ASYMPTOTIC DISTRIBUTION OF SERIAL STATISTICS AND APPLICATIONS TO PROBLEMS OF NONPARAMETRIC TESTS OF HYPOTHESES

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Summary. The asymptotic distribution of a class of statistics, which has been called serial statistics, has been obtained, for permutations of the observed sample values. Specific instances of the use of such statistics, for the test of randomness of a sequence, have been given and the large sample power functions have been considered, when the alternative is a Markov process.

1. Introduction. In testing for the randomness of a linearly ordered set of observations $x_1 \cdots x_n$ (such as time series), a plausible alternative hypothesis is frequently the existence of either cyclical or other periodic fluctuations, with varying amplitudes, including time series of the moving average or the autoregressive type as investigated by Yule [16] and Kendall [5]. The whole class of such alternatives may be characterised by the absence of a strong monotonic trend and predominance of periodicity in the general sense.

Yule [16] and Kendall [5] have considered the general autoregressive model,

$$x_i = f(x_{i-1} \cdots x_{i-k}) + \epsilon_i \tag{1.1}$$

where ϵ_i's are independent random variables and $x_1 \cdots x_n$ is the observed series. Especially the linear autoregressive model has been successfully applied to various situations by Yule [16], Kendall [5], Walker [15], and others. The theoretical model in such cases is determined by a law of succession, involving at most k successive observations. Thus the relation between neighbouring observations is more important for the test of the hypothesis of randomness against such alternatives.

Where the model does not specify the distributions of the random elements, the test of significance must be nonparametric. The nonparametric serial correlation test based upon the permutations of the observations $x_1 \cdots x_n$, suggested by Wald and Wolfowitz [13] as a test for randomness, seems to be suitable in such cases. In the case where a strong and persistent trend exists, for example in growth processes, the relations of any observation with all other observations in the series are obviously important. A test of randomness suggested by H. B. Mann [9] seems to be better suited in such cases. An investigation carried out by G. E. Noether [11] strengthens this conclusion, although it is difficult to decide between the two types of tests when neither the monotonic trend nor the periodic element is predominant.

A general class of nonsymmetric statistics $S(x_1 \cdots x_n)$ of the serial correlation type will be considered in this paper. These depend upon the relation between

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neighbouring observations in the ordered sequence and will be called serial statistics. Serial correlation, number of runs up and down (Wolfowitz [12]), etc., are instances of serial statistics. In fact most of the existing nonparametric tests of randomness are either of the serial statistic type or Hoeffding's [4] U-statistic type.

A serial statistic $S(x_1 \cdots x_n)$ is defined as

$$S(x_1 \cdots x_n) = \frac{1}{n} \sum_{i=1}^{n} f_i(x_i \cdots x_{i+k-1}) \tag{1.2}$$

where $f_i(x_i \cdots x_{i+k-1})$ are functions of variables $x_i, x_{i+1}, \cdots, x_{i+k-1}$ only. In the nonparametric method, the conditional distribution of a nonsymmetric statistic for fixed sample values, when only permutations of the sample values are considered, is used. Such a distribution we have called a nonparametric distribution.

The nonparametric distribution of a nonsymmetric statistic, such as $S(x_1 \cdots x_n)$, depends upon the unordered sample values $\{x_i\}$ and is thus a random distribution function. It has been shown that the nonparametric distribution of $S(x_1 \cdots x_n)$ converges stochastically to a normal distribution when the absolute moments of the functions $f_i(x_1 \cdots x_k)$ are uniformly bounded for all values of i and a certain relation (B_1) holds between the product-moments, for large samples.

This result has been generalised to the case of p serial statistics $S_1(x_1 \cdots x_n) \cdots S_p(x_1 \cdots x_n)$. It is shown that their joint distribution converges stochastically to the multivariate normal distribution. Under the hypothesis H_1, for which the variables $x_1 \cdots x_n$ form a Markov process of order p, a stochastically asymptotic expression for the power function has been obtained, on the basis of which a nonparametric test for randomness may be chosen, which for large samples discriminates against this type of alternatives. Thus we get a nonparametric test of randomness which has asymptotically optimum properties for alternatives H_1. As pointed out by Wolfowitz [2], no test of randomness discriminates against all alternatives and thus any such test has to be designed to discriminate strongly against a class of alternatives.

In the case of a stationary Gaussian process, in which the variables are circularly ordered, the method of this paper gives a lower limit for the power of the uniformly most powerful test. The existence of an uniformly most powerful test has been proved by Lehmann and Stein [7] for the nonparametric case and by Lehmann [6] and T. W. Anderson [1] for the parametric case.

Notations and terminology. Throughout this paper we consider stochastic convergence of random variables and random distribution functions. These concepts are explained below.

DEFINITION. Two sequences of random variables, $\{x_n\}$ and $\{y_n\}$, will be called asymptotically stochastically equal, denoted by $x_n =_p y_n$, when $\Pr \{| x_n - y_n | > \epsilon\} \to 0$ as $n \to \infty$ for any given $\epsilon > 0$.

If in particular $y_n = y$, independent of n, then we have the usual notion of convergence in probability and we write Plim $x_n = y$, a notation due to Wald and Mann. This also includes the case when y is a constant.

We also consider stochastic order relations.

DEFINITION. For a sequence of random variables $\{x_n\}$:

(i) $x_n = O_p(n^\alpha)$ implies $\Pr\{n^{-\alpha} \mid x_n \mid > A\} < \epsilon$ holds for $A > A_\epsilon$ and $n > n_\epsilon$, for given ϵ,

(ii) $x_n = o_p(n^\alpha)$ implies $\Pr\{n^{-\alpha} \mid x_n \mid > \delta\} < \epsilon$ holds for $n > n(\epsilon, \delta)$, for given ϵ and δ.

(iii) For a set of random variables $x_1 \cdots x_m$ we write $\{x_1 \cdots x_m\} = o_p(n^\alpha)$, when

$$\Pr\{n^{-\alpha} \mid x_1 \mid < \delta \cdots n^{-\alpha} \mid x_m \mid < \delta\} > 1 - \epsilon \quad \text{for } n > n(\epsilon, \delta)$$

where m may depend upon n, say $m = \phi(n)$.

We now introduce the concept of stochastic asymptote, which is very useful for the purpose of this paper.

DEFINITION. Two sequences of random variables, $\{x_n\}$ and $\{y_n\}$, will be called stochastically asymptotic, denoted by $x_n \simeq_p y_n$ if Plim $x_n/y_n = 1$. Since Plim $x_n/y_n = 1$ also implies Plim $y_n/x_n = 1$, the stochastic asymptotic relationship is symmetric.

The following results will be found to be useful.

LEMMA C.

$$(1.3)\quad \begin{cases} \text{(i) } x_n =_p y_n \quad \textit{and} \quad 1/y_n = O_p(1) & \textit{implies } x_n \simeq_p y_n \\ \text{(ii) } x_n \simeq_p y_n \quad \textit{and} \quad \phi_n \simeq_p \psi_n & \textit{implies } x_n/\phi_n \simeq_p y_n/\psi_n \\ \text{(iii) } x_n \simeq_p y_n \quad \textit{implies} \quad x_n^r \simeq y_n^r & \textit{for any positive } r \\ \text{(iv) } x_n =_p y_n \quad \textit{and} \quad \phi_n =_p \psi_n & \textit{implies } x_n + \phi_n =_p y_n + \psi_n. \end{cases}$$

PROOF. (ii), (iii) and (iv) are obvious. From $1/y_n = O_p(1)$,

$$\Pr\{1/y_n > A\} < \eta \qquad \text{for } n > n_0.$$

Let now $x_n = y_n + \epsilon_n$; then from $x_n =_p y_n$,

$$\Pr\{\epsilon_n > \epsilon/A\} < \eta \qquad \text{for } n > n_1 > n_0.$$

Thus

$$\Pr\{\mid x_n/y_n - 1 \mid < \epsilon\} > 1 - 2\eta, \qquad \text{when } n > n_1.$$

Hence (i) follows.

We consider now a sequence of random distribution functions.

DEFINITION. A sequence of random distribution functions $\{F_n(x, \alpha_n)\}$, where the α_i's are random variables defined in a probability space, will be considered to converge stochastically to a distribution function $F(x)$, if, for $n > n_0(\epsilon, \delta)$,

$$\Pr\{\mid F_n(x, \alpha_n) - F(x) \mid > \epsilon\} < \delta$$

at all points of continuity of $F(x)$. We denote this as Plim $F_n(x) = F(x)$.

A relevant theorem for the stochastic convergence of a sequence of distribution functions, from the stochastic convergence of the set of all moments, has been proved by Ghosh [3] and will be used. An extension of this result to p-dimensional Euclidean spaces is possible by the same method and we shall make use of this, without giving a formal proof. For a test of significance based on nonparametric distribution, the power function is itself a random function. The usual notions of consistency, asymptotically most powerful test, etc., have to be defined in the sense of stochastic convergence. We shall consider these notions for the case of nonparametric distributions.

Let $\{C_n\}$ be a sequence of critical regions, corresponding to a test T of the hypothesis H_0 in the universe of permutations of sample values, $\Gamma_n(x_1 \cdots x_n)$, and let $P_H\{C_n \mid x_1 \cdots x_n\}$ denote the conditional probability of C_n in $\Gamma_n(x_1 \cdots x_n)$ under the hypothesis H.

DEFINITION. The test T will be called stochastically consistent against an alternative H_1 when both

$$\text{Plim}\, P_{H_0}\{C_n \mid x_1 \cdots x_n\} = \alpha(\text{constant}) \qquad \text{Plim}\, P_{H_1}\{C_n \mid x_1 \cdots x_n\} = 1.$$

DEFINITION. If for a class of alternative hypothesis H_ω(ω lying in a space Ω) there exists a function $F(C_n, \omega)$, depending only on permutations, but not on actual values of $x_1 \cdots x_n$, such that

$$\text{Plim}\, P_{H_\omega}\{C_n \mid x_1 \cdots x_n\}/F(C_n, \omega) = 1 \qquad \omega \subset \Omega,$$

then $F(C_n, \omega)$ will be called a stochastically asymptotic power function of the test, for $\omega \subset \Omega$, and denoted by

$$P_{H_\omega}\{C_n \mid x_1 \cdots x_n\} \simeq_p F(C_n, \omega).$$

Obviously, for a class of tests $T_1, T_2, T_3, \cdots$ which possess stochastically asymptotic power functions $F_1, F_2, F_3, \cdots$ the notion of most powerful, uniformly most powerful, etc., tests corresponds stochastically to similar notions for the functions $F_1, F_2, F_3, \cdots$ which are independent of the unordered sample.

2. Distribution of serial statistic. Let $x_1 \cdots x_n$ be a sequence of independent random variables with absolutely continuous distribution function $F(x)$ and $\{f_t(x_1 \cdots x_k)\}$ be a sequence of functions of $x_1 \cdots x_k$ such that

$$\text{(A)} \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f_t(x_1 \cdots x_k)|^s \, dF(x_1) \cdots dF(x_k) < C_s$$

holds uniformly for $t = 1, 2, 3, \cdots$, and for $s = 1, 2, 3, \cdots$ We shall consider the distribution of

$$S(x_1 \cdots x_n) = \frac{1}{n} \sum_{t=1}^{n} f_t(x_t \cdots x_{t+k-1}) \tag{2.1}$$

where we put $x_{n+j} = x_j$ for $j > 0$.

Now $S(x_1, \cdots x_n)$ is a nonsymmetric statistic of $x_1 \cdots x_n$, which assumes the values $S_1 \cdots S_N$ for $N = n!$ permutations of $x_1 \cdots x_n$. For given values of $x_1 \cdots x_n$, we shall consider the repartition function (Von Mises [8]) of the variable $S(x_1 \cdots x_n)$ in the universe of permutations $\Gamma_n(x_1 \cdots x_n)$. On the hypothesis H_0 that the variables $x_1 \cdots x_n$ are independently distributed with the absolutely continuous distribution function $F(x)$, the conditional probability of each point of $\Gamma_n(x_1 \cdots x_n)$ exists and is equal to $1/N$. The repartition of the variable $S(x_1 \cdots x_n)$ in $\Gamma_n(x_1 \cdots x_n)$ thus gives the conditional distribution of $S(x_1 \cdots x_n)$ in $\Gamma_n(x_1, \cdots x_n)$ under the hypothesis H_0. The conditional distribution of a nonsymmetric statistic $S(x_1 \cdots x_n)$ in $\Gamma_n(x_1 \cdots x_n)$ will also be called its nonparametric distribution, and the distribution function of $S(x_1 \cdots x_n)$ in $\Gamma_n(x_1 \cdots x_n)$ will be denoted by $G_n(S, x_1 \cdots x_n)$. The expectation of the nonsymmetric statistic $\phi(x_1 \cdots x_n)$ for the conditional distribution in $\Gamma_n(x_1 \cdots x_n)$ under the hypothesis H_0 is given by

$$\text{(2.2)} \qquad \begin{aligned} E'\{\phi(x_1 \cdots x_n)\} &= \frac{\sum_p \phi(x_1 \cdots x_n)}{n!} \\ &= \int_{\Gamma_n(x_1 \cdots x_n)} \phi(x_1 \cdots x_n)\, dG_n(S, x_1 \cdots x_n) \end{aligned}$$

where $\sum_p$ denotes summation for permutations of $x_1 \cdots x_n$. Thus $E'\{\phi(x_1 \cdots x_n)\}$ is a symmetric function of $x_1 \cdots x_n$. In particular when $\phi(x_1 \cdots x_n)$ is a function of k variables only, we shall write

$$\text{(2.3)} \qquad E'\{\phi(x_1 \cdots x_k)\} = \frac{\sum_p \phi(x_{i_1} \cdots x_{i_k})}{n(n-1) \cdots (n-k+1)}.$$

Here the symbol $\sum_p$ stands for summation for all sets of different suffixes $(i_1 \cdots i_k)$ from 1 to n. When there is no ambiguity we shall merely write $\sum_p \phi(x_1 \cdots x_k)$ instead of $\sum_p \phi(x_{i_1} \cdots x_{i_k})$.

We shall denote by $E\{\phi(x_1 \cdots x_n)\}$ the expectation of the function $\phi(x_1 \cdots x_n)$, for the distribution of $\phi(x_1 \cdots x_n)$ in the sample space R_n. Let

$$\text{(2.4)} \qquad M_1 = E'\{S(x_1 \cdots x_n)\} = \frac{1}{n} \sum_{t=1}^{n} \frac{\sum_p f_t(x_1 \cdots x_k)}{n^{[k]}}$$

where $n^{[k]}$ denotes $n(n-1) \cdots (n-k+1)$. Then M_1 is the conditional expectation of $S(x_1 \cdots x_n)$ in $\Gamma_n(x_1 \cdots x_n)$. Let also

$$\text{(2.5)} \qquad M_r = E'\{[S(x_1, \cdots x_n) - M_1]^r\} \qquad r = 2, 3, 4, \cdots$$

be the rth moment of $S(x_1 \cdots x_n)$, for the conditional distribution in Γ_n. We shall show that when condition (A) and a further condition (B_1), to be stated later, hold

$$\text{(2.6)} \qquad \operatorname{Plim} \frac{M_r}{M_2^{r/2}} = \begin{cases} 0 & \text{when } r \text{ is odd,} \\ r!/2^{r/2}(r/2)! & \text{when } r \text{ is even,} \end{cases}$$

so that the nonparametric distribution of

$$\{S(x_1 \cdots x_n) - M_1\}/\sqrt{M_2} \tag{2.7}$$

converges stochastically to the normal distribution with mean zero and variance unity, from a theorem of Ghosh [3]. Let

$$T(x_1 \cdots x_n) = \frac{1}{n} \sum_{t=1}^{n} g_t(x_t \cdots x_{t+k-1}) \tag{2.8}$$

where

$$g_t(x_t \cdots x_{t+k-1}) = f_t(x_t \cdots x_{t+k-1}) - M_1$$

so that $E'\{T(x_1 \cdots x_n)\} = 0$. We have

$$M_r = E'\{[T(x_1 \cdots x_n)]^r\} = \frac{1}{n^r} E'\left\{\left[\sum_{t=1}^{n} g_t(x_1 \cdots x_{t+k-1})\right]^r\right\}. \tag{2.9}$$

In the expansion of $[\sum_{t=1}^{n} g_t(x_t \cdots x_{t+k-1})]^r$ we get products

$$\prod_{t=1}^{n} [g_t(x_t \cdots x_{t+k-1})]^{p_t}.$$

Since $\sum p_t = r < n$ for sufficiently large n at most r out of $p_1 \cdots p_n$ are different from zero. The g-factors $g_t(x_t \cdots x_{t+k-1})$ with nonzero indices may be divided into subsets, such that in each subset, a g-factor $g_t(x_t \cdots x_{t+k-1})$ has at least one common x-suffix with another g-factor of the same subset but no common x-suffix with g-factors belonging to a different subset. We shall call these subsets P-sets.

Generally, if $\prod_{i=1}^{e} [g_{q_i}(x_{i_1} \cdots x_{i_k})]^{p_i}$ be any product of g-factors with nonzero indices, we can divide them into subsets with the above property and these again we call P-sets. The grouping of g-factors into P-sets is obviously invariant for permutations of $x_1 \cdots x_n$. The essential character of a P-set depends upon the relations between nonzero values $\alpha_1 \cdots \alpha_l$ of the p's and the x-suffixes of the g-factors, which are invariant for permutations of $(x_1 \cdots x_n)$.

Two P-sets will be considered to have the same structure if one can be derived from the other by a permutation of $x_1 \cdots x_n$, the suffixes $q_1 \cdots q_l$ of $g_{q_j}(x_{j_1} \cdots x_{j_k})$ being ignored. Thus the structure of a P-set is invariant for permutations of $x_1 \cdots x_n$. The number of different g-factors in a P-set will be called its length. A P-set of length one will be called a linear P-set. For given indices $\alpha_1 \cdots \alpha_l$, the number of P-sets with different structures is obviously bounded (independent of n), since P-sets of different structures may be obtained from the product,

$$[g_1(x_1 \cdots x_k]^{\alpha_1}[g_2(x_{k+1} \cdots x_{2k})]^{\alpha_2} \cdots [g_l(x_{(l-1)k+1} \cdots x_{lk})]^{\alpha_l}$$

by proper identification of the x-suffixes, that is, by replacing groups of variables $x_{i_1} \cdots x_{i_s}$ by x_{i_1} etc.

We shall consider two special types of P-sets. A P-set is of type I when, by a suitable permutation of $(x_1 \cdots x_n)$, it can be expressed as

$$\prod_{t=1}^{l} [g_{q_t}(x_{q_t} \cdots x_{q_t+k-1})]^{\alpha_t} \qquad \begin{cases} |q_t - q_{t+1}| < k \\ q_1 < q_2 < \cdots < q_l . \end{cases}$$

All P-sets in the expansion of M_r (2.9) are of type I.

A P-set is of type II when, by a suitable permutation of $(x_1 \cdots x_n)$, it can be expressed as

$$g_{q_1}(x_1 \cdots x_k) g_{q_2}(x_{\sigma_1} \cdots x_{\sigma_k})$$

where $\sigma_j = i(i, j \leqq k)$ and the other suffixes $\sigma_1 \cdots \sigma_k$ are different from 1 to k. A P-set of type II is also a P-set of type I when $j = 1$ and $i = k$ and $|q_1 - q_2| < k$.

We may now write

$$(2.10) \qquad M_n = \frac{1}{n^r} \sum{}' C_r(\alpha, l) \sum_q{}_s E' \left\{ \sum_{t=1}^{l} [g_{q_t}(x_{q_t} \cdots x_{q_t+k-1})]^{\alpha_t} \right\}$$

where $l_1 + \cdots + l_m = e$. The summation $\sum_q{}_s$ is taken for all P-sets of type I and given indices $(\alpha_1 \cdots \alpha_{l_1}) \cdots (\alpha_{l_1+\cdots+l_{m-1}+1} \cdots \alpha_{l_1+\cdots+l_m})$ with different structures corresponding to the terms in the expansion of $[\sum g_t(x_t \cdots x_{t+k-1})]^n$. The summation $\sum'$ is taken for all sets of values of $\alpha_1 \cdots \alpha_{l_1+\cdots+l_m}$ such that $\alpha_1 + \cdots + \alpha_{l_1+\cdots l_m} = r$, and for different lengths of P-sets, with appropriate coefficients $C_r(\alpha, l) = C_r\{\alpha \cdots \alpha_{l_1}) \cdots (\alpha_{l_1+\cdots l_{m-1}+1} \cdots \alpha_{l_1+\cdots l_m})\}$, which are equal to the number of ways of grouping r factors in (2.9) in sets of $(\alpha_1 \cdots \alpha_{l_1})$, etc.

Before proving the main theorem we shall establish a number of lemmas.

LEMMA I. *Let $z_1 \cdots z_p$ be real numbers. Then*

$$(2.11) \qquad | z_1^{\lambda_1} \cdots z_p^{\lambda_p} | \begin{cases} \leqq |z_1|^\lambda + \cdots + |z_p|^\lambda \\ \leqq 1 + |z_1|^{\lambda+1} + \cdots + |z_p|^{\lambda+1} \end{cases}$$

where $\lambda = \lambda_1 + \cdots + \lambda_p$ and $\lambda_i > 0$.

PROOF. Let $|z_M|$ be the largest of the numbers $|z_1| \cdots |z_p|$, then by replacing them by $|z_M|$ the product $|z_1^{\lambda_1}| \cdots |z_p|$, only increases. Thus $|z_1^{\lambda_1} \cdots z_p^{\lambda_p}| \leqq |z_M|^\lambda \leqq \sum_{i=1}^p |z_i|^\lambda$,

$$| z_1^{\lambda_1} \cdots z_p^{\lambda_p} | \leqq \begin{cases} |z_M|^\lambda \leqq 1 & |z_M| \leqq 1 \\ |z_M|^{\lambda+1} \leqq \sum_{i=1}^{p} |z_i|^{\lambda+1} & |z_M| > 1. \end{cases}$$

Hence

$$| z_1^{\lambda_1} \cdots z_p^{\lambda_p} | \leqq 1 + \sum_{i=1}^{p} |z_i|^{\lambda+1}.$$

LEMMA II. *Let*

$$M'_{\lambda,t} = E'\{[f_t(x_1 \cdots x_k)]^\lambda\}.$$

Then

$$\left| E'\left\{\prod_{j=1}^{s} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{\alpha_j}\right\} \right|$$

$$\leqq \begin{cases} \sum_{j=1}^{s} M_{1,q_j}'^{r+1} + d_2 \sum_{j=1}^{s} M_{1,q_j}'^{r-1} M_{2,q}' + \cdots + d_{r+1} \sum_{j=1}^{s} M_{r+1,q_j}' + 1 & r \text{ odd}, \\ d_{r+2} \sum_{j=1}^{s} M_{1,q_j}'^{r} + \cdots + d_{2r+2} \sum_{j=1}^{s} M_{r,q_j}' & r \text{ even}, \end{cases}$$

where the x_{ij} *belong to* $x_1 \cdots x_n$ *(the double suffix being used for convenience only), and* $d_1 \cdots d_{2r+2}$ *are independent of n and* $x_1 \cdots x_n$.

PROOF. From Lemma I, for $\alpha_1 + \cdots + \alpha_s = r$,

$$\left| \prod_{j=1}^{s} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{\alpha_j} \right| \leqq \begin{cases} \sum_{j=1}^{s} [g_{q_j}\, x_{j,1} \cdots x_{j,k})]^r & r \text{ even}, \\ 1 + \sum_{j=1}^{s} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{r+1} & r \text{ odd}. \end{cases}$$

Taking expectations E' for both sides, the inequality holds for expected values and

$$\left| E'\left\{\prod_{j=1}^{s} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{\alpha_j}\right\} \right| \leqq \begin{cases} \sum_{j=1}^{s} E'[g_{q_j}(x_{j,1} \cdots x_{j,k})]^r & r \text{ even}, \\ 1 + \sum_{j=1}^{s} E'[g_{q_j}(x_{j,1} \cdots x_{j,k})]^{r+1} & r \text{ odd}. \end{cases}$$

The result now follows immediately from

$$E'[g_{q_j}(x_{j,1} \cdots x_{j,k})]^r = \sum_{\lambda=0}^{r} (-1)^{\lambda}\, {}^rC_{\lambda} M_{1,q_j}'^{\lambda} M_{r-\lambda,q_j}',$$

etc.

LEMMA III. *Let* $U = E'\{\phi(x_1 \cdots x_k)\}$ *be a U-statistic of Hoeffding* [4]. *Then we have* $\text{Var}(U) \leqq \frac{1}{2}k(3k-1)\,\text{Var}[\phi]/n$ *where* $x_1 \cdots x_n$ *are independent random variables with the same distribution.*

PROOF.

$$\begin{aligned} \text{Var}(U) &= E\{[\textstyle\sum_p (\phi(x_{i_1} \cdots x_{i_k} - \bar{\phi})/n^{[k]}]^2\} \\ &= (n^{[k]})^{-2} \textstyle\sum_p' E\{[\phi(x_i \cdots x_{i_k} - \bar{\phi}][\phi(x_j \cdots x_{jk}) - \bar{\phi}]\} \end{aligned}$$

where $\sum_p'$ represents summation for all different sets of $i_1 \cdots i_k$ and $j_1 \cdots j_k$, and $\bar{\phi} = E[\phi(x_1 \cdots x_k)]$. The only nonzero terms in this expression are those for which at least one of the j-suffixes is equal to a i-suffix. The number of such terms is not greater than

$$(n^{[k]})^2 - n^{[k]}(n-k)^{[k]}$$

$$\leqq (n^{[k]})^2 \left\{1 - \frac{(n-k) \cdots (n-2k+1)}{n(n-1) \cdots (n-k+1)}\right\}$$

$$\leqq (n^{[k]})^2 \left\{1 - \left(1 - \frac{k}{n}\right) \cdots \left(1 - \frac{2k-1}{n}\right)\right\}$$

$$\leqq (n^{[k]})^2 \left\{\frac{k + (k+1) + \cdots + (2k-1)}{n}\right\} = (n^{[k]})^2 \frac{k(3k-1)}{2}.$$

Also

$$E\{[\phi(x_{i_1} \cdots x_{i_k}) - \bar{\phi}][\phi(x_{j_1} \cdots x_{j_k} - \bar{\phi}]\} \leqq \text{Var}\,[\phi(x_1 \cdots x_k)]$$

for all sets $i_1 \cdots i_k$ and $j_1 \cdots j_k$. Hence the result.

LEMMA IV. *For any given* $\delta > 0$,

$$\{M'_{1,1}, \cdots, M'_{1,n}, M'_{2,1}, \cdots, M'_{2,n}, \cdots, M'_{r,n}\} = o_p(n^{\delta}). \tag{2.12}$$

PROOF. From Lemma III,

$$\text{Var}\,[M'_{s,t} \mid \leqq \tfrac{1}{2}k(3k-1)n^{-1}\,\text{Var}\,\{[f_t(x_1 \cdots x_k)]^s\} \leqq \tfrac{1}{2}k(3k-1)C_{2s}/n.$$

Hence

$$\Pr\{\mid M'_{s,t} - E(M'_{s},t) \mid > n^{\delta/2}\} \leqq \tfrac{1}{2}k(3k-1)\cdot C_{2s}/n^{1+\delta}.$$

Again from assumption (A), $\mid E(M'_{s,t}) \mid \leqq C_s$, so that

$$\Pr\{\mid M'_{s,t} \mid < 2n^{\delta/2}, s = 1 \cdots r; t = 1 \cdots n\} > 1 - \tfrac{1}{2}k(3k-1)[C_2 + \cdots C_{2r}]/n^{\delta}$$

for sufficiently large n. Hence the result.

In all that follows, whenever we consider a sum $\sum_{i=1}^{n}$ in which a suffix $i + p > n$ occurs, we shall take its value to be $i + p - n$. The same interpretation will hold for pairs i and j for which $\mid i - j \mid < k$, which will be considered to hold for values of i and j so that $i - j (\text{mod}\, n) < k$. We now obtain an asymptotic expression for M_2.

LEMMA V.

$$nM_2 =_p [\Lambda_1 - \Lambda_2 - \Lambda_3]/n \tag{2.13}$$

where

$$\begin{cases} \Lambda_1 = \sum_{\mid i-j \mid < k} E'\{g_i(x_i \cdots x_{i+k-1})\, g_j(x_j \cdots x_{j+k-1})\} \\ \Lambda_3 = \sum_{\mid i-j \mid < k} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\} \\ \Lambda_2 = \frac{1}{n} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{k} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{\sigma_1} \cdots x_{\sigma_k})\} = \frac{1}{n} \sum_{\alpha,\beta=1}^{k} Z_{\alpha,\beta} \end{cases} \tag{2.14}$$

$$Z_{\alpha,\beta} = \sum_{i,j=1}^{n} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{\sigma_1} \cdots x_{\sigma_k})\}$$

where $\sigma_\beta = \alpha (\alpha, \beta \leqq k)$ that is *the* βth *suffix* σ_β *is equal to* α, *where of course both* β *and* α *are less than or equal to* k, *and other suffixes do not belong to* $1, 2, \cdots, k$. *The summation is taken for all such values of* α *and* β.

PROOF.

$$M_2 = \frac{1}{n^2} E'\{\textstyle\sum g_i(x_i \cdots x_{i+k-1})\}^2$$

$$= \frac{1}{n^2} \sum_{|i-j|<k} E'\{g_i(x_i \cdots x_{i+k-1})\, g_j(x_j \cdots x_{j+k-1})\}$$

$$+ \frac{1}{n^2} \sum_{|i-j| \geqq k} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\}$$

$$\sum_{|i-j|\geqq k} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\}$$

$$= \sum_{i,j=1}^{n} \frac{\sum_p\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\}}{n^{[2k]}}$$

$$- \sum_{|i-j|<k} \frac{\sum_p\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\}}{n^{[2k]}}$$

$$(2.15) \qquad = -\frac{\sum_p}{n^{[2k]}} \left\{ \sum_{i,j=1}^{n} g_i(x_1 \cdots x_k) \textstyle\sum'' g_j(x_{\sigma_1} \cdots x_{\sigma_k}) + \displaystyle\sum_{|i-j|<k} g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k}) \right\}$$

$$+ \frac{\sum_p}{n^{[2k]}} \left\{ \sum_{i=1}^{n} g_i(x_1 \cdots x_k) \times \textstyle\sum_p \displaystyle\sum_{j=1}^{n} g_j(x_{k+1} \cdots x_{2k}) \right\}.$$

Here $\sum''$ represents summation for sets of values of $\sigma_1 \cdots \sigma_k$, at least one of these x-suffixes being common with an x-suffix of the first g-factor $g_i(x_1 \cdots x_k)$. The last term in the expression (2.15) is zero. In the summation

$$\frac{\sum_p}{n^{[2k]}} \left\{ \sum_{i,j} g_i(x_1 \cdots x_k) \textstyle\sum'' g_j(x_{\sigma_1} \cdots x_{\sigma_k}) \right\}$$

when two or more of $\sigma_1 \cdots \sigma_k$ are common with the suffixes 1 to k, the number of these terms is of order $O(n^{2k-2})$ and, considering the values of i and j in the summation $\sum_{ij}$, the number of such terms is $O(n^{2k})$. Also from Lemmas II and IV, all these terms are simultaneously of the order $o_p(n^{\delta})$. Thus the sum of these terms is of order $o_p(n^{2k+\delta})$, while the denominator in nM_2 is $n^{[2k]+1}$. Thus we consider, in the above summation, only those terms for which exactly one of $\sigma_1 \cdots \sigma_k$ is equal to one of the suffixes $1, 2, \cdots, k$; hence the result.

LEMMA VI.

$$nM_2 =_p \frac{1}{n} \sum_{|i-j|<k} E\{f_i(x_i \cdots x_{i+k-1})\, f_j(x_j \cdots x_{j+k-1})\}$$

$$- \frac{1}{n} \sum_{|i-j|<k} E\{f_i(x_1 \cdots x_k) f_j(x_{k+1} \cdots x_{2k})\}$$

$$- \frac{1}{n^2} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{k} E\{f_i(x_1 \cdots x_k)\, f_j(x_{\sigma_1} \cdots x_{\sigma_k})\}$$

$$+ k^2 \left\{ \frac{1}{n} \sum_{t=1}^{n} E[f_t(x_1 \cdots x_k)] \right\}^2 \qquad = \mu_{2\,n}.$$

where $\sigma_1 \cdots \sigma_k$ is a fixed set of numbers different from 1 to k, except $\sigma_\beta = \alpha(\alpha, \beta \leqq k)$ as before.

PROOF.

$$\frac{1}{n}\sum_{t=1}^{n} E'\{g_t(x_1 \cdots x_k)\, g_{t+p}(x_{p+1} \cdots x_{p+k})\}$$

$$= E'\left\{\frac{1}{n}\sum_{t=1}^{n} f_t(x_1 \cdots x_k)\, f_{t+p}(x_{p+1} \cdots x_{p+k})\right\} - M_1^2 = U_1 - M_1^2$$

Here U_1 is a U-statistic and

$$\operatorname{var}\left\{\frac{1}{n}\sum_{t=1}^{n} f_t(x_1 \cdots x_k)\, f_{t+p}(x_{p+1} \cdots x_{p+k})\right\}$$

$$\leqq \frac{1}{n^2}\sum_{s,t=1}^{n} E\{f_s(x_1 \cdots x_k)\, f_t(x_1 \cdots x_k)\, f_{s+p}(x_{p+1} \cdots y_{p+k})\, f_{t+p}(x_{p+1} \cdots x_{p+k})\}.$$

From assumption (A), by the application of Schwarz's inequality,

$$E\{f_s(x_1 \cdots x_k)\, f_t(x_1 \cdots x_k)\, f_{s+p}(x_{p+1} \cdots x_{p+k})\, f_{t+p}(x_{p+1} \cdots x_{p+k})\}$$

is bounded and the bound depends on the constants C_j only. Thus from Lemma III,

$$E'\left\{\frac{1}{n}\sum_{t=1}^{n} f_t(x_1 \cdots x_k)\, f_{t+p}(x_{p+1} \cdots x_{p+k})\right\}$$

$$=_p \frac{1}{n}\sum_{t=1}^{n} E\{f_t(x_1 \cdots x_k)\, f_{t+p}(x_{p+1} \cdots x_{p+k})\}$$

$$M_1^2 =_p \left\{\frac{1}{n}\sum_{t=1}^{n} E[f_t(x_1 \cdots x_k)]\right\}^2 = \mu_1^2.$$

Hence

$$\frac{1}{n}\sum_{|i-j|<k} E'\{g_i(x_i \cdots x_{i+k-1}) g_j(x_j \cdots x_{j+k-1})\}$$

$$=_p \frac{1}{n}\sum_{|i-j|<k} E[f_i(x_i \cdots x_{i+k-1})\, f_j(x_j \cdots x_{j+k-1})] - (2k-1)\mu_1^2.$$

$$\frac{1}{n}\sum_{|i-j|<k} E'\{g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k})\}$$

$$=_p \frac{1}{n}\sum_{|i-j|<k} E\{f_i(x_1 \cdots x_k)\, f_j(x_{k+1} \cdots x_{2k})\} - (2k-1)\mu_1^2.$$

Consider the U-statistic

$$E'\left\{\frac{1}{n^2}\sum_{i,j=1}^{n} f_i(x_1 \cdots x_k)\, f_j(x_{\sigma_1} \cdots x_{\sigma_k})\right\}.$$

Since Var $\{n^{-2} \sum f_i(x_1 \cdots x_k) f_j(x_{\sigma_1} \cdots x_{\sigma_k})\}$ is bounded from assumption (A), we have from Lemma III

$$\frac{Z\alpha,\beta}{n^2} =_p \frac{1}{n^2} \sum_{i,j=1}^{n} E\{f_i(x_1 \cdots x_k) f_j(x_{\sigma_1} \cdots x_{\sigma_k})\} - \left\{\frac{1}{n} \sum_{t=1}^{n} E[f_t(x_1 \cdots x_k)]\right\}^2.$$

Hence the result.

The condition (B_1) may now be stated as

$$(B_1) \qquad \liminf \mu_{2,n} > 0.$$

This implies $1/nM_2 = O_p(1)$ and hence from Lemma C (1.3) we have

$$(2.16) \qquad nM_2 \simeq_p (1/n)[\Lambda_1 - \Lambda_2 - \Lambda_3].$$

Reduction of linear P-sets. In the expansion of M_r, from (2.10), all the terms consist in P-sets of type I. For given values of the indices $\alpha_1 \cdots \alpha_{l_1+\cdots+l_m}$, we may group together terms with different suffixes for the g-factors such that there are m P-sets of type I and lengths $l_1 \cdots l_m$ and with given structures. Such a sum is represented by

$$(2.17) \qquad \sum_{q}{}_1 E'\left\{\prod_{j=1}^{s} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j}\right\}.$$

Let q_1 be the q-suffix of a linear P-set in (2.17). For fixed values $\pi_2 \cdots \pi_s$ of $q_2 \cdots q_s$, we consider the summation $\sum_q{}_1$, for q_1, satisfying the above conditions that is $|q_1 - \pi_i| > k$ $(i = 2, 3, \cdots s)$:

$$(2.18) \qquad \begin{aligned} &\sum_{q_1}{}_1 \sum{}_p \prod_{j=1}^{s} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \\ &\quad = \sum{}_p \prod_{j=2}^{s} [g_{\pi_j}(x_{\pi_j} \cdots x_{\pi_j+k-1})]^{\alpha_j} \times \sum_{q_1}{}_1 \sum{}' g_{q_1}(x_{i_1} \cdots x_{i_k}) \end{aligned}$$

where $\sum'$ denotes summation for all $(i_1 \cdots i_k)$, none of the suffixes belonging to $\pi_i + j$ $(i = 2 \cdots s; j = 0, 1, \cdots, k - 1)$. Thus

$$(2.19) \qquad \begin{aligned} \sum_{q_1}{}_1 \sum{}' g_{q_1}(x_{i_1} \cdots x_{i_k}) &= \sum_{q_1=1}^{n} \sum{}_p g_{q_1}(x_1 \cdots x_k) \\ &\quad - \sum_{q_1=1}^{n} \sum{}'' g_{q_1}(x_{\sigma_1} \cdots x_{\sigma_k}) - \sum_{q_1}{}_2 \sum{}_p g_{q_1}(x_{i_1} \cdots x_{i_k}). \end{aligned}$$

Here $\sum''$ denotes summation for x-suffixes $\sigma_1 \cdots \sigma_k$ at least one of which is common with the suffixes $\pi_i + j$, and $\sum_{q_1}{}_2$ is the summation for all values of q_1 such that $|q_1 - \pi_i| < k$ for some i $(i = 2 \cdots s)$ that is, q_1 is tied to one of the q-suffixes $\pi_2 \cdots \pi_s$. Since

$$\sum_{q_1=1}^{n} \sum{}_p g_{q_1}(x_1 \cdots x_k) = 0,$$

$\sum_{q_1} \sum' g_{q_1}(x_{i_1} \cdots x_{i_k})$ is given by the last two sums of (2.19). By applying the process of reduction to $\sum_{q 1} \sum_p \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j}$ we get summations of two kinds. In the first kind of summation, $e - 1$ g-factors form P-sets of type I as in (2.17) with the same structures of P-sets and a g-factor $g_{q_1}(x_{\sigma_1} \cdots x_{\sigma_k})$, where $\sigma_1 \cdots \sigma_k$ have at least one element common with the x-suffixes of $e - 1$ other g-factors, and q_1 assumes all values from 1 to n. In this case the number of P-sets is reduced by one. In the second kind of summation, $e - 1$ g-factors form $m - 1$ P-sets of type I as in (2.17), with the same structures and a g-factor $g_{q_1}(x_{i_1} \cdots x_{i_k})$ such that q_1 is tied to a q-suffix of the $m - 1$ P-sets of type I. These two kinds of summation correspond to summations $\sum_{q_1=1}^{n} \sum''$ and $\sum_{q_1 2} \sum_p$ of (2.19).

We may apply the reduction process again and by such successive reductions we get the following type of summation. Let

$$M(\rho, t, v, \beta, \gamma) = \sum_p \sum_{q 1} \prod_{j=1}^{e} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{\alpha_j}. \tag{2.20}$$

The factors in (2.20) belong to three sets A, B and C and have ρ different x-suffixes altogether. The e-β-γ g-factors of A form t P-sets of type I (ignoring the other g-factors of B and C). The γ g-factors of C are tied to the g-factors of A, such that γ_1 g-factors are tied to g_{q_1}, γ_2 to g_{q_2}, etc., with $\sum \gamma_i = \gamma$, but have no common x-suffixes. The β g-factors of B have common x-suffixes with other g-factors forming P-sets which always contain a g-factor of A or C. The g-factors of A, B and C are so related in the M-sets $M(\rho, t, v, \beta, \gamma)$ that there are v linear P-sets of A, unconnected with g-factors of B and with no g-factors of C tied to them, which we call free linear P-sets. The summation $\sum_{q 1}$ is taken for q-suffixes, so that the q-suffixes of A may assume any values between 1 and n, subject to the restriction that these correspond to t P-sets of type I, with given structures and given values of α_j's. The q-suffixes of the g-factors of C assume all possible values consistent with their relation to the g-factors of A. The q-suffixes of the g-factors of B may assume any value be between 1 and n.

The M-set $M(\rho, t, v, \beta, \gamma)$ depends upon the structures of P-sets, which is the same for all terms of M, and on the way in which the q-suffixes of C are tied to the g-factors of A, which together determine the structural relations of an M-set. The structural relations of an M-set may be of any kind, except that the g-factors of A form t P-sets of type I and there are v free linear P-sets. Since the number of x-variables in any term of the M-set is at most $\rho + k(\beta + \gamma)$, the number of M-sets with different structural relations (including tied q-suffixes) is bounded and independent of n. The process of reduction, outlined above, may be applied to any M-set $M(\rho, t, v, \beta, \gamma)$, where we reduce a free linear P-set of A, given by the suffix q_1 (say). If the sum $\sum_{q_1 1} \sum_p g_{q_1}(x_{q_1} \cdots x_{q_1+k-1})$ is expressed as in (2.19), $M(\rho, t, v, \beta, \gamma)$ can be expressed linearly in a finite number of M-sets of the type

(2.21) $M(\rho - \omega_s, t-1, v_1, \beta+1, \gamma)$, $\quad M(\rho - \omega_s, t-1, v_2, \beta, \gamma+1)$,

where the suffixes $\sigma_1 \cdots \sigma_k$ have ω_s suffixes common with other g-factors of A or C. Also

$$v_1 \geqq v - \omega_s - 1 \quad \text{and} \quad v_2 \geqq v - \omega_s - 2 \tag{2.22}$$

hold in this case. The minimum of v_1 corresponds to the case when all ω_s x-suffixes are from different free linear P-sets. The minimum of v_2 corresponds to the case when, moreover, the reduced P-set becomes a g-factor of C and is tied to another linear P-set. We shall call a pair of g-factors, where one is tied to the other, a Q-set, when they have no common x-suffix.

We have already defined in (2.17) and (2.20) the summation $\sum_{q} {}_1$, which is a q-sum retaining the structural relations of the g-factors. We now consider another kind of summation $\sum_{q} {}_s$, where the summation is taken for all possible structural relations consistent with t, v and for given sets of indices $(\alpha_1 \cdots \alpha_{l_1})$, etc., of the P-sets of A. We shall now prove the following lemma.

LEMMA VII. *When v is the number of linear P-sets and m the number of P-sets of type I,*

$$\frac{1}{n^{r/2}} \sum_{q} {}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} = o_p(n^{m-[(v+1)/2]+\delta-r/2}) \tag{2.23}$$

where the summation $\sum_{q} {}_s$ has the same sense as in (2.10), and $\alpha_1 \cdots \alpha_e$ are constants such that $\sum \alpha_i = r$ and $[(v+1)/2]$ is the integral part of $(v+1)/2$.

PROOF. We may write

$$\sum_{q}{}' E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} = \frac{M(\rho_0, m, v, 0, 0)}{n^{[\rho_0]}}$$

where ρ_0 is the number of different x-suffixes. By successive application of the reduction process to free linear P-sets we may express $M(\rho_0, m, v, 0, 0)$ linearly as

$$\begin{aligned} &M(\rho_0, m, v, 0, 0) \\ &\quad = \sum (-1)^j M(\rho_0 - \omega_1 - \cdots - \omega_j, m-j, v_j, \beta, \gamma) \quad (j = \beta + \gamma) \end{aligned} \tag{2.24}$$

where β of the reduced P-sets belong to B and γ to C, and $\omega_1 \cdots \omega_s$ represent the number of x-suffixes reduced at different stages. Each term of $M(\rho_0 - \omega_1 - \cdots - \omega_j, m-j, v_j, \beta, \gamma)$ corresponds to different combinations of the $\rho_0 - \omega_1 - \cdots - \omega_j$ x-suffixes from 1 to n and to variations of the q-suffixes with proper restrictions. Thus from Lemma IV, for fixed values of the q-suffixes, the sum of terms in the M-set is of the order $o_p(n^{\rho_0-\omega_1-\cdots-\omega_s+\delta})$. Again the number of such sums in the M-set is of the order $O(n^{m-\gamma})$, since the number of ways in which the q-suffixes of A can be chosen is $O(n^{m-\beta-\gamma})$, while the number of ways in which the q-suffixes of B can be chosen independently of the choice of q-suffixes of A is of the order $O(n^{\beta})$, and the q-suffixes of C are tied to those of A.

Thus $M(\rho_0 - \omega_1 - \cdots - \omega_j, m - j, v_j, \beta, \gamma)$ is of the order $o_p(n^{\rho_0 - \omega_1 - \cdots - \omega_j + m - \gamma + \delta})$. If at the ith stage of reduction, a linear P-set is reduced to a g-factor of B, then the corresponding $\omega_i > 0$ and thus

$$\omega_1 + \omega_2 + \cdots + \omega_j \geqq \beta. \tag{2.25}$$

Also from (2.22)

$$v_j \geqq v - \omega_1 - \cdots - \omega_j - j - \gamma. \tag{2.26}$$

The process of reduction may be continued until $v_j = 0$. Then from (2.26)

$$\omega_1 + \omega_2 + \cdots + \omega_j + j + \gamma \geqq v, \qquad 2j \geqq v, \qquad j \geqq [(v + 1)/2],$$

$$\omega_1 + \omega_2 + \cdots + \omega_j + \gamma \geqq \beta + \gamma \geqq [(v + 1)/2].$$

Hence $M(\rho_0 - \omega_1 - \cdots - \omega_j, m - j, v_j, \beta, \gamma)$ is of the order $o_p(n^{\rho_0 + m - [(v+1)/2] + \delta})$ and

$$\frac{1}{n^{r/2}} \sum_q{}_1 E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} = o_p(n^{m - [(v+1)/2] + \delta - r/2}). \tag{2.27}$$

Since the number of P-sets with different structures is bounded, (2.23) follows from (2.27).

It follows from above that we need consider only such M-sets for which ω_i assumes only the values one and zero, since in all other cases $m - [(v + 1)/2] < r/2$ and the sum of all these terms converges stochastically to zero, from what has just been proved. These correspond to cases where a linear P-set is reduced to a g-factor of B, which is connected with a free linear P-set of A by common x-suffix forming a P-set of type II, or it is reduced to a g-factor of C, when it is tied to a free linear set of A, forming a Q-set. We shall now prove the following theorem.

THEOREM I. *Let $x_1 \cdots x_n$ be a sequence of independent random variables with the absolutely continuous distribution function $F(x)$, and let $\{f_t(x_1 \cdots x_k)\}$ be a sequence of functions of $x_1 \cdots x_k$ such that the conditions* (A) *and* (B_1) *hold. Then the nonparametric distribution of the nonsymmetric statistic*

$$\left[\frac{1}{n} \sum_{t=1}^{n} f_t(x_t, \cdots x_{t+k-1}) - M_1\right] \Big/ \sqrt{M_2} \tag{2.28}$$

converges stochastically to the normal distribution with mean zero and variance unity.

PROOF. From Lemma VII, in the expression for M_r (2.10) the term

$$\frac{1}{n^{r/2}} \sum_q{}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} \qquad \sum \alpha_j = r$$

is of order $o_p(n^{m - [(v+1)/2] - r/2 + \delta})$ when the number of P-sets is m, of which v are linear. Now

$$2(m - v) + v \leqq r \quad \text{or} \quad m - v/2 \leqq r/2$$

where equality holds when and only when all nonlinear P-sets are of length two. Thus $m - [(v + 1)/2] < r/2$ when r is odd and in this case, from (B_1),

$$\text{Plim } M_r/M_2^{r/2} = 0. \tag{2.29}$$

When r is even, we need consider only terms with $2u$ linear P-sets and $r/2 - u$ P-sets of length two, u assuming all values from 0 to $r/2$. Let $v = 2u$. Then, considering highest order terms in $M(\rho, m, 2u, 0, 0)$, we get two kinds of M-sets after the first stage of reduction, for example, $M(\rho - 1, m - 1, 2(u - 1), 1, 0)$, and $M(\rho, m - 1, 2(u - 1), 0, 1)$, both of which have negative signs and contain $2(u - 1)$ free (untied) linear P-sets. In $M(\rho - 1, m - 1, 2(u - 1)\ 1, 0)$, there is one P-set of length two and type II, corresponding to each way of combination of a linear P-set given by q_1 (say), with any other linear P-set. In $M(\rho, m - 1, 2(u - 1), 0, 1)$ there is a Q-set corresponding to each way of tying up a linear P-set with another one.

Proceeding in this manner, we get, after the uth stage, a sum of M-sets with the coefficient $(-1)^u$, for which there are β P-sets of length two and type II, one g-factor of such a P-set belonging to B and the other to A, and γ tied pairs of linear P-sets forming Q-sets of length two. These β P-sets of type II and γ Q-sets are derived in all possible manners of grouping the $2u$ linear P-sets. Thus we may write, considering highest order terms in n only, for $a = e - 2u$ and $b = e - 2u + \beta$,

$$\begin{aligned}
&\frac{1}{n^{r/2}} \sum_q{}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} \\
&= \frac{1}{n^{[\rho_0]+r/2}} \sum_\beta \sum{}_p \sum_q{}_s \left\{ \prod_{j=1}^{a} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right. \\
&\quad \times \prod_{i=1}^{\beta} (-1) g_{\xi_i}(x_{\xi_i(1)} \cdots x_{\xi_i(k)}) g_{q_{i+a}}(x_{q_{i+a}} \cdots x_{q_{i+a}+k-1}) \\
&\quad \left. \times \prod_{i=1}^{\gamma} (-1) g_{\zeta_i}(x_{\zeta_i(1)} \cdots x_{\zeta_i(k)}) g_{q_{i+b}}(x_{q_{i+b}} \cdots x_{q_{i+b}+k-1}) \right\}
\end{aligned} \tag{2.30}$$

where the q-suffixes q_j belong to A and the ξ_i belong to B and the ζ_i to C. Since $\beta + \gamma = 2u$ and ρ_0 is reduced to $\rho_0 - \beta$, the summation $\sum_q{}_s$ is taken for different structural relations of the g-factors such that the x-suffixes $\xi_i(1) \cdots \xi_i(k)$, $\zeta_i(1) \cdots \zeta_i(k)$ are all different for all i, j $(1 \leqq i \leqq \beta,\ 1 \leqq j \leqq \gamma)$ and different from the x-suffixes of A except that for sets of type II,

$$\xi_i(x) = q_{e-2u+i} + y - 1$$

holds for just one pairs of values of x and y between 1 and k, for each term in the summation. Also β and γ have all possible values satisfying $\beta + \gamma = u$. We may thus write, considering highest order in terms in n only,

$$\frac{1}{n^{r/2}} \sum_q{}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\}$$

$$(2.31) \quad =_p \frac{(-1)^u C_{u,\beta}}{n^{[\rho_0]+r/2}} \left\{ \sum_p \sum_q{}_s \prod_{j=1}^{e-2u} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} \times \left\{ \sum_{x,y=1}^{k} \sum_{i,j=1}^{n} \sum_p g_i(x_1 \cdots x_k)\, g_j(x_{\sigma_1} \cdots x_{\sigma_k}) \right\}^\beta \times \left\{ \sum_p \sum_{|i-j|<k} g_i(x_1 \cdots x_k)\, g_j(x_{k+1} \cdots x_{2k}) \right\}^\gamma$$

where $\sigma_1 \cdots \sigma_k$ have just one element common with $1, 2, \cdots k$ and where $C_{u,\beta}$ is the number of ways in which $2u$ linear P-sets can be divided into two groups of 2β and 2γ g-factors, forming P-sets of type II and Q-sets, respectively. Thus

$$(2.32) \qquad C_{u,\beta} = \frac{(2u)!}{(2\beta)!(2\gamma)!} \frac{(2\beta)!}{2^\beta(\beta)!} \frac{(2\gamma)!}{2^\gamma(\gamma)!} = \frac{(2u)!}{2^{\beta+\gamma}(\beta)!(\gamma)!}.$$

We thus have

$$(2.33) \quad \frac{1}{n^{r/2}} \sum_q{}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} =_p \frac{1}{n^{r/2}} \frac{(2u)!}{2^{\beta+\gamma}(\beta)!(\gamma)!} \cdot \sum_q{}_s E' \left\{ \prod_{j=1}^{e-2u} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\} \times [-\Lambda_2]^\beta \times [-\Lambda_3]^\gamma$$

where $g_{q_1} \cdots g_{q_{e-2u}}$ denote the g-factors of nonlinear P-sets of type I.

The nonlinear P-sets are of type I and are obtained from every manner of grouping $r - 2u$ factors in $[\sum g_i(x_i \cdots x_{i+k-1})]^r$ into pairs. Thus from Lemma VII, considering highest order terms only, we have

$$\frac{1}{n^{r/2}} \sum_q{}_s E' \left\{ \prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \right\}$$

$$(2.34) \quad \begin{aligned} &=_p \frac{(2u)!}{n^{r/2}(\beta!)(\gamma!)2^{\beta+\gamma}} \left[\sum_q{}_s E' \prod_{j=1}^{(r/2)-u} g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})\, g_{q'_j}(x_{q'_j} \cdots x_{q'_j+k-1}) \right] \\ &\times [-\Lambda_2]^\beta \times [-\Lambda_3]^\gamma \qquad \text{where } |q_j - q'_j| < k \\ &=_p \frac{(2u)!}{n^{r/2}(\beta!)(\gamma!)2^{\beta+\gamma}} [E'\{ \sum_{|i-j|<k} g_i(x_i \cdots x_{i+k-1})\, g_j(x_j \cdots x_{j+k-1}) \}]^{(r/2)-u} \\ &\times [-\Lambda_2]^\beta \times [-\Lambda_3]^\gamma \\ &=_p \frac{1}{n^{r/2}} \frac{(2u)!}{(\beta!)(\gamma!)2^u} [\Lambda_1]^{(r/2)-u} [-\Lambda_2]^\beta [-\Lambda_3]^\gamma, \end{aligned}$$

neglecting terms of lower order in n which tend stochastically to zero.

We now find the coefficient $C_r\{(1), (1), \cdots, (1, 1), \cdots, (1, 1)\}$ with $2u$ linear P-sets. These $2u$ factors can be chosen from r factors in ${}^rC_{2u}$ ways, and the remaining $r - 2u$ factors can be grouped into pairs corresponding to P-sets of length two and type I in $(r - 2u)!/[(r/2) - u]!2^{(r/2)-u}$ ways. The number of terms of the type (2.34) in M_r is

$$\frac{r!}{(2u)!(r-2u)!}\frac{(r-2u)!}{[(r/2)-u]!2^{(r/2)-u}} = \frac{r!}{(2u)![(r/2)-u]!2^{(r/2)-u}}.$$

Thus

$$\begin{aligned} n^{r/2}M_r &=_p \frac{1}{n^{r/2}}\sum_{\beta,\gamma}\frac{r!}{\beta!\gamma![(r/2)-u]!2^{r/2}}[\Lambda_1]^{(r/2)-u}[-\Lambda_2]^{\beta}[-\Lambda_3]^{\gamma} \\ &=_p \frac{1}{n^{r/2}}\frac{r!}{2^r(r/2)!}(\Lambda_1-\Lambda_2-\Lambda_3)^{r/2} \\ &\simeq_p \frac{1}{n^{r/2}}\frac{r!}{2^r(r/2)!}(\Lambda_1-\Lambda_2-\Lambda_3)^{r/2} \end{aligned} \tag{2.35}$$

from assumption (B_1). Hence from (2.16) and Lemma C (1.3),

$$\text{Plim } M_r/M_2^{r/2} = r!/2^r(r/2)!. \tag{2.36}$$

Theorem I follows from (2.29), (2.36) and the theorem on stochastic convergence of distribution functions [3].

3. Joint distribution of two or more serial statistics. We shall now generalise Theorem I to two sets of functions $\{f_t^{(1)}(x_1 \cdots x_k)\}$ and $\{f_t^{(2)}(x_1 \cdots x_k)\}$ satisfying condition (A). Let

$$\begin{cases} F_n^{(1)} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} f_i^{(1)}(x_i \cdots x_{i+k-1}), \qquad F_n^{(2)} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} f_i^{(2)}(x_i \cdots x_{i+k-1}), \\ \qquad M_{r,s} = E'\{[F_n^{(1)} - \bar{F}_n^{(1)}]^r[F_n^{(2} - \bar{F}_n^{(2)}]^s\}. \end{cases} \tag{3.1}$$

where $E'(F_t^{(1)}) = \bar{F}_n^{(1)}$ and $E'(F_n^{(2)}) = \bar{F}_n^{(2)}$. When condition ($B_1$) is satisfied by the functions $\{f_t^{(1)}\}$ and $\{f_t^{(2)}\}$, and (B_2), to be stated later, is satisfied, the limiting form of the joint nonparametric distribution of $(F_n^{(1)} - \bar{F}_n^{(1)})/\sqrt{M_{2,0}}$ and $(F_n^{(2)} - \bar{F}_n^{(2)})/\sqrt{M_{0,2}}$, for permutations of $x_1 \cdots x_n$, is a bivariate normal distribution with means zero, variances unity, and correlation coefficient ρ (defined in B_2). Let

$$G_n = (1/n)\sum g_i(x_i \cdots x_{i+k-1}) \quad \text{and} \quad H_n = (1/n)\sum h_i(x_i \cdots x_{i+k-1}), \tag{3.2}$$

where

$$g_i(x_i \cdots x_{i+k-1}) = f_i^{(1)}(x_i \cdots x_{i+k-1}) - \bar{F}_n^{(1)},$$
$$h_i(x_i \cdots x_{i+k-1}) = f_i^{(2)}(x_i \cdots x_{i+k-1}) - \bar{F}_n^{(2)},$$

so that $E'(G_n) = 0$ and $E'(H_n) = 0$ and $M_{r,s} = E'\{G_n^r H_n^s\}$ for $r, s = 2, 3, 4, \cdots$ In the expression for $M_{r,s}$ we get products of g-factors and h-factors. As in Section 2, we define P-sets consisting of g-factors and h-factors connected by common x-suffixes. The P-sets consisting of g-factors only will be called $P(g)$-sets and those containing both g-factors and h-factors will be called $P(g, h)$-sets or mixed P-sets.

As in Section 2, we have P-sets of type I and II. A P-set is of type I when, by a permutation of $(x_1 \cdots x_n)$, it can be written as

$$\prod_{j=1}^{e} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j}[h_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha'_j}$$

where $| q_{j-q_{j+1}} | < k$ holds (either α_j or α'_j, may be zero but not both).
All P-sets in the expansion of $M_{r,s}$ are obviously of type I.

A mixed P-set is of type II when, by a permutation of $x_1 \cdots x_n$, it can be expressed as $g_{q_1}(x_1 \cdots x_k)h_{q_2}(x_{\sigma_1} \cdots x_{\sigma_k})$, where $\sigma_j = i$ for $i, j \leqq k$ and the other suffixes $\sigma_1 \cdots \sigma_k$ are different from 1 to k.

The relations of the g and h factors and the x-suffixes of a P-set which are invariant for permutations of $x_1 \cdots x_n$ will be called its structure. Two P-sets will be considered to have identical structure if one can be derived from the other by a permutation of $x_1 \cdots x_n$, both the q-suffixes and the distinction between g-factors and h-factors being ignored.
We may now write,

$$M_{r,s} = \frac{1}{n^{r+s}} \sum{}' C_{r,s}(\alpha, l) \sum_{q,q'}{}_{s} E' \left\{ \prod_{g=1}^{e_1} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \times \prod_{j=1}^{e_2} [h_{q'_j}(x_{q'_j} \cdots x_{q'_j+k-1})]^{\alpha'_j} \right\}, \tag{3.3}$$

where $l_1 + \cdots + l_m = e_1$ and $l'_1 + \cdots + l'_m = e_2$. The summation $\sum_{q,q'}{}_s$ is taken for all $P(g, h)$-sets of type I with indices $(\alpha_1 \cdots \alpha_{l_1}, \alpha'_1 \cdots \alpha'_{l'_1}) \cdots (\alpha_{l_1+\cdots+l_{m-1}+1} \cdots \alpha_{l_1+\cdots+l_m}, \alpha'_{l'_1+\cdots+l'_{m-1}+1} \cdots \alpha'_{l'_1+\cdots+l'_m})$, where $\sum\alpha_i = r$, $\sum\alpha'_i = s$, $\sum l_i = e_1$, and $\sum l'_i = e_2$. The summation $\sum'$ is taken for all systems of values of $\alpha_1 \cdots \alpha_{e_1}$ and $\alpha'_1 \cdots \alpha_{e'_2}$ satisfying $\sum\alpha_i = r$ and $\sum\alpha'_i = s$ and $l_1, \cdots, l_m, l'_1, \cdots, l'_m$, with appropriate coefficients

$$C_{r,s}(\alpha, l) = C_{r,s}\{(\alpha_1 \cdots \alpha_{l_1}, \alpha'_1 \cdots \alpha'_{l'_1}) \cdots (\alpha_{l_1+\cdots+l_{m-1}+1} \cdots \alpha'_{l'_1+\cdots l'_m})\},$$

corresponding to the number of ways of grouping r g-factors and s h-factors into P-sets with given indices.

It is easy to see that the Lemmas I to IV may easily be generalised to the present case.

Considering products of g-factors only or of h-factors only, stochastic asymptotic expressions for $M_{2,0}$ and $M_{0,2}$ are given by Lemma V. We now find the asymptotic expression for $M_{1,1}$.

LEMMA V*

$$nM_{1,1} =_p [\Lambda_{1,g,h} - \Lambda_{2,g,h} - \Lambda_{3,g,h}]/n \tag{3.4}$$

where

$$(3.5)\quad \begin{cases} \Lambda_{1,g,h} = \sum_{|i-j|<k} E'\{g_i(x_i \cdots x_{i+k-1})\, h_j(x_j \cdots x_{j+k-1})\} \\ \Lambda_{3,g,h} = \sum_{|i-j|<k} E'\{g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\} \\ \Lambda_{2,g,h} = \sum_{x,y=1}^{k} \frac{Z_{x,y,g,h}}{n} = \sum_{x,y=1}^{k} \sum_{i,j=1}^{n} E'\left\{\frac{g_i(x_1 \cdots x_k)\, h_j(x_{\sigma_1} \cdots x_{\sigma_k})}{n}\right\} \end{cases}$$

where x, y in $\sum_{x,y}$ assume all values from 1 to k and $\sigma_y = x$ $(x, y \leqq k)$ and other suffixes $\sigma_1 \cdots \sigma_k$ are different from 1 to k.

PROOF.

$$M_{1,1} = \frac{1}{n^2} E'\{[\sum_i g_i(x_i \cdots x_{i+k-1})][\sum_j h_j(x_j \cdots x_{j+k-1})]\}$$

$$= \frac{1}{n^2} \sum_{|i-j|<k} E'\{g_i(x_i \cdots x_{i+k-1})\, h_j(x_j \cdots x_{j+k-1})\}$$

$$+ \frac{1}{n^2} \sum_{|i-j|\geqq k} E'\{g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\}.$$

$$\sum_{|i-j|\geqq k} E'\{g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\}$$

$$= \sum_{i,j=1}^{n} \frac{\sum_p \{g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\}}{n^{[2k]}}$$

$$- \sum_{|i-j|<k} \frac{\sum_p \{g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\}}{n^{[2k]}}.$$

Since

$$\sum_{j=1}^{n} \sum_p h_j(x_{k+1} \cdots x_{2k}) = 0,$$

the above expression can be written as

$$- \frac{1}{n^{[2k]}} \{\sum_{i,j} \sum_p g_i(x_1 \cdots x_k) \sum{}'' h_j(x_{\sigma_1} \cdots x_{\sigma_k})$$

$$+ \sum_{|i-j|<k} \sum_p g_i(x_1 \cdots x_k)\, h_j(x_{k+1} \cdots x_{2k})\}$$

where $\sum''$ denotes, as before, a sum for sets of values of $\sigma_1 \cdots \sigma_k$ at least one of which belongs to 1 to k. As in Lemma V, considering highest order terms in n, the result follows, since the sum of all other terms converges stochastically to zero.

LEMMA VI.*

$$nM_{1,1} =_p \mu_{1,1}$$

where

$$\mu_{1,1} = \frac{1}{n} \sum_{|i-j|<k} E\{f_i^{(1)}(x_i \cdots x_{i+k-1})\, f_j^{(2)}(x_j \cdots x_{j+k-1})\}$$

$$(3.6) \qquad - \frac{1}{n} \sum_{|i-j|<k} E\{f_i^{(1)}(x_1 \cdots x_k)\, f_j^{(2)}(x_{k+1} \cdots x_{2k})\}$$

$$- \frac{1}{n^2} \sum_{i,j=1}^{n} \sum_{x,y=1}^{k} E\{f_i^{(1)}(x_1 \cdots x_k)\, f_j^{(2)}(x_{\sigma_1} \cdots x_{\sigma_k})\} + k^2 \mu_{1,0}\, \mu_{0,1},$$

where $\sigma_1 \cdots \sigma_k$ *have just one element common with* 1 *to* k, $\sigma_x = y$ $(x, y \leqq k)$, *and*

$$\mu_{1,0} = \frac{1}{n} \sum_{t=1}^{n} E\{f_t^{(1)}(x_1 \cdots x_k)\}, \qquad \mu_{0,1} = \frac{1}{n} \sum_{t=1}^{n} E\{f_t^{(2)}(x_1 \cdots x_k)\}.$$

The proof of this lemma is analogous to that of Lemma VI and will be omitted. We now have the condition

$$(3.7)\ (\mathrm{B}_2) \qquad \lim \mu_{1,1}/\sqrt{\mu_{2,0} \times \mu_{0,2}} = \rho$$

exists and is less than one in absolute value.

Reduction of P-sets. As in Section 2, we define the sum

$$(3.8) \qquad M(\rho, t, v, \beta, \gamma) = \textstyle\sum_P \sum_{q,q'} \displaystyle\prod_{j=1}^{e_1} [g_{q_j}(x_{j,1} \cdots x_{j,k})]^{\alpha_j} \times \prod_{j=1}^{e_2} [h_{q'_j}(x_{j,k+1} \cdots x_{j,2k})]^{\alpha'_j}.$$

Here there are three sets of g and h factors: $e_1 + e_2 - \beta - \gamma$ g or h factors of A form t P (g, h)-sets of type I, ignoring g or h factors of B or C; γ g or h factors of C are tied to g or h factors of A; and β g or h factors of B have common x-suffixes with other g or h factors forming $P(g, h)$-sets, which always contain a g or h factor of A or C and such that there are v free linear P-sets. The summation is for q-suffixes so that the q-suffixes of the g or h factors of A may assume any value between 1 and n, subject to the restriction that these correspond to t $P(g, h)$-sets of type I with given structures and indices of P-sets, while the q-suffixes of the g or h factors of C are tied to the q-suffixes of A and the q-suffixes of B are free and may assume any value between 1 and n.

The structural relation between these factors must be such that there are v linear P-sets among them which are not connected with the g or h factors of B by common x-suffixes and have no g or h factors of C tied to them, and which we call free linear P-sets.

We may now apply the reduction process described in Section 2 to free linear sets in a M-set. We get M-sets of type

$$(3.9) \qquad M(\rho - \omega_s, t - 1, v, \beta + 1, \gamma), \qquad M(\rho - \omega'_s, t - 1, v_2, \beta, \gamma + 1),$$

where $\omega_s \geqq 1$, $v_1 \geqq v - \omega_s - 1$, $\omega'_s \geqq 0$, and $v_2 \geqq v - \omega'_s - 2$. Also it may be shown that, retaining terms of highest order in n, we need consider only such

M-sets for which either ω_s assumes the value one, corresponding to the case where the free linear P-set is reduced to a g or h factor of B and is connected with another untied (free) linear P-set forming a P-set of type II, or ω'_s assumes the value zero when the free linear P-set is reduced to a g or h factor of C and is tied to a free linear P-set forming a Q-set.

Now in the expression of $M_{r,s}$ all the terms consist of P-sets of type I. For given values of the indices $\alpha_1 \cdots \alpha_{e_1}$ and $\alpha'_1 \cdots \alpha'_{e_2}$, we group together all terms with different q-suffixes, so that there are m $P(g, h)$-sets of type I with any structures, but with indices

$$(\alpha_1 \cdots \alpha_{l_1}, \alpha'_1 \cdots \alpha'_{l'_1}) \cdots (\alpha_{l_1+\cdots+l_{m-1}+1} \cdots \alpha_{l_1+\cdots+l_m}, \alpha'_{l'_1+\cdots+l'_{m-1}+1} \cdots \alpha'_{l'_1+\cdots+l'_m}).$$

Such a sum is represented as

$$(3.10) \qquad \sum_{q,q'}{}_s E'\left\{\prod_{j=1}^{e_1} [g_{q_j}(x_{q_j} \cdots x_{q_j+k-1})]^{\alpha_j} \prod_{j=1}^{e_2} [h_{q'_j}(x_{q'_j} \cdots x_{q'_j+k-1})]^{\alpha'_j}\right\}.$$

Proceeding exactly as in Section 2 and from (3.9), we can prove

LEMMA VII*. *Let m be the total number of P-sets and v linear P-sets in the expression* (3.10). *Then the sum* (3.10) *is of the order* $o_p(n^{m-[(v+1)/2]+\delta})$.

We shall now prove the following theorem.

THEOREM II. *Let $\{f_i^{(1)}(x_1 \cdots x_k)\}$ and $(f_i^{(2)}(x_1 \cdots x_k)\}$, for $i = 1, 2, 3, \cdots$, be two sequences of functions satisfying the conditions* (A), (B₁) *and* (B₂). *Then the joint nonparametric distribution of*

$$(F_n^{(1)} - \bar{F}_n^{(1)})/\sqrt{M_{2,0}} \quad \textit{and} \quad (F_n^{(2)} - \bar{F}_n^{(2)})/\sqrt{M_{0\,2}}$$

converges stochastically to the bivariate normal distribution with means zero, variances unity, and correlation coefficient ρ given by (3.7).

PROOF. In the expression for $M_{r,s}$ we get sums of the type (3.10) with coefficients $1/n^{r+s}$. If in any such sum (3.10) there are m' P-sets of which v' are linear, then

$$2(m' - v') + v' \leqq r + s \quad \text{or} \quad m' - v'/2 \leqq (r + s)/2.$$

where equality holds only when all the nonlinear $P(g, h)$-sets are of length two. Thus $m' - [(v' + 1)/2] < (r + s)/2$ when $r + s$ is odd. When $r + s$ is even, we need consider sums (3.10) with $2u'$ linear P-sets and $(r + s)/2 - u'$ $P(g, h)$ sets of length two. These nonlinear $P(g, h)$-sets are of type I and are obtained from every manner of grouping $r + s - 2u'$ factors in

$$[\sum g_i(x_i \cdots x_{i+k-1})]^r[\sum h_i(x_i \cdots x_{i+k-1})]^s$$

into pairs. Also by the process of reduction described before, we get a P-set of type II for each way combining a g or h factor of B with another linear P-set, while we get a Q-set for each way of tying up a g or h factor of C with another linear P-set.

After complete reduction of free linear P-sets in an expression of the form (3.10), with ξ $P(g)$-sets and $Q(g)$-sets of length two, ζ $P(h)$-sets and $Q(h)$-sets of length two, and η mixed P-sets of length two, we get, exactly in the same manner as in Section 2, terms like

$$(3.11)\quad [\Lambda_{1,g}]^{\omega_1}[\Lambda_{1,h}]^{\omega_2}[\Lambda_{1,g,h}]^{\omega_3}[-\Lambda_{2,g}]^{\beta_1} [-\Lambda_{2,h}]^{\beta_2}[-\Lambda_{2,g,h}]^{\beta_3}[-\Lambda_{3,g}]^{\gamma_1}[-\Lambda_{3,h}]^{\gamma_2}[-\Lambda_{3,g,h}]^{\gamma_3},$$

$$(3.12)\quad \omega_1+\omega_2+\omega_3 = (r+s)/2 - u', \quad \beta_1+\beta_2+\beta_3=\beta, \quad \gamma_1+\gamma_2+\gamma_3=\gamma, \qquad u'=\beta+\gamma$$

$$(3.13)\quad \omega_1+\beta_1+\gamma_1=\xi, \qquad \omega_2+\beta_2+\gamma_2=\zeta, \quad \omega_3+\beta_3+\gamma_3=\eta, \qquad 2\xi+\eta=r, \qquad 2\zeta+\eta=s.$$

The $\Lambda_{1,g}$, $\Lambda_{2,g}$, $\Lambda_{3,g}$ and $\Lambda_{1,h}$, $\Lambda_{2,h}$, $\Lambda_{3,h}$ are given by (2.14) for functions of $f_i^{(1)}$ and $f_i^{(2)}$, respectively, while $\Lambda_{1,g,h}$, $\Lambda_{2,g,h}$, and $\Lambda_{3,g,h}$ are given by (3.5).

A term like (3.11) occurs in the reduced form of $M_{r,s}$ for each way of pairing $r+s$ factors in

$$[\sum g_i(x_i, \cdots x_{i+k-1})]^r[\sum h_i(x_i, \cdots x_{i+k-1})]^s$$

in appropriate manner. Thus the number of terms (3.11) in the reduced form of $M_{r,s}$ is given by

$$(3.14)\qquad K(r, s, \omega_1, \omega_2, \omega_3; \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3).$$

This is the number of ways of choosing 2ξ factors from r g-factors and 2ζ factors from s h-factors to form: ω_1 $P(g)$-sets, ω_2 $P(h)$-sets, and ω_3 $P(g, h)$-sets of type I; β_1 $P(g)$-sets, β_2 $P(h)$-sets, and β_3 $P(g, h)$-sets of type II; and γ_1 Q-sets of g-factors, γ_2 Q-sets of h-factors, and γ_3 Q-sets of mixed type. Taking the sum of all values of ω_1, ω_2, ω_3, β_1, β_2, β_3, γ_1, γ_2, γ_3, satisfying (3.12) and (3.13) we have, from assumptions (B_1), (B_2) and Lemma VI*.

$$(3.15)\quad n^{(r+s)/2}M_{r,s} =_p \frac{1}{n^{(r+s)/2}}\sum L(r, s, \xi, \eta, \zeta) \times (\Lambda_{1,g}-\Lambda_{2,g}-\Lambda_{3,g})^{\xi} \times (\Lambda_{1,h}-\Lambda_{2,h}-\Lambda_{3,h})^{\zeta} \times (\Lambda_{1,g,h}-\Lambda_{2,g,h}-\Lambda_{3,g,h})^{\eta}$$
$$\begin{cases} \simeq_p \sum_{\xi\eta\zeta} L(r, s, \xi, \eta, \zeta)[M_{2,0}]^{\xi}[M_{1,1}]^{\eta}[M_{0,2}]^{\zeta} & r+s \text{ even}, \\ =_p 0 & r+s \text{ odd}. \end{cases}$$

Where $L(r, s, \xi, \eta, \zeta)$ is the number of ways in which: 2ξ factors can be selected from r g-factors; 2ζ factors can be selected from s h-factors; and ξ pairs of g-factors, ζ pairs of h-factors, and η mixed pairs of g and h factors may be formed. Thus

$$(3.16)\quad L(r, s, \xi, \eta, \zeta) = \frac{r!}{(2\xi)!(r-2\xi)!}\cdot\frac{s!}{(2\zeta)!(s-2\zeta)!}\cdot\frac{(2\xi)!}{2^{\xi}.\xi!}\cdot\frac{(2\zeta)!}{2^{\zeta}.\zeta!}\cdot\eta! = r!\,s!/\xi!\,\eta!\,\zeta!2^{\xi+\zeta}.$$

From (3.15), (3.7), (1.3), and (B_2)

$$(3.17)\qquad \frac{M_{r,s}}{[M_{2,0}]^{r/2}[M_{0,2}]^{s/2}} \begin{cases} \simeq_p \sum_t \frac{r!\,s!}{2^{((r+s)/2)-t}}\,\rho^t \Big/ \left(\frac{r-t}{2}\right)!\,t!\left(\frac{s-t}{2}\right)! & r+s \text{ even}, \\ =_p 0 & r+s \text{ odd}, \end{cases}$$

where t assumes even values only when r and s are even, otherwise assuming odd values only.

Obviously the right-hand side of (3.17) is the coefficient of $z_1^r z_2^s/r!s!$ in the expansion of

$$(3.18)\qquad \exp\{\tfrac{1}{2}(z_1^2 + 2\rho z_1 z_2 + z_2^2)\}$$

which is the moment-generating function of the distribution

$$1/2\pi\sqrt{(1-\rho^2)}\exp\{-(1/2(1-\rho^2))(x^2 - 2\rho xy + y^2)\}.$$

Thus the result follows from an extension of Ghosh's theorem [3] to two dimensions.

It is possible to extend these results to the case of sequences of functions,

$$(3.19)\qquad \{f_i^{(1)}(x_1 \cdots x_k)\} \cdots \{f_i^{(p)}(x_1 \cdots x_k)\}$$

satisfying conditions (A) and (B_1) and (B_p) stated below. Let

$$(3.20)\qquad \begin{cases} F_n^{(t)} = \frac{1}{n}\sum_{i=1}^{n} f_i^{(t)}(x_i \cdots x_{i+k-1}) & t = 1, 2, \cdots, p \\ \bar{F}_n^{(t)} = E'(F_n^{(t)}) \\ M_2^{(t)} = E'\{[F_n^{(t)} - \bar{F}_n^{(t)}]^2\} \\ M_{1,1}^{(t_1,t_2)} = E'\{[F_n^{(t_1)} - \bar{F}_n^{(t_1)}][F_n^{(t_2)} - \bar{F}_n^{(t_2)}]\}. \end{cases}$$

Since the functions $\{f_i^{(t)}(x_1 \cdots x_k)\}$ satisfy the condition (B_1),

$$(3.21)\qquad nM_2^{(t)} \simeq_p \mu_{2,n}^{(t)} \quad\text{and}\quad \liminf \mu_{2,n}^{(t)} > 0,$$

for all t. Let

$$(3.22)\qquad \mu_1^{(t)} = E(F_n^{(t)}).$$

It can be shown, as in Lemma VI, that when (B_p) holds,

$$(3.23)\qquad nM_{1,1}^{(t_1,t_2)} \simeq_p \mu_{1,1}^{(t_1,t_2)},$$

where

$$(3.24)\qquad \begin{aligned} \mu_{1,1}^{(t_1,t_2)} &= \frac{1}{n}\sum_{|i-j|<k} E\{f_i^{(t_1)}(x_i \cdots x_{i+k-1})\, f_j^{(t_2)}(x_j \cdots x_{j+k-1})\} \\ &\quad - \frac{1}{n}\sum_{|i-j|<k} E\{f_i^{(t_1)}(x_1 \cdots x_k)\, f_j^{(t_2)}(x_{k+1} \cdots x_{2k})\} \\ &\quad - \frac{1}{n^2}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{k} E\{f_i^{(t_1)}(x_1 \cdots x_k)\, f_j^{(t_2)}(x_{\sigma_1} \cdots x_{\sigma_k})\} \\ &\quad + k^2\,\mu_1^{(t_1)}\,\mu_1^{(t_2)} \end{aligned}$$

where $\sigma_1 \cdots \sigma_k$ have just one element common with $1 \cdots k$ and $\sigma_\beta = \alpha$ $(\alpha, \beta \leqq k)$. We shall assume condition (B_p) that the matrix

$$\text{(3.25) } (B_p) \qquad \alpha_{ij} = \begin{cases} \lim_{n\to\infty} \mu_{1,1}^{(i,j)}/\sqrt{\mu_2^{(i)}\,\mu_2^{(j)}} & i \neq j \\ 1 & i = j \end{cases}$$

exists and is a nonsingular correlation matrix. We may then state

THEOREM III. *Let* $\{f_i^{(t)}(x_1 \cdots x_k)\}$, $(t = 1 \cdots p)$ *be sequences of functions satisfying conditions* (A), (B_1) *and* (B_p). *Then the joint nonparametric distribution of*

$$(F_n^{(1)} - \bar{F}_n^{(1)})/\sqrt{M_2^{(1)}} \cdots (F_n^{(p)} - \bar{F}_n^{(p)})/\sqrt{M_2^{(p)}}$$

converges stochastically to the multivariate normal distribution with means zero and correlation matrix (α_{ij}) *given by* (3.25).

4. Randomised distribution of serial statistics and power function. When the variables $x_1 \cdots x_n$ are independently and identically distributed, the conditional distribution in the universe of permutations $\Gamma_n(x_1 \cdots x_n)$ is uniform. When, however, the variables are correlated or have different distributions, the conditional distribution in $\Gamma_n(x_1 \cdots x_n)$ is not in general uniform. For any nonsymmetric statistic $T(x_1 \cdots x_n)$ which assumes the values

$$\text{(4.1)} \qquad T_1, T_2, \cdots, T_N \qquad N = n!$$

for different points of $\Gamma_n(x_1 \cdots x_n)$, let the conditional probabilities associated with these values be $\pi_1 \cdots \pi_N$ $(\sum \pi_i = 1)$. By a randomisation (random permutation) of the sample $x_1 \cdots x_n$, we can make the probabilities of $T_1 \cdots T_N$ all equal to $1/N$, whatever be the alternative hypothesis. The repartition (Von Mises [8]) of T also gives the probability distribution of T, when the probabilities of $T_1 \cdots T_N$ are equal, that is for a randomised sample. It will be called the randomised distribution function of the nonsymmetric statistic $T(x_1 \cdots x_n)$. When the variables $x_1 \cdots x_n$ form a Markov process of order p (stationary or not), we shall find the stochastic limiting form of the randomised distribution of a nonsymmetric statistic

$$\text{(4.2)} \qquad S(x_1 \cdots x_n) = \frac{1}{n} \sum_{i=1}^{n} f_i(x_i \cdots x_{i+k-1})$$

where the functions $\{f_t(x_1 \cdots x_k)\}$ satisfy the condition

$$\text{(A}'\text{)} \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f_t(x_{i_1} \cdots x_{i_k})|^s \, dF_{i_1 \cdots i_k}(x_{i_1} \cdots x_{i_k}) < c_s$$

for all systems of values of $i_1 \cdots i_k$ and for all t and s, $F_{i_1 \cdots i_k}(x_{i_1} \cdots x_{i_k})$ being the joint distribution function of $x_{i_1} \cdots x_{i_k}$.

For the randomised distribution, the expected value $E'(S)$ of the nonsymmetric function S is given by

$$M_1 = E'(S) = \frac{1}{n} \sum_{t=1}^{n} \frac{\sum_p f_t(x_1 \cdots x_k)}{n^{[k]}} \tag{4.3}$$

and the moments M_r are given by

$$M_r = E'\{(S - M_1)^r\} \qquad r = 2, 3, 4, \cdots \tag{4.4}$$

Except for Lemmas III and VI, all the results derived in Section 2, that is, Lemmas V, VII, and Theorem I, may be applied to randomised distributions. We shall prove Lemmas III(a) and VI(a), generalising III and VI.

LEMMA III(a). *Let* $U = E'\{\phi(x_1 \cdots x_k)\}$. *Then for a Markov process of order p, we have, when* (A′) *holds,*

$$\text{Var}(U) < kc_2/n \tag{4.5}$$

where k is a constant free from n and the function $\phi(x_{i_1} \cdots x_{i_k})$, *and* c_2 *is the upper bound of the second moment of* $\phi(x_{i_1} \cdots x_{i_k})$.

PROOF.

$$\begin{aligned}\text{Var}(U) &= E\{[\textstyle\sum_p[\phi(x_{i_1}\cdots x_{i_k}) - \bar{\phi}_{i_1\cdots i_k}]/n^{[k]}]^2\}\\ &= (n^{[k]})^{-2}\textstyle\sum_p' E\{[\phi(x_{i_1} \cdots x_{i_k}) - \bar{\phi}_{i_1\cdots i_k}][\phi(x_{j_1} \cdots x_{j_k}) - \bar{\phi}_{j_1\cdots j_k}]\}\end{aligned}$$

where $\bar{\phi}_{i_1\cdots i_k} = E[\phi(x_{i_1} \cdots x_{i_k})]$ and $\sum_p'$ is a summation for all systems $i_1 \cdots i_k$ and $j_1 \cdots j_k$.

We need only consider such terms in the summation $j_1 \cdots j_k$ for which at least one lies in the ranges $(i_1 - p, i_1 + p) \cdots (i_k - p, i_k + p)$. The number of terms satisfying this condition is $O(n^{2k-1})$. Again

$$\begin{aligned}&E\{[\phi(x_{i_1} \cdots x_{i_k}) - \bar{\phi}_{i_1\cdots i_k}][\phi(x_{j_1} \cdots x_{j_k}) - \bar{\phi}_{j_1\cdots j_k}]\}\\ &\qquad \leqq \sqrt{\text{Var}[\phi(x_{i_1} \cdots x_{i_k})]\text{Var}[\phi(x_{j_1} \cdots x_{j_k})]} \leqq c_2\end{aligned}$$

since, from assumption (A′), the second moment of $\phi(x_{i_1} \cdots x_{i_k})$ is bounded by c_2; hence the result.

From Lemma III(a), proceeding exactly as in Section 2, we have Lemma VI(a).

LEMMA VI(a).

$$\begin{aligned}nM_2 =_p &\frac{1}{n} \sum_{|i-j|<k} \frac{\sum_p E\{f_i(x_i \cdots x_{i+k-1})\, f_j(x_j \cdots x_{j+k-1})\}}{n^{[k+|i-j|]}}\\ &- \frac{1}{n} \sum_{|i-j|<k} \frac{\sum_p E\{f_i(x_{i_1} \cdots x_{i_k})\, f_j(x_{i_{k+1}} \cdots x_{i_{2k}})\}}{n^{[2k]}}\\ &- \frac{1}{n^2} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{k} \frac{\sum_p E\{f_i(x_1 \cdots x_k)\, f_j(x_{\sigma_1} \cdots x_{\sigma_k})\}}{n^{[2k-1]}}\\ &+ k^2 \left\{\frac{1}{n} \sum_{t=1}^{n} \frac{\sum_p E[f_t(x_{i_1} \cdots x_{i_k})]}{n^{[k]}}\right\}^2\\ &= \mu_{2,n}\end{aligned} \tag{4.6}$$

where $i_1 \cdots i_k$, $i_{k+1} \cdots i_{2k}$ *are any set of* $2k$ *different numbers from* 1 *to* n, *and* $\sigma_1 \cdots \sigma_k$ *are any fixed set of different numbers, from* 1 *to* n, *which have exactly one element in common with* $1 \cdots k$.

We now assume that the variables $x_1 \cdots x_n$ and the functions $\{f_i(x_1 \cdots x_k)\}$ satisfy the condition

$$\text{(B}'\text{)} \qquad \liminf \mu_{2,n} > 0.$$

THEOREM IV. *Let* $x_1 \cdots x_n$ *be a sequence of random variables forming a Markov process of order* p *(stationary or not) and let* $\{f_t(x_1 \cdots x_k)\}$ *be a sequence of functions satisfying conditions* (A′) *and* (B′). *Then the randomised distribution function of*

$$\left[\frac{1}{n}\sum_{i=1}^{n} f_i(z_i \cdots z_{i+k-1}) - M_1\right] \Big/ \sqrt{M_2} \tag{4.7}$$

converges stochastically to the normal distribution with mean zero and variance unity, where $(z_1 \cdots z_n)$ *is a random permutation of* $x_1 \cdots x_n$.

From the randomised distribution of T we shall derive the stochastic asymptotic expression for the power of a test in $\Gamma_n(x_1 \cdots x_n)$, for the alternative hypothesis H_1, according to which $x_1 \cdots x_n$ forms a Markov process of order p.

Let the conditional probability density of x_i for given values of $x_{i-1} \cdots x_{i-p}$ be $g_i(x_i \mid x_{i-1} \cdots x_{i-p})$. When only $x_{i-1} \cdots x_{i-j}$ $(j < p)$ are given, let the conditional probability density of x_i be $g_i(x_i \mid x_{i-1} \cdots x_{i-j})$. The joint probability distribution of $x_1 \cdots x_n$ is then given by

$$\prod_{i=1}^{p} g_i(x_i \mid x_{i-1} \cdots x_1) \prod_{i=p+1}^{n} g_i(x_i \mid x_{i-1} \cdots x_{i-p})\, dx_1 \cdots dx_n \tag{4.8}$$

We shall assume that the functions $\{\log g_i(x_{j+1} \mid x_j \cdots x_1)\}$ $(i = 1 \cdots n,$ $j = 1 \cdots p)$ satisfy the conditions (A′) and (B′).

A sufficient condition that the functions $\{\log g_i(X_{j+1} \mid X_j \cdots X_1)\}$ satisfy the condition (A′) is that there exists a polynomial

$$P(X_1 \cdots X_{j+1}) = \sum A_{k_1 \cdots k_{j+1}} X_1^{k_1} \cdots X_{j+1}^{k_{j+1}} \quad j = 1, \cdots, p \tag{4.9a}$$

such that

$$\mid \log g_i(X_{j+1} \mid X_j \cdots X_1) \mid < P(X_1 \cdots X_{j+1}) \tag{4.9b}$$

and also that

$$\int \mid x \mid^s g_i(x)\, dx < A'_s \tag{4.9c}$$

for all s, independent of i, $g_i(X)$ being the probability density of x_i. These conditions are usually satisfied for exponential type of populations considered in statistical theory.

Thus the randomised distribution of the nonsymmetric statistic

$$T = \left\{\frac{1}{n}\left[\sum_{i=1}^{p} \log g_i(z_i \mid z_{i-1} \cdots z_1) + \sum_{i=p+1}^{n} \log g_i(z_i \mid z_{i-1} \cdots z_{i-p})\right] - M_1\right\} \Big/ \sqrt{M_2} \tag{4.10}$$

converges stochastically to the normal distribution with mean zero and variance unity, $(z_1 \cdots z_n)$ being a random permutation of $x_1 \cdots x_n$ and M_1, M_2 being defined by (4.3) and (4.4) for $\{\log g_i(X_i \mid X_{i-1} \cdots X_1)\}$ when $i = 1 \cdots p$ and for $\{\log g_i(X_i \mid X_{i-1} \cdots X_{i-p})\}$ when $i > p$.

At any point $(z_1 \cdots z_n)$ in Γ_n, the probability according to H_1 is

$$\frac{1}{n!C'_n} \prod_{i=1}^{p} g_i(z_i \mid z_{i-1} \cdots z_1) \prod_{i=p+1}^{n} g_i(z_i \mid z_{i-1} \cdots z_{i-p})$$

C'_n independent of the order of $z_1 \cdots z_n$,

$$(4.11) \qquad = \begin{cases} \dfrac{1}{n!C'_n} \exp\{n(\sqrt{M_2}\,T + M_1)\}, \\ \dfrac{1}{n!C_n} \exp\{n\sqrt{M_2}\,T\}, \qquad C_n = E'\{\exp(n\sqrt{M_2}\,T)\}. \end{cases}$$

For two constants, T' and T'',

$$(4.12) \qquad \begin{aligned} \Pr\{T' < T \leqq T'' \mid x_1 \cdots x_n\} &= \Pr\{T' < T \leqq T'' \mid X_n\} \\ &= \frac{1}{C_n} \sum_{T' < T \leqq T''} \frac{\exp(n\sqrt{M_2}\,T)}{n!} = \frac{1}{C_n} \int_{T'}^{T''} \exp(n\sqrt{M_2}\,T)\, dG_n(T) \end{aligned}$$

where $G_n(T)$ is the randomised distribution function of T in $\Gamma_n(x_1 \cdots x_n)$. Now

$$C_n = E'(\exp n\sqrt{M_2}\,T) \geqq \int_a^b \exp(n\sqrt{M_2}\,T)\, dG_n(T)$$

$$\geqq \exp(n\sqrt{M_2}\,a)[G_n(b) - G_n(a)]$$

where a and b are constants. Since $G_n(x)$ converges stochastically to

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp - \xi^2/2\, d\xi,$$

which is uniformly continuous,

$$(4.13) \qquad \left| G_n(x) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp - \xi^2/2\, d\xi \right| < \epsilon$$

with probability greater than $1 - \delta$, whenever $n > n_0$, uniformly for x. Thus

$$G_n(b) - G_n(a) \geqq (2\pi)^{-\frac{1}{2}} \int_a^b \exp(-\xi^2/2)\, d\xi - \epsilon > k \qquad k > 0$$

with probability greater than $1 - \delta$ when $n > n_0$. Hence

$$\Pr\{C_n \geqq \exp(na\sqrt{M_2})k\} > 1 - \delta.$$

Again

$$\int_{T'}^{T''} \exp(n\sqrt{M_2}\,T)\, dG_n(T) \leqq \exp(n\sqrt{M_2}\,T'').$$

Thus

$P\{T' < T \leqq T'' \mid X_n\} \leqq 2 \exp (n\sqrt{M_2}T'')/C_n \leqq 2 \exp (n\sqrt{M_2}T'')/k \exp (n\sqrt{M_2}a)$ with probability greater than $1 - \delta$ Since $\exp (n\sqrt{M_2}(T'' - a)) \to 0$ as $n \to \infty$, when $a > T''$, from (B′), we find that $P\{T' < T \leqq T'' \mid X_n\}$ converges stochastically to zero, for fixed (T, T'), as $n \to \infty$. Thus we have

THEOREM V. *For the test of randomness of a sequence $X_1 \cdots X_n$ against an alternative H_1, for which the sequence forms a Markov process of order p, and where the logarithms of the conditional probabilities satisfy conditions* (A′) *and* (B′), *the acceptance region $T' < T \leqq T''$ is stochastically consistent, T being given by* (4.10).

The randomised distribution of a statistic T may be used to find a stochastic asymptotic form of the power function, so that a nonparametric test criterion may be selected, having desirable properties for large samples, on the basis of such power functions.

Two problems will be considered below for illustrative purposes. The first problem is concerned with the test for positive circular serial correlation in a sequence of random variables $x_1 \cdots x_n$. This problem has been solved by Lehmann and Stein [7], in which they obtain the most powerful randomised test function. We shall find the stochastic asymptotic power function of the corresponding nonrandomised test for large samples. The second problem is concerned with a more general type of stochastic pattern. In this case also it may be possible to get a most stringent test for small samples, along the lines of Lehmann and Stein [7], though this has not been considered before. From considerations of the stochastic asymptotic power functions we get an asymptotically most stringent test in this case.

Consider a sequence of random variables $x_1 \cdots x_n$ with circular serial correlation so that the conditional probability density function of x_{i+1} is

$$(2\pi)^{-\frac{1}{2}} \exp \{-(x_{i+1} - bx_i)^2/2\sigma^2\} = g(x_{i+1} \mid x_i).$$

We have

$$\begin{aligned}(1/n) \sum \log g(x_{i+1} \mid x_i) &= -\tfrac{1}{2} \log 2\pi - \log \sigma - \tfrac{1}{2}n\sigma^2 \sum (x_{i+1} - bx_i)^2 \\ &= -\tfrac{1}{2} \log 2\pi - \log \sigma - \tfrac{1}{2}\sigma^2\{(1 + b^2) \sum x_i^2/n - (2b/n) \sum x_i x_{i+1}\},\end{aligned}$$

which depends on the nonsymmetric statistic $T = \sum x_i x_{i+1}/n$ only, since $\sum x_i^2$, being a symmetric function of $X_1 \cdots X_n$, has the same value for all points in $\Gamma_n(x_1 \cdots x_n)$.

As shown by Lehmann and Stein [7], a uniformly most powerful randomised test function exists in this case for values of $b > 0$ or $b < 0$ and depends only upon values of $T = \sum x_i x_{i+1}/n$.

We consider only a nonrandomised test. Obviously for a test of significance with first kind of error α, the randomised test is equivalent to a nonrandomised test for some values of α.

Let $M_1 = E'(T)\, M_2 = E'[(T - M_1)^2]$ and $G_n(T)$ be the randomised distribution of $(T - M_1)/\sqrt{M_2}$. Then

$$P_b(T' < T \leqq T'' \mid X_n) = \Pr_{H_1(b)}(T' < T \leqq T'' \mid X_n)$$

(4.14)

$$= \frac{\int_{T'}^{T''} \exp\,[bn\sigma^{-2}(\sqrt{M_2}\,T + M_1)]\, dG_n(T)}{\int_{-\infty}^{\infty} \exp\,[bn\sigma^{-2}(\sqrt{M_2}\,T + M_1)]\, dG_n(T)} = \frac{\int_{T'}^{T''} \exp\,[bn\sigma^{-2}(\sqrt{M_2}\,T]\, dG_n(T)}{\int_{-\infty}^{\infty} \exp\,[bn\sigma^{-2}(\sqrt{M_2}\,T]\, dG_n(T)}.$$

We note that Theorem IV can be immediately extended to a set of circularly correlated variables. For any given $b(\neq 0)$, $P_b(T' < T \leqq T'' \mid X_n)$ converges stochastically to zero. Thus we study the power function for sufficiently small values of b and we consider $b_n = 0(1/\sqrt{n})$; say $\lim_{n\to\infty} \sqrt{n}b_n = \lambda$, $\lambda \neq 0$. In this case we have

$$\text{(4.15)} \qquad P\lim nM_2 = \sigma^4,$$

so that the condition (B′) holds, and $G_n(T)$ converges stochastically to the normal distribution with mean zero and variance unity. We have also

$$\text{(4.16)} \qquad \lambda_n = b_n n\sigma^{-2}\sqrt{M_2} \quad \text{and} \quad P\lim \lambda_n = \lambda.$$

Now

$$\int_{T}^{T''} \exp\,(\lambda_n T)\, dG_n(T) = [\exp\,(\lambda_n T) G_n(T)]_{T'}^{T''} - \int_{T'}^{T''} \lambda_n \exp\,(\lambda_n T) G_n(T)\, dT,$$

$$(2\pi)^{-\frac{1}{2}} \int_{T'}^{T''} \exp\,(\lambda_n T - T^2/2)\, dT = [\exp\,(\lambda_n T)\phi(T)]_{T'}^{T''}$$

$$- \int_{T'}^{T''} \lambda_n \exp\,(\lambda_n T)\phi(T)\, dT,$$

where $\phi(T) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{T} \exp\,(-\xi^2/2)\, d\xi$. Thus

$$\int_{T'}^{T''} \exp\,(\lambda_n T)\, dG_n(T) - (2\pi)^{-\frac{1}{2}} \int_{T'}^{T''} \exp\,(\lambda_n T - T^2/2)\, dT$$

$$= [G_n(T'') - \phi(T'')] \exp\,(\lambda_n T'') - [G_n(T') - \phi(T')] \exp\,(\lambda_n T')$$

$$- \int_{T'}^{T''} [G_n(T) - \phi(T)]\lambda_n \exp\,(\lambda_n T)\, dt.$$

Since $\Pr \{ \mid G_n(T) - \phi(T) \mid < \epsilon\} > 1 - \delta$, for $n > n_0(\epsilon, \delta)$,

$$\left| \int_{T'}^{T''} \exp (\lambda_n T)\, dG_n(T) - (2\pi)^{-\frac{1}{2}} \int_{T'}^{T''} \exp (\lambda_n T - T^2/2)\, dT \right|$$

$$\leqq \epsilon[\exp (\lambda_n T'') + \exp(\lambda_n T')] + \epsilon \int_{T'}^{T''} \lambda_n \exp (\lambda_n T)\, dT$$

$$\leqq 3\epsilon \exp (\lambda_n T'') \leqq 3\epsilon' \int_{T'}^{T''} \exp (\lambda_n T)\, dT$$

holds with probability greater than $1 - \delta$. As $\int_{T'}^{T''} \exp (\lambda_n T - T^2/2)\, dT$ is a uniformly continuous function of λ_n in (T', T''), we get

$$(4.17) \quad \begin{aligned} &P_{\lambda/n\sqrt{n}}\{T' < T \leqq T'' \mid X_n\} \\ &\qquad = ((1 + \epsilon_n'')/C_n\sqrt{2\pi}) \int_{T'}^{T''} \exp (\lambda T - T^2/2)\, dT \end{aligned}$$

with probability $>1 - \delta$ for $n > n_0(\epsilon'')$, where ϵ_n'' is a small quantity $\leqq \epsilon''$, and $C_n = \int \exp (\lambda_n T)\, dG_n(T)$. Hence

$$(4.18) \quad \begin{aligned} &P_{\lambda_n/\sqrt{n}}^{!}\{T' < T \leqq T'' \mid X_n\} \\ &\qquad \simeq_p [\exp (\tfrac{1}{2}\lambda^2)/C_n\sqrt{2\pi}] \int_{T'}^{T''} \exp [-\tfrac{1}{2}(T - \lambda)^2]\, dT. \end{aligned}$$

In order to show that when the parameter $b \sim \lambda/\sqrt{n}$ $(\lambda \neq 0)$ the hypothesis $H_1(b = O(1\sqrt{n}))$ may be discriminated against the hypothesis $H_0(b = 0)$, on the basis of a sample of size n, we prove the following inequality.

THEOREM VI.

$$\Pr\left\{P_{\lambda_n/\sqrt{n}}(T' < T \leqq T'' \mid X_n) < [(1 + \epsilon)/\sqrt{2\pi}] \int_{T'}^{T''} \exp\left[-\frac{(\xi - \lambda)^2}{2}\right] d\xi\right\} > 1 - \delta$$

holds for $n > n_0(\epsilon, \delta)$.

PROOF.

$$C_n = \int_{-\infty}^{\infty} \exp (\lambda_n T)\, dG_n(T) \geqq \int_{T_1}^{T_2} \exp (\lambda_n T)\, dG_n(T).$$

As shown above,

$$\int_{T_1}^{T_2} \exp (\lambda_n T)\, dG_n(T) \simeq_p [\exp (\tfrac{1}{2}\lambda^2)/\sqrt{2\pi}] \int_{T_1}^{T_2} \exp [-\tfrac{1}{2}(\xi - \lambda)^2]\, d\xi$$

that is,

$$\int_{T_1}^{T_2} \exp(\lambda_n T)\, dG_n(T) > \exp(\tfrac{1}{2}\lambda^2)(1 - \epsilon_1)$$

holds with probability greater than $1 - \delta/2$ for n sufficiently large. Thus

$$\Pr\{C_n > \exp(\tfrac{1}{2}\lambda^2)(1 - \epsilon_1)\} > 1 - \delta/2.$$

From (4.17) the result follows when ϵ_1 and ϵ'' are sufficiently small.

In the case of a single parameter, the optimum test procedure is obtained without the help of (4.18), the stochastically asymptotic power function, but we shall see that in the multiparameter case the stochastically asymptotic power function is a useful tool for the purpose of finding a test with good power properties for large samples.

Consider a more general stochastic pattern, in which the probability density of x_i depends upon $x_{i-1} \cdots x_{i-p}$ and is given by

$$(4.19)\quad g(x_i \mid x_{i-1} \cdots x_{i-p}) = (1/\sqrt{2\pi}\sigma) \exp\{-(x_i - b_1 x_{i-1} - \cdots - b_p x_{i-p})^2/2\sigma^2\}$$

the variables $x_1 \cdots x_n$ being considered in circular order.

$$\frac{1}{n}\sum \log g(x_i \mid x_{i-1} \cdots x_{i-p})$$

$$= -\frac{\log 2\pi}{2} - \log\sigma - \frac{1}{2n\sigma^2}\sum_{i=1}^{n}(x_i - b_1 x_{i-1} \cdots - b_p x_{i-p})^2$$

$$= -\frac{\log 2\pi}{2} - \log\sigma - \frac{1}{2\sigma^2}\left\{(1 + b_1^2 + \cdots + b_p^2)\frac{\sum x_i^2}{n} - 2\sum_{s \neq t=0}^{p} b_s b_t r_{s-t}\right\}$$

where $r_j = \sum x_i x_{i+j}/n$ and $b_0 = 1$.

Since $\sum x_i^2$ is a symmetric function of $x_1 \cdots x_n$, the probability distribution in $\Gamma_n(x_1 \cdots x_n)$ depends upon $r_1 \cdots r_p$ only and thus we need only consider the space $\bar{R}_p$, consisting of all points $(r_1 \cdots r_p)$.

Let $r_1' \cdots r_p'$ be standardised variables corresponding to $r_1 \cdots r_p$, that is, $r_i = (r_i - M_1^{(i)})/\sqrt{M_2^{(i)}}$ where $M_1^{(i)} = E'(r_i)$ and $M_2^{(i)} = E'[(r_i - M_1^{(i)})^2]$. Let $G_n(r_1' \cdots r_n')$ be the randomsed joint distribution of $r_1' \cdots r_p'$ in $\Gamma_n(x_1 \cdots x_n)$. Then $G(r_1' \cdots r_p')$ converges stochastically to the normal distrbution with means zero and dispersion matrix (γ_{ij}) and, as in Theorem V, any bounded region of acceptance C in $\bar{R}_p$ gives a stochastically consistent test. In particular, the region of acceptance

$$(4.20)\qquad \sum_{i,j=1}^{p} \gamma_{ij} r_i' r_j' < \alpha$$

gives a stochastically consistent test, for given values of $b_1 \cdots b_p$.

In order to find a stochastically asymptotic expression for the power function, we consider $b_i = O(1/\sqrt{n})$, say $b_i = \lambda_n^{(i)}/\sqrt{n}$, and $\lim \lambda_n^{(i)} = \lambda^{(i)}$. We have now

$$\text{Plim } nM_2^{(i)} = \sigma^4, \qquad \text{Plim } nM_{1,1}^{(i,j)} = 0 \qquad i \neq j \tag{4.21}$$

so that (B_p) holds and $\alpha_{ij} = \delta_j^i$, where δ_j^i is the Kronecker δ symbol. For any bounded region C in $\bar{R}_p$, we may show, as in (4.18),

$$P_{(\lambda_n^{(1)}/\sqrt{n}\cdots\lambda_n^{(p)}/\sqrt{n})}(C) \simeq_p \frac{1}{(2\pi)^{m/2}\sigma^m C_n} \int_C \exp\left(\sum \lambda_i r'_i\right) \exp\left(-\sum_{i=1}^{p} \frac{r_i'^2}{2\sigma^2}\right) dr'_1 \cdots dr'_p \tag{4.22}$$

where

$$C_n = \int_{\bar{R}_p} \exp\left(\frac{n}{\sigma^2} \sum_{s\neq t=0}^{p} b_s b_t r'_{s-t} \sqrt{M_2^{(s-t)}}\right) dG_n(r'_1 \cdots r'_p).$$

Corresponding to Theorem VI we have

THEOREM VI(a).

$$\Pr\left\{P_{(\lambda_n^{(1)}/\sqrt{n}\cdots\lambda_n^{(p)}/\sqrt{n})}(C) < \frac{1+\epsilon}{(2\pi)^{m/2}\sigma^m} \int_C \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{p}(\xi_i - \lambda^{(i)})^2\right) d\xi_1 \cdot d\xi_p\right\} > 1-\delta \tag{4.23}$$

for any bounded region C in $\bar{R}_p$ and for sufficiently large value of $n > n_0(\epsilon, \delta)$.

For fixed values of $b_1 \cdots b_p$ there exists a most powerful region, but there is no uniformly most powerful region for values of $b_1 \cdots b_p$. But we may apply the well known methods of multivariate normal theory to the stochastically asymptotic power function and select an optimum test. In the present case we may consider the most stringent test (Wald [14]) or the most powerful test on the average (Nandi [10]), the asymptotic power function being averaged over the spheres $(\lambda^{(1)})^2 + \cdots + (\lambda^{(p)})^2 = \rho^2$. The region of acceptance with these properties is given by

$$(r'_1)^2 + \cdots + (r'_p)^2 < \alpha. \tag{4.24}$$

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