae and also the sample sizes required by the fixed sample tests having the same values of α' and α''. In the cases shown in the table $\alpha' = \alpha''$ and in the present instance this implies that the ASN will be the same when either H' or H'' is true; this common value is shown in the table.

Comparing (6) with (8), or (7) with (9), we see that the ASN for (II) with $r = 2$ will be almost double that for (III). This is to be expected since (II) with

TABLE I

Procedure	$\alpha' = \alpha'' = 0.05$			$\alpha' = \alpha'' = 0.01$		
	$\theta''/\theta' = 2$	2.5	3	$\theta''/\theta' = 2$	2.5	3
(II) $r = 2$	355.0	203.8	142.3	602.8	346.0	241.6
(II) $r = 3$	200.2	115.2	80.6	339.9	195.6	136.9
(II) $r = 4$	158.4	91.3	64.0	269.1	155.1	108.7
(II) $r = 6$	128.7	74.3	52.2	218.6	126.3	88.6
(II) $r = 12$	106.7	61.7	43.4	181.3	104.9	73.8
(II) $r = \infty$	90.0	52.2	36.8	152.8	88.7	62.6
(III)	179.5	103.9	73.1	303.4	175.0	122.8
Fixed sample	184	108	76	328	212	150
Girshick	89.3	51.6	36.2	150.5	86.3	60.2

$r = 2$ neglects all comparisons between successive groups of 2 observations from each population. As r increases, the proportion of neglected comparisons diminishes and the approximate ASN also diminishes. When the ASN is not large compared with $2r$, the number of additional observations obtained at each stage, the values have of course only a formal significance. It is however of interest to note that as r tends to infinity, the approximate ASN tend respectively to limiting values of

$$(10) \qquad -4a/\log \frac{(\theta' + \theta'')^2}{4\theta'\theta''}; \; 4b/\log \frac{(\theta' + \theta'')^2}{4\theta'\theta''}.$$

These values are shown in Table I on the line $r = \infty$. It may be conjectured that these values might, in fact, be related to the approximate ASN for Procedure (I). It will be noted that they are considerably lower than the figures for (II) with r equal to 2 or 3, or for (III). These latter are in fact only slightly lower than the fixed sample sizes, and it is likely that when θ lies between θ' and θ'' the ASN for these schemes would substantially exceed the fixed sample sizes.

Girshick [3] has constructed a test discriminating between the hypotheses

$$\tilde{H}'(\sigma_1 = \sigma_1^0, \sigma_2 = \sigma_2^0) \text{ and } \tilde{H}''(\sigma_1 = \sigma_2^0, \sigma_2 = \sigma_1^0).$$

This test is based on the inequalities

$$\log A < \tfrac{1}{2}\left[(\sigma_1^0)^{-2} - (\sigma_2^0)^{-2}\right] \sum_{j=2}^{n} (y_{1j}^2 - y_{2j}^2) < \log B.$$

The application of this test requires a knowledge of the actual values of σ_1^0 and σ_2^0, not only the ratio $\theta^0 = (\sigma_1^0/\sigma_2^0)^2$. However the formula for the approximate ASN depends only on this ratio. It is

$$2 - 4a/[(\theta^0)^2 + (\theta^0)^{-2} - 2] \text{ if } \tilde{H}' \text{ is true,}$$

and $-b$ replaces a if $\tilde{H}''$ is true.

The ratio

$$\left(\frac{\bar{\theta}''}{\bar{\theta}'}\right) = \frac{(\sigma_1/\sigma_2)^2 \text{ when } \tilde{H}'' \text{ is true}}{(\sigma_1/\sigma_2)^2 \text{ when } H' \text{ is true}}$$

is equal to $(\theta^0)^2$. Hence if we take $\theta^0 = \sqrt{\theta''/\theta'}$ we can compare the approximate ASN for Girshick's test with those for the various procedures described above. Such ASN values are shown in the last line of Table I. It is to be expected that they will be smaller than corresponding ASN for the other procedures, since (i) σ_1^0 and σ_2^0 must be known to apply Girshick's test, and (ii) the test is a sequential probability ratio test based on the independent pairs of random variables (y_{1j}, y_{2j}). The closeness of the ASN's for Girshick's test and for Procedure (II) with $r = \infty$ is noteworthy.

3. Comparison of variance components. We will consider a one-way classification by groups and denote the internal (within-group) variance by σ^2 and the external (between-group) variance by σ_0^2. This means that if x_{ti} is the ith observation from the tth group then $x_{ti} = A + u_t + z_{ti}$ where the u's and z's are mutually independent normal variables each with expected value zero and variances σ_0^2, σ^2 respectively; A is a constant. It is desired to discriminate between the hypotheses $H'(\sigma_0^2/\sigma^2 = \delta')$ and $H''(\sigma_0^2/\sigma^2 = \delta'')$ with risks of error α', α'' as defined in Section 2.

Procedures analogous to (I) and (II) above have been discussed in [4]. There are two simple alternative ways of constructing sequential procedures in this problem: (a) taking a fixed number, k, of groups and, at each stage taking one further additional observation from each group; (b) at each stage selecting (at random) a further group (or set of r groups) and taking a fixed number, m, of observations from each group. It was found in [4] that Procedure (I) used in conjunction with (a) will not terminate with probability one unless δ' (or δ'') is equal to zero. Procedures (II) and (III) are not applicable in conjunction with (a) as the successive sets of observations are not independent of each other. This section will be concerned exclusively with sequential procedures constructed according to system (b).

PROCEDURE (I). "Start by taking two groups and m observations in each group. At each subsequent stage (if required) one further group is chosen and m observations taken in it.

"At the $(n - 1)$st stage calculate

$$G_n = m \sum_{t=1}^{n} (\bar{x}_t - \bar{x})^2 \Big/ \sum_{t=1}^{n} \sum_{i=1}^{m} (x_{ti} - \bar{x}_t)^2. \tag{11}$$

Accept H' if $\quad p(G_n \mid \theta'')/p(G_n \mid \theta') < A.$

Accept H'' if $\quad p(G_n \mid \theta'')/p(G_n \mid \theta') > B.$

Otherwise proceed to the next stage."

In the above statement $\theta' = 1 + m\delta'$, $\theta'' = 1 + m\delta''$.

and

$$p(G_n \mid \theta) = \frac{\theta^{\frac{1}{2}n(m-1)}}{B(\frac{1}{2}(n-1), \frac{1}{2}n(m-1))} \cdot \frac{G_n^{(\frac{1}{2}n-3)}}{(\theta + G_n)^{\frac{1}{2}(nm-1)}} \tag{12}$$

so that

$$p(G_n \mid \theta'')/p(G_n \mid \theta') = [\theta''/\theta']^{\frac{1}{2}n(m-1)} [(\theta' + G_n)/(\theta'' + G_n)]^{\frac{1}{2}(nm-1)}.$$

The validity of Procedure (I) can be demonstrated by showing that it is equivalent to a sequential likelihood ratio test based on the sequence $\{G_n\}$. As in Section 1, Wald's approximate formulae cannot be applied in this case.

PROCEDURE (II). "At each stage take a sample of r groups and take m observations in each group. Calculate G_r by (11) for these mr values and call it $G_{r,n}$ if based on the observations taken at the nth stage.

Accept H' if $\quad \prod_{j=1}^{n} [p(G_{r,j} \mid \theta'')/p(G_{r,j} \mid \theta')] < A.$

Accept H'' if $\quad \prod_{j=1}^{n} [p(G_{r,j} \mid \theta'')/p(G_{r,j} \mid \theta')] > B.$

Otherwise proceed to the $(n + 1)$st stage."

In the above statement $p(G_{r,j} \mid \theta)$ is given by (12) with G_n replaced by $G_{r,j}$ and n by r. Hence

$$\prod_{j=1}^{n} [p(G_{r,j} \mid \theta'')/p(G_{r,j} \mid \theta')] = (\theta''/\theta')^{\frac{1}{2}nr(m-1)} \prod_{j=1}^{n} [(\theta' + G_{r,j})/(\theta'' + G_{r,j})]^{\frac{1}{2}(rm-1)}.$$

Procedure (III). "Start by taking two groups and m observations in each group. At each subsequent stage (if required) one further group is chosen and m observations taken in it.

"At the $(n - 1)$st stage calculate

$$G'_n = m(\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_{n-1} - (n-1)\bar{x}_n)^2/n(n-1) \sum_{i=1}^{m} (x_{ni} - \bar{x}_n)^2.$$

Accept H' if $\prod_{j=2}^{n} [p(G'_j \mid \theta'')/p(G'_j \mid \theta')] < A.$

Accept H'' if $\prod_{j=2}^{n} [p(G'_j \mid \theta'')/p(G'_j \mid \theta')] > B.$

Otherwise proceed to the nth stage."

In the above statement

$$(13) \qquad p(G'_j \mid \theta) = \frac{\theta^{\frac{1}{2}(m-1)}}{B(\frac{1}{2}, \frac{1}{2}(m-1))} \cdot \frac{G_j'^{-\frac{1}{2}}}{(\theta + G'_j)^{\frac{1}{2}m}}$$

so that

$$\prod_{j=2}^{n} \left[\frac{p(G'_j \mid \theta'')}{p(G'_j \mid \theta')}\right] = \left(\frac{\theta''}{\theta'}\right)^{\frac{1}{2}(n-1)(m-1)} \prod_{j=2}^{n} \left[\frac{(\theta' + G'_j)}{(\theta'' + G'_j)}\right]^{\frac{1}{2}m}.$$

(Note that the equation analogous to (4) does not hold.)

In Procedure (II) the $G_{r,j}$'s are evidently mutually independent and have a common distribution. In Procedure (III) the quantities

$$y_j = \sqrt{m}\,(\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_{j-1} - (j-1)\bar{x}_j)/\sqrt{j(j+1)}$$

are mutually independent normal variables each with expected value zero and variance $\sigma^2 + m\sigma_0^2$. Hence the G'_j's are mutually independent and have the common distribution (13). In both (II) and (III) therefore we can use Wald's approximate formulae for the ASN. The results of applying these formulae are given below.

For Procedure (II), the approximate ASN are

$$(14) \qquad mra/E(r-1, r(m-1); \theta'), \qquad \text{when } H' \text{ is true,}$$

$$(15) \qquad mrb/E(r-1, r(m-1); \theta''), \qquad \text{when } H'' \text{ is true.}$$

For Procedure (III), the approximate ASN are

$$(16) \qquad m + ma/E(1, m-1; \theta'), \qquad \text{when } H' \text{ is true,}$$

$$(17) \qquad m + mb/E(1, m-1; \theta''), \qquad \text{when } H'' \text{ is true.}$$

Table II shows the ASN calculated by these formulae and also the sample sizes required by the fixed sample tests having the same values of α' and α''.

As r increases the ASN of (II) decreases. As r tends to infinity the values given by (14) and (15) for the approximate ASN tend respectively to

$$\frac{2ma}{(m-1)\log\frac{\theta''}{\theta'} + m\log\frac{m\theta'}{(m-1)\theta''+\theta'}}, \qquad \frac{2mb}{(m-1)\log\frac{\theta''}{\theta'} + m\log\frac{(m-1)\theta'+\theta''}{m\theta''}} \tag{18}$$

These values are shown in Table II on the lines $r = \infty$.

TABLE II

δ'	δ''	True Hypothesis	Procedure	$\alpha' = \alpha'' = 0.05$						$\alpha' = \alpha'' = 0.01$					
				$m = 2$	3	4	5	6	7	$m = 2$	3	4	5	6	7
0	1	H'	(II) $r = 2$	93.0	69.0	64.0	63.6	64.8	—	158.0	117.2	108.7	107.9	110.1	—
			(II) $r = 3$	65.4	49.6	46.7	46.8	—	—	111.0	84.2	79.3	79.4	—	—
			(II) $r = \infty$	36.8	30.4	29.6	30.1	31.1	32.4	62.6	51.6	50.2	51.1	52.9	55.0
			(III)	73.1	48.2	42.5	41.5	42.0	43.4	122.7	79.8	69.5	66.9	67.2	68.7
		H''	(II) $r = 2$	84.7	51.2	40.1	34.6	—	—	143.8	86.9	68.0	58.8	—	—
			(II) $r = 3$	60.8	36.9	28.6	25.1	—	—	103.3	62.6	48.6	42.6	—	—
			(II) $r = \infty$	36.8	22.9	18.2	15.8	14.4	13.4	62.6	39.0	31.0	26.9	24.4	22.7
			(III)	73.1	42.8	33.8	29.9	28.1	27.2	122.7	70.5	54.5	47.3	43.5	41.3
		Any	Fixed	76	57	52	55	54	56	150	111	100	95	96	105
0	2	H'	(II) $r = 2$	45.5	37.9	38.0	39.7	—	—	77.3	64.4	64.5	67.4	—	—
			(II) $r = \infty$	18.0	17.0	17.8	19.0	20.4	21.7	30.6	28.8	30.2	32.3	34.6	36.9
			(III)	35.6	27.1	26.2	27.2	—	—	59.0	43.9	41.8	42.8	—	—
		H''	(II) $r = 2$	39.3	25.2	20.6	18.2	—	—	66.7	42.9	35.0	31.0	—	—
			(II) $r = \infty$	18.0	11.8	9.6	8.6	7.9	7.4	30.6	20.0	16.4	14.5	13.4	12.6
			(III)	35.6	23.2	19.8	18.6	—	—	59.0	37.2	30.8	28.2	—	—
		Any	Fixed	38	33	32	35	36	42	72	57	56	55	60	63
1	2	H'	(II) $r = 2$	417.6	379.7	404.0	443.6	—	—	709.1	644.8	686.0	753.3	—	—
			(II) $r = 3$	291.4	271.5	292.7	323.9	—	—	494.9	461.1	497.1	550.0	—	—
			(II) $r = 4$	248.9	235.8	255.6	284.1	—	—	423.0	400.6	434.4	482.7	—	—
			(II) $r = \infty$	164.3	163.7	181.7	204.5	229.2	254.7	278.9	277.9	308.6	347.2	389.1	432.5
			(III)	327.9	264.8	263.0	276.3	295.7	—	555.3	447.5	443.8	465.7	497.9	—
		H''	(II) $r = 2$	397.7	335.4	338.4	357.3	—	—	675.4	569.6	574.7	606.8	—	—
			(II) $r = 3$	280.3	240.0	244.2	259.2	—	—	476.1	407.5	414.7	440.1	—	—
			(II) $r = 4$	241.4	208.2	212.7	226.5	—	—	410.2	353.8	361.4	384.8	—	—
			(II) $r = \infty$	164.3	159.3	149.8	161.0	174.6	189.4	278.9	270.5	254.5	274.3	296.5	321.6
			(III)	327.9	251.2	238.9	242.2	—	—	555.3	424.5	402.9	407.9	—	—
		Any	Fixed	332	312	336	370	—	—	664	624	672	740	—	—
1	3	H'	(II) $r = 2$	155.1	148.3	162.5	181.9	—	—	263.4	251.8	275.9	308.8	—	—
			(II) $r = \infty$	60.8	64.4	73.8	84.6	96.0	107.7	103.2	109.4	125.3	143.7	163.1	182.9
			(III)	121.0	103.0	105.8	113.8	—	—	204.1	172.7	176.8	189.7	—	—
		H''	(II) $r = 2$	143.2	121.2	122.1	128.6	—	—	243.1	205.9	207.3	218.3	—	—
			(II) $r = \infty$	60.8	52.7	54.3	57.9	62.4	67.4	103.2	90.0	92.2	98.3	106.0	114.5
			(III)	121.0	94.7	90.9	92.7	—	—	204.1	158.8	151.6	153.9	—	—
		Any	Fixed	124	120	132	150	—	—	246	234	256	285	—	—

It may be conjectured that these values might, in fact, be related to the approximate ASN for procedure (I).

For any given value of m, comparisons between Procedures (II) and (III) and the fixed sample procedure are mostly similar to the comparisons in Section 2, where variances in two different populations were being compared. However there is, in the present case, the possibility of choosing a most suitable value for m. Except for Procedure (II) when H' specifies $\delta' = 0$ and the counter-hypothesis H'' is true, the figures shown in Table II indicate the existence of a minimum ASN corresponding to a fairly small value of m. In the exceptional cases just described, the approximate ASN formulae give results which decrease as m increases. Since the minimum possible sample number r is in fact mr there will be, however, a value of m for which the ASN is actually minimized in these cases also, though it may be expected to be rather higher than in the other cases of Table II.

Usually the cost of choosing a further group and the cost of taking an observation within a group will both enter the calculations so that the minimization of the ASN may not be a primary objective. If, for example, the second of these two costs is much smaller than the first, then values of m rather larger than those minimizing the ASN would be preferable.

4. Formulae used in calculation of approximate ASN. The evaluation of formulae (6) to (9) and (14) to (17) (leading to Tables I and II respectively) involved the calculation of the quantities $E(\nu_1, \nu_2; \theta)$. The formulae used in these calculations are recorded in this Section. From (5)

$$E(\nu_1, \nu_2; \theta) = \frac{\nu_2}{2} \log \frac{\theta''}{\theta'} + \frac{\nu_1 + \nu_2}{2} \frac{\theta^{\frac{1}{2}\nu_2}}{B(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \cdot \int_0^\infty \frac{g^{\frac{1}{2}\nu_1 - 1}}{(\theta + g)^{\frac{1}{2}(\nu_1+\nu_2)}} \log \frac{\theta' + g}{\theta'' + g} \, dg. \tag{19}$$

Expanding the logarithm in the integrand in various ways, the following expressions for $E(\nu_1, \nu_2; \theta)$ are obtained:

$$\frac{\nu_2}{2} \log \frac{\theta''}{\theta'} + \frac{\nu_1 + \nu_2}{2} \sum_{j=1}^{\infty} \frac{1}{j} \cdot \frac{\nu_2(\nu_2 + 2) \cdots (\nu_2 + 2j - 2)}{(\nu_1 + \nu_2) \cdots (\nu_1 + \nu_2 + 2j - 2)} \cdot \left[\left(1 - \frac{\theta''}{\theta}\right)^j - \left(1 - \frac{\theta'}{\theta}\right)^j\right] \quad \text{for } 0 < \frac{\theta''}{\theta} \leqq 2; \;\; 0 < \frac{\theta'}{\theta} \leqq 2. \tag{20}$$

$$\frac{\nu_1}{2} \log \frac{\theta'}{\theta''} + \frac{\nu_1 + \nu_2}{2} \sum_{j=1}^{\infty} \frac{1}{j} \cdot \frac{\nu_1(\nu_1 + 2) \cdots (\nu_1 + 2j - 2)}{(\nu_1 + \nu_2) \cdots (\nu_1 + \nu_2 + 2j - 2)} \cdot \left[\left(1 - \frac{\theta}{\theta''}\right)^j - \left(1 - \frac{\theta}{\theta'}\right)^j\right] \quad \text{for } 0 < \theta/\theta'' \leqq 2; \;\; 0 < \theta/\theta' \leqq 2. \tag{21}$$

With $\theta = \theta'$, (20) gives an expression for $E(\nu_1, \nu_2; \theta')$ valid for $0 < \theta''/\theta' \leqq 2$; (21) gives an expression valid for $0 < \theta'/\theta'' \leqq 2$. Thus one or the other will always be valid, sometimes both. When both are valid, one of the two is usually much more rapidly convergent than the other. In some cases it is advantageous to expand log (θ''/θ') in powers of $(1 - \theta''/\theta')$ or $(1 - \theta'/\theta'')$ and to combine the two infinite series into one. For example, one formula obtained from (20) in this way is

$$E(\nu_1, \nu_2; \theta') \tag{22}$$
$$= -\tfrac{1}{2}\nu_2 \sum_{j=1}^{\infty} \frac{1}{j}\left[1 - \frac{(\nu_2+2)(\nu_2+4)\cdots(\nu_2+2j-2)}{(\nu_1+\nu_2+2)\cdots(\nu_1+\nu_2+2j-2)}\right]\left(1 - \frac{\theta''}{\theta'}\right)^j$$

(for $0 < \theta''/\theta' \leqq 2$).

5. Comments. The following remarks are intended to elucidate certain points of detail which have been omitted in Sections 2 and 3, where the development was more or less formal.

In the successive stages of each of the procedures described it would usually be more convenient to take logarithms in the inequalities which are used. Thus, for example, in Procedure (III) of Section 2 the function

$$\sum_{j=2}^{n} [\log(\theta' + g_j') - \log(\theta'' + g_j')]$$

can be compared with the critical limits

$$\log A - \tfrac{1}{2}(n-1)\log(\theta''/\theta'), \quad \log B - \tfrac{1}{2}(n-1)\log(\theta''/\theta').$$

In Section 3, Procedure (III) could be modified by using $\sum_{j=1}^{2} \sum_{i=1}^{m} (x_{ji} - \bar{x}_j)^2$ in the denominator of G_2' in place of $\sum_{i=1}^{m} (x_{2i} - \bar{x}_2)^2$ and altering $p(G_2' \mid \theta)$ accordingly. The approximate ASN formulae (16) and (17) would not apply, of course, but the ASN of the modified test presumably would be less than that of the original test.

It is likely that Procedure (I) will be preferable to either (II) or (III) even if the ASN are not as small as the values given by the conjectural formulae (10) and (18). Procedures (II) and (III) are not here proposed as serious alternatives to (I) but rather to provide a background to help in assessing (I).

In the case $\delta' = 0$ of Section 3 there is the further alternative of using a sampling scheme based on a fixed number of groups and an increasing number of observations in each group, following a sequential procedure analogous to (I). While often this will be preferable, for practical reasons, it appears intuitively that the ASN of such a procedure will exceed that of procedure (I) of Section 3 based on increasing the number of groups.

For higher order hierarchal classifications (e.g. observations x_{ijk} of structure $x_{ijk} = A + v_i + w_{ij} + z_{ijk}$) the problems are similar to those studied in this paper. There will, of course, be more variables in the choice of procedure. Just as the number of observations per group was an added variable in Section 3 as compared with Section 2, so there will now be the added variable of the number of second-order (w) groups per first order (v) groups, and similarly for higher order classifications.

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