A NOTE ON PARTIALLY BALANCED DESIGNS

By Marvin Zelen

National Bureau of Standards

It is well known [1], [2] that a singular group divisible design containing two associate classes can be derived from a balanced incomplete block design by replacing each treatment by n treatments. In this paper it is shown that a partially balanced design with $(m + 1)$ associate classes can be derived from a partially balanced design with m associate classes by replacing each treatment by n treatments.

The definition of a partially balanced incomplete block design with m associate classes can briefly be described as an experimental plan

(i) having v treatments arranged in b blocks such that each block contains k experimental units,

(ii) where each treatment is replicated r times and no treatment occurs more than once in any block,

(iii) such that with respect to any treatment t, the remaining treatments can be divided into m associate classes such that the ith class contains n_i treatments and t occurs in λ_i blocks with each of the treatments in the ith class $(i = 1, 2, \cdots, m)$,

(iv) and if two treatments are kth associates, the number of treatments common to the ith associates of one and the jth associates of the other treatment is p^k_{ij} (for $i, j, k = 1, 2, \cdots, m$, with $p^k_{ij} = p^k_{ji}$), and is independent of the particular pair of treatments.

It has been shown [3] that the following relations hold between the parameters of the design.

$$bk = vr \tag{1}$$

$$\sum_{i=1}^{m} n_i = v - 1 \tag{2}$$

$$\sum_{i=1}^{m} n_i \lambda_i = r(k - 1) \tag{3}$$

$$\sum_{j=1}^{m} p^k_{ij} = \begin{cases} n_i, & \text{for } i \neq k \\ n_i - 1 & \text{for } i = k \end{cases} \tag{4}$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}. \tag{5}$$

The main result of this paper can be stated in the following theorem.

Received 9/14/53.

Main Theorem. *If, in a partially balanced incomplete block design having m associate classes and parameters*

$$v^*,\ b^*,\ r^*,\ k^*,\ \lambda_i^*,\ n_i^*,\ p_{ij}^{*k} \qquad (i, j, k = 1, 2, \cdots, m), \tag{6}$$

such that $\lambda_i^* \neq r^* (i = 1, 2, \cdots, m)$, *each treatment is replaced by n different treatments, the derived design will be a partially balanced incomplete block design with* $(m + 1)$ *associate classes having parameters*

$$\left\{\begin{array}{llll} v = nv^*, & b = b^*, \quad k = nk^*, & & r = r^*, \\ \lambda_i = \lambda_i^*, & n_i = nn_i^*, & & i = 1, 2, \cdots, m, \\ \lambda_{m+1} = r^*, & n_{m+1} = n - 1, & & \\ p_{ij}^k = np_{ij}^{*k} & & & i, j, k = 1, 2, \cdots, m, \\ p_{k,m+1}^k = n - 1 & & & k = 1, 2, \cdots, m, \\ p_{i,m+1}^k = 0 & & i \neq k, & i, k = 1, 2, \cdots, m, \\ p_{ij}^{m+1} = 0 & & i \neq j,\ p_{ii}^{m+1} = nn_i^* & \text{for } i = 1, 2, \cdots, m, \\ p_{m+1,m+1}^{m+1} = n - 2. & & & \end{array}\right. \tag{7}$$

Proof. Let t be a treatment in the original design and denote the remaining treatments by $t_j^{(i)}$ for $j = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, m$, where $t_j^{(i)}$ are the ith associate treatments of t. Denote the treatments replacing t and $t_j^{(i)}$ in the derived design by the row vectors $\underline{t} = (t_1, t_2, \cdots, t_n)$ and $\underline{t}_j^{(i)} = t_{j1}^{(i)}, t_{j2}^{(i)}, \cdots, t_{jn}^{(i)})$ respectively; then the row vector $T_i = (\underline{t}_1^{(i)}, \underline{t}_2^{(i)}, \cdots, \underline{t}_{n_i}^{(i)})$ denotes the ith associates of any treatment element of $\underline{t}$. If t and t' are two kth associate treatments in the original design, and if t_p is any treatment element of $\underline{t}$ and t_q' is any treatment element of $\underline{t}'$, then t_p and t_q' will be kth associates. With respect to each of these treatments, the remaining treatments can be divided into $(m + 1)$ associate classes in the following manner:

Associate Classes

	$1 \quad \cdots k \cdots \quad m$	$m + 1$
t_p	$T_1 \cdots T_k \cdots T_m$	$t_1, \cdots, t_{p-1}, t_{p+1}, \cdots, t_n$
t_q'	$T_1' \cdots T_k' \cdots T_m'$	$t_1', \cdots, t_{q-1}', t_{q+1}', \cdots, t_n'.$

Upon replacing each treatment in the original design by n different treatments, the new design will have $v = nv^*$, $b = b^*$, $k = nk^*$, $r^* = r$. From the array we see that $n_i = nn_i^*$ for $i = 1, 2, \cdots, m)$ and $n_{m+1} = n - 1$. Since any treatment in the original design occurred in λ_i^* blocks with each of its ith associates, the new treatments will occur in $\lambda_i = \lambda_i^*$ blocks with each of their ith associates for $i = 1, 2, \cdots, m$. Also each treatment will occur in r blocks with each of its $(m + 1)$ associates, that is, $\lambda_{m+1} = r$.

Since t and t' have p_{ij}^{*k} treatments in common which are ith associates (say) of t and jth associates of t', then t_p and t'_q will have $p^k_{ij} = np^{*k}_{ij}$ treatments in common which are ith associates of t_p and jth associates of t'_q for $i, j, k = 1, 2, \cdots, m$. It is readily seen that the number of treatments in common between the kth associates of t_p and the $(m + 1)$ associates of t'_q is $p^k_{k,m+1} = (n - 1)$ for $k = 1, 2, \cdots, m$, and that $p^k_{i,m+1} = 0$ for $i \neq k$ and $i, k = 1, 2, \cdots, m$. Similarly if, with respect to a pair of treatments which are $(m + 1)$ associates, the remaining treatments were put in an array, it can be demonstrated that $p^{m+1}_{ij} = 0$ for $i \neq j$, while $p^{m+1}_{ii} = nn^*_i$ for $i = 1, 2, \cdots, m$, and $p^{m+1}_{m+1,m+1} = n - 2$.

It is now necessary to show that the relations (1) through (5) are satisfied. For (1) we have

$$bk = b^*nk^* = nr^*v^* = rv.$$

For (2) we have

$$\sum_{i=1}^{m+1} n_i = n(v^* - 1) + (n - 1) = v - 1.$$

For (3) we have

$$\sum_{i=1}^{m+1} n_i\lambda_i = nr^*(k^* - 1) + (n - 1)r = r(k - 1).$$

For (4) we have

$$\sum_{i=1}^{m+1} p^k_{ij} = \begin{cases} nn^*_j & j \neq k; j = 1, 2, \cdots, m; k = 1, \cdots, m + 1, \\ n(n^*_j - 1) + n - 1 = n_j - 1 & j = k; j, k = 1, 2, \cdots, m, \end{cases}$$

$$\sum_{i=1}^{m+1} p^k_{i,m+1} = n - 1 = n_{m+1} \qquad k = 1, 2, \cdots, m,$$

$$\sum_{i=1}^{m+1} p^{m+1}_{i,m+1} = n - 2 = n_{m+1} - 1.$$

For (5) we have

$$n_k p^k_{ij} = n^2n^*_kp^{*k}_{ij} = n_ip^i_{jk} = n^2n^*_ip^{*k}_{ij} = n_jp^j_{ik} = n^2n^*_jp^{*j}_{ik} \quad i, j, k = 1, 2, \cdots, m,$$

$$n_{m+1}p^{m+1}_{ij} = n_ip^i_{j,m+1} = n_jp^j_{i,m+1} = 0 \qquad i, j \neq m + 1,$$

$$n_ip^i_{i,m+1} = nn^*_i(n - 1) = n_{m+1}p^{m+1}_{ii} \qquad i = 1, 2, \cdots, m.$$

The condition that $\lambda^*_i \neq r^*$ arises from the fact that if this condition is not true, then with respect to a particular treatment t there will exist a group of (say) mth associate treatments which will occur with t in exactly r blocks. Since every treatment is replicated r^* times, t will always appear with the same group of treatments. Thus if a treatment occurs in a certain block, then every mth associate treatment will also occur in that block. Therefore it is possible to replace each group of treatments by a single treatment to derive a new design.

It can be shown, using an argument similar to that used to prove the main theorem, that the derived design will be a partially balanced incomplete block design with $(m - 1)$ associate classes having parameters

$$(8) \quad \begin{cases} v = v^*/n_m^*, \quad b = b, \quad r = r^*, \quad k = k^*/n_m^*, \\ \lambda_i = \lambda_i^*, \quad n_i = n_i^*/n_m^* \quad i = 1, 2, \cdots, m - 1, \\ p_{ij}^k = p_{ij}^{*k}/n_m^* \quad i, j, k = 1, 2, \cdots, m - 1. \end{cases}$$

This last result can be summarized in the following theorem.

THEOREM: *If in a partially balanced design having m associate classes and parameters* (6) *such that (say)* $\lambda_m^* = r$, *then the treatments can be divided into* v^*/n_m^* *groups so that each treatment occurs in a block with all the treatments of its group* r^* *times. Also it is possible to replace each group of treatments by one treatment to derive a partially balanced design with* $(m - 1)$ *associate classes and parameters given by* (8).

A large number of partially balanced incomplete block designs with two associate classes are available [4], [5]. These designs can be used to construct three associate class designs having parameters

$$v = nv^*, \quad b = b^*, \quad k = nk^*, \quad r = r^*,$$

$$\lambda_i = \lambda_i^*, \quad n_i = nn_i^*, \quad i = 1, 2,$$

$$\lambda_3 = r, \quad n_3 = n - 1,$$

$$p_{ij}^1 = \begin{pmatrix} np_{11}^{*1} & np_{12}^{*1} & n-1 \\ np_{12}^{*1} & np_{22}^{*1} & 0 \\ n-1 & 0 & 0 \end{pmatrix}, \quad p_{ij}^2 = \begin{pmatrix} np_{11}^{*2} & np_{12}^{*2} & 0 \\ np_{12}^{*2} & np_{22}^{*2} & n-1 \\ 0 & n-1 & 0 \end{pmatrix},$$

$$p_{ij}^3 = \begin{pmatrix} nn_1^* & 0 & 0 \\ 0 & nn_2^* & 0 \\ 0 & 0 & n-2 \end{pmatrix}.$$

REFERENCES

[1] R. C. BOSE AND W. S. CONNOR, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.

[2] R. C. BOSE AND T. SHIMAMOTO, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.

[3] R. C. BOSE AND K. R. NAIR, "Partially balanced incomplete block designs," *Sankhya*, Vol. 4 (1939), pp. 337–372.

[4] R. C. BOSE, S. S. SHRIKHANDE AND K. N. BHATTACHARYA, "On the construction of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 167–195.

[5] R. C. BOSE, W. H. CLATWORTHY AND S. S. SHRIKHANDE, "*Tables of Partially Balanced Designs With Two Associate Classes*," Univ. North Carolina, Mimeo. Series 77, (1953).