As the level of significance increases, the power efficiency of the rank sum test increases slightly whereas the power efficiencies of the median and maximum deviation tests decrease.

TABLE II. When tests for samples of size 5 are randomized to the single level of significance $\alpha = .025$, it is easy to compare the tests and note that the rank sum test has greater power than the median and maximum deviation tests. Particularly for near alternatives, the maximum deviation test has greater power than the median test.

TABLE III. The local power efficiencies for the rank sum test are very high. For all cases computed they are greater than $3/\pi$, the limiting local power efficiency for large samples.

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Addendum

Papers on this topic appearing since submission of this paper include:
[6] E. L. LEHMANN, "The power of rank tests," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 23–43.
[7] B. L. VAN DER WAERDEN, "Order tests for the two sample problem," *Nederl. Akad. Wetensch. Proc. Ser. A.*, Vol. 56 (1953), pp. 303–316.

A REMARK ON THE JOINT DISTRIBUTION OF CUMULATIVE SUMS

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Let X_k, $k = 1, \cdots, n$, be any finite number n of independent random variables with respective distribution functions $F_k(x) = \Pr[X_k \leqq x]$. Let $T_k = X_1 + \cdots + X_k$ be the successive cumulative sums of the X_k, with individual distribution functions $G_k(t) = \Pr[T_k \leqq t]$ and joint distribution function $G(t_1, \cdots, t_n) = \Pr[T_1 \leqq t_1, \cdots, T_n \leqq t_n]$. Since the T_k are not in general stochastically independent, the function $G(t_1, \cdots, t_n)$ will not in general be equal to the product of the n functions $G_k(t_k)$, but we shall show that the *inequality*

$$G(t_1, \cdots, t_n) \geqq \prod_1^n G_k(t_k) \tag{1}$$

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always holds. This is intuitively more or less obvious, but the proof is not entirely trivial.

LEMMA 1 (Tchebycheff[1]). *Let X be a random variable and let $u(x)$ and $v(x)$ be any two nonincreasing functions of x for which $Eu(X)$ and $Ev(X)$ are finite; then*

$$E[u(X)v(X)] \geqq Eu(X) \cdot Ev(X). \tag{2}$$

LEMMA 2. *Using the notation of the first paragraph, let A, B denote any partition of the set of integers $1, \cdots, n$ into two disjoint subsets; then*

$$\begin{aligned} &\Pr[T_k \leqq t_k \text{ for all } k = 1, \cdots, n] \\ &\qquad \geqq \Pr[T_k \leqq t_k \text{ for all } k \in A] \cdot \Pr[T_k \leqq t_k \text{ for all } k \in B]. \end{aligned} \tag{3}$$

PROOF. Induction on n. The theorem is trivially true for $n = 1$, since one of the sets A, B must be empty. If it is true for $n - 1$, then for any fixed $x \leqq t_1$,

$$\begin{aligned} \Pr[T_k \leqq t_k \text{ for all } k = 1, \cdots, n \mid X_1 = x] &= \Pr[X_2 + \cdots + X_n \\ &\qquad \leqq t_k - x \text{ for all } k = 2, \cdots, n] \\ \geqq \Pr[X_2 + \cdots + X_k \leqq t_k - x &\text{ for all } k \in A - \{1\}] \cdot \Pr[X_2 + \cdots + X_k \\ &\qquad \leqq t_k - x \text{ for all } k \in B - \{1\}] \\ = \Pr[T_k \leqq t_k \text{ for all } k \in A &- \{1\} \mid X_1 = x] \cdot \Pr[T_k \\ &\qquad \leqq t_k \text{ for all } k \in B - \{1\} \mid X_1 = x] \\ = \Pr[T_k \leqq t_k \text{ for all } k \in A &\mid X_1 = x] \cdot \Pr[T_k \\ &\qquad \leqq t_k \text{ for all } k \in B \mid X_1 = x]. \end{aligned} \tag{4}$$

The inequality between the first and last members of (4) remains valid even for $x > t_1$, since then the first member and one of the two factors of the last member is 0. Thus, setting

$$\begin{aligned} u(x) &= \Pr[T_k \leqq t_k \text{ for all } k \in A \mid X_1 = x], \\ v(x) &= \Pr[T_k \leqq t_k \text{ for all } k \in B \mid X_1 = x], \end{aligned} \tag{5}$$

we have for all x $\Pr[T_k \leqq t_k \text{ for all } k = 1, \cdots, n \mid X_1 = x] \geqq u(x) \cdot v(x)$. Integrating from $-\infty$ to ∞ with respect to the distribution function $F_1(x)$ of X_1, we obtain

$$\Pr[T_k \leqq t_k \text{ for all } k = 1, \cdots, n] \geqq \int_{-\infty}^{\infty} u(x)v(x)\, dF_1(x). \tag{6}$$

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[1] See Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge (1934), p. 43; also M. Biernacki, "Sur une inégalité entre les intégrales due à Tchebycheff," *Annales Univ. Mariae Curie-Sklodowska*, Lublin, Sect. A, Vol. 5 (1951), pp. 23–29.

It is clear from the definitions (5) that both $u(x)$ and $v(x)$ are bounded and nonincreasing functions of x, and hence by (6) and (2)

$$\Pr[t_k \leqq T_k \text{ for all } k = 1, \cdots, n] \geqq \int_{-\infty}^{\infty} u(x)\, dF_1(x) \cdot \int_{-\infty}^{\infty} v(x)\, dF_1(x)$$
$$= \Pr[T_k \leqq t_k \text{ for all } k \varepsilon A] \cdot \Pr[T_k \leqq t_k \text{ for all } k \varepsilon B],$$

which proves (3).

THEOREM. *If $A_1, \cdots, A_r$ form a partition of the set of integers* $1, \cdots, n$ *into any number r of disjoint subsets, then*

$$\Pr[T_k \leqq t_k \textit{ for all } k = 1, \cdots, n] \geqq \prod_{j=1}^{r} \Pr[T_k \leqq t_k \textit{ for all } k \varepsilon A_j].$$

In particular, setting $r = n$, $A_j = \{j\}$, $j = 1, \cdots, n$, (1) holds.

PROOF. Induction on r, using Lemma 2.

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Pasadena meeting of the Institute, June 18–19, 1954)

1. The Integral of a Symmetric Unimodal Function over a Symmetric Convex Set and Some Probability Inequalities. T. W. ANDERSON, Columbia University and Stanford University.

The integral over an interval of fixed length of a symmetric unimodal function is maximized if the interval is centered at the origin; in fact, the value of the integral is a nonincreasing function of the distance of the midpoint of the interval from the origin. A generalization of this result to n-space is the following: *Theorem* 1. *Let E be a convex set in n-space, symmetric about the origin. Let $f(x) \geqq 0$ be a function such that $f(x) = f(-x)$, $\{x \mid f(x) \geqq u\}$ is convex for every $u(0 \leqq u \leqq \infty)$, and $\int_E f(x)\, dx < \infty$ (in the Lesbesgue sense). Then $\int_E f(x + ky)\, dx \geqq \int_E f(x + y)\, dx$ for $0 \leqq k \leqq 1$.* A direct consequence is that the distribution of $X + Y$ is more spread out than the distribution of X. *Theorem* 2. *Let X be a random vector with density $f(x)$ satisfying the conditions of Theorem* 1; *let Y be an independent random vector*; *and let E be a convex set, symmetric about the origin. Then* $\Pr\{X + kY \varepsilon E\} \geqq \Pr\{X + Y \varepsilon E\}$ *for* $0 \leqq k \leqq 1$. Inequalities are derived for distributions of functions of random variables such as ΣX_i^2 and $\max_{1 \leqq i \leqq n} |X_i|$ and corresponding functionals of stochastic processes. Another application is to show that certain tests of location parameters are unbiased. (Work supported by the Office of Naval Research.)

2. The Spectral Method of Hypothesis Testing Concerning Continuous Gaussian Stationary Random Processes. R. C. DAVIS, Hughes Tool Company.

Present rigorous methods of hypothesis testing concerning Gaussian stationary random functions depending upon a continuous parameter—in which a process is observed only during a finite time interval of duration T—have been based upon an analysis carried out in the time domain. In order to determine the sample decision function by this method for testing even a simple hypothesis against a single alternative, it is necessary to solve