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THE DISTRIBUTION OF DISTANCE IN A HYPERSPHERE

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1. In a note with the above title, Hammersley [2] has used ad hoc methods to deal with the distribution of the distance AB, when A and B are points uniformly distributed in a sphere of radius a in s dimensions. I show here how this question may be treated by general methods which I have developed elsewhere [3] for random vectors with spherical distributions. A random vector $\mathbf{r}$ will be said to have a *spherical distribution* if its probability function is a function of $|\mathbf{r}|$ only.

I start with the observation that the problem is in fact one of the addition of independent random vectors with spherical distributions. We require the distribution of $\mathbf{r}_1 - \mathbf{r}_2$ where $\mathbf{r}_1$ and $\mathbf{r}_2$ are random vectors with the same uniform spherical distribution. But on account of the spherical symmetry, $-\mathbf{r}_2$ has the same distribution as $\mathbf{r}_2$, so that the problem is equivalent to finding the distribution of $\mathbf{r}_1 + \mathbf{r}_2$. It will be dealt with in this form in what follows.

2. The first method uses the polar form of the characteristic function. For any spherical distribution in s dimensions let

$$P(r)\, dr = Pr\{r < |\mathbf{r}| < r + dr\}.$$

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The characteristic function of the distribution of r is $E(e^{i\mathbf{r}\cdot\boldsymbol{\rho}})$. On changing to polar coordinates it is found (as in [1] or [3]) to be a function of $\rho = |\boldsymbol{\rho}|$ only, and is

$$\Phi(\rho) = \int_0^\infty P(r)\Lambda_{s/2-1}(r\rho)\, dr, \tag{1}$$

where

$$\begin{aligned}\Lambda_\alpha(x) &= \Gamma(\alpha + 1)(\tfrac{1}{2}x)^{-\alpha}J_\alpha(x)\\ &= 1 - \frac{(\frac{1}{2}x)^2}{1\cdot(\alpha+1)} + \frac{(\frac{1}{2}x)^4}{1\cdot 2(\alpha+1)(\alpha+2)} - \cdots,\end{aligned} \tag{2}$$

with inversion formula

$$P(r) = 2^{-s/2+1}\{\Gamma(\tfrac{1}{2}s)\}^{-1}\int_0^\infty (r\rho)^{s/2}J_{s/2-1}(r\rho)\Phi(\rho)\, d\rho. \tag{3}$$

It should be emphasised that $\Phi(\rho)$ is the characteristic function of the s-dimensional distribution of $\mathbf{r}$ and not of the one-dimensional distribution of $r = |\mathbf{r}|$.

For a distribution uniform in a sphere of radius a

$$P(r) = \begin{cases} sr^{s-1}a^{-s}, & 0 \leqq r \leqq a;\\ 0, & r > a,\end{cases} \tag{4}$$

$$\Phi(\rho) = \Lambda_{s/2}(a\rho). \tag{5}$$

Multiplying characteristic functions and inverting, we obtain the probability function for $\mathbf{r}_1 + \mathbf{r}_2$ as

$$P_2(r) = s\Gamma(s/2 + 1)(2r/a^2)^{s/2}\int_0^\infty \rho^{-s/2}J_{s/2}^2(a\rho)J_{s/2-1}(r\rho)\, d\rho.$$

This integral is not completely evaluated by Watson [4], but we merely need to make simple substitutions (in line 6 of sec. 13.46 and in equation (2) of sec. 13.4). The result is

$$P_2(r) = \frac{s\Gamma(\frac{1}{2}s + 1)}{\Gamma(\frac{1}{2}s + \frac{1}{2})\Gamma(\frac{1}{2})} r^{s-1}a^{-s}\int_A^\pi \cos^2 \tfrac{1}{2}\phi\, d\phi,$$

where $0 \leqq A \leqq \pi$ and $\sin \frac{1}{2}A = r/2a$. Putting $t = \cos^2 \frac{1}{2}\phi$, we obtain Hammersley's form

$$P_2(r) = sr^{s-1}a^{-s}I_\mu(\tfrac{1}{2}s + \tfrac{1}{2}, \tfrac{1}{2}), \tag{6}$$

where $\mu = 1 - r^2/4a^2$ and $I_x(p, q)$ is the incomplete Beta function defined by

$$B(p, q)I_x(p, q) = \int_0^x t^{p-1}(1 - t)^{q-1}\, dt.$$

3. In the second method the distributions are treated as projections of spherical distributions in a space of a higher number of dimensions. It is clear that from a spherical distribution of random vectors with O for center (i.e. O is the point $\mathbf{r} = 0$) we obtain another spherical distribution with center O if we project the vectors orthogonally onto a space of lower dimensions through O. A simple calculation [3] shows that if any spherical distribution in space of $(s + 2m)$ dimensions is projected onto a space of s dimensions, the corresponding probability functions $P^{(s+2m)}(r)$ and $P^{(s)}(r)$, satisfy

$$(7) \qquad P^{(s)}(r) = \frac{2\Gamma(\frac{1}{2}s + m)}{\Gamma(\frac{1}{2}s)\Gamma(m)} r^{s-1} \int_r^\infty P^{(s+2m)}(t)(t^2 - r^2)^{m-1} t^{-s-2m+2}\, dt.$$

If the distribution in the higher space is uniform over the surface of a sphere of radius a, then

$$(8) \qquad P^{(s)}(r) = \begin{cases} \dfrac{2\Gamma(\frac{1}{2}s+m)}{\Gamma(\frac{1}{2}s)\Gamma(m)} a^{-s-2m+2}(a^2 - r^2)^{m-1} r^{s-1}, & r \leqq a, \\ 0, & r > a. \end{cases}$$

When $m = 1$, this reduces to (4).

This shows that a uniform distribution through the volume of an s-dimensional sphere can be obtained by projection from a uniform distribution over the surface of an $(s + 2)$-dimensional sphere, each sphere having radius a. In the case $s = 1$, we see that a distribution uniform over a diameter can be obtained by projection from a distribution uniform over the surface of a sphere. This is essentially Archimedes' theorem on the surface area of a sphere.

Now for the sum of two vectors, each with a distribution uniform over the surface of an $(s + 2)$-dimensional sphere, we can appeal to a special case of Kluyver's original solution of the problem of random flights, or rather to the generalisation to any number of dimensions given by Watson [4]. From his results ([4], secs. 13.48 and 13.46 (3)), it follows that the probability function is

$$(9) \qquad P_2^{(s+2)}(r) = \begin{cases} 2^{1-s} \dfrac{\Gamma(\frac{1}{2}s + 1)}{\Gamma(\frac{1}{2}s + \frac{1}{2})\Gamma(\frac{1}{2})} a^{-2s} r^s (4a^2 - r^2)^{(s-1)/2}, & r \leqq 2a; \\ 0, & r > 2a. \end{cases}$$

Substituting in (7) with $m = 1$, we obtain $P_2(r)$ as a multiple of $\int_r^{2a} (4a^2 - t^2)^{(s-1)/2}\, dt$, and then (6) follows.

If the distribution of each $\mathbf{r}_1$ and $\mathbf{r}_2$ is according to (8), with m not necessarily equal to 1, then it is the projection from space of dimensions $(s + 2m)$ of a distribution uniform over the surface of a sphere. The argument just used is applicable and shows that

$$P_2(r) = k \int_r^{2a} (4a^2 - t^2)^{m+(s-3)/2}(t^2 - r^2)^{m-1} t^{-2m+2}\, dt,$$

where k is a constant. When $m = 1$ this reduces to (4), but is otherwise more complicated. The result is still true when $2m$ is not an integer.

4. Hammersley proves that for large values of s the distance between two points in the sphere is nearly always equal to $a\sqrt{2}$, the diagonal of the rectangle determined by orthogonal radii. He does this by showing that as s tends to infinity, $|\mathbf{r}_1 + \mathbf{r}_2|$ is asymptotically distributed in a normal distribution with mean $a\sqrt{2}$ and variance $a^2/2s$.

From the characteristic function it is seen that Hammersley's result is a corollary of a more general one, namely that the s-dimensional distribution given by (8) is asymptotically normal with second moment $a^2s(s + 2m)^{-1}$. Here a normal distribution has the probability function

$$P(r) = C_s r^{s-1} \exp\left(-\tfrac{1}{2}sr^2/\mu_2\right),$$

where μ_2 is the second moment and C_s a constant, and has the characteristic function

$$\Phi(\rho) = \exp\left(-\tfrac{1}{2}\mu_2\rho^2/s\right).$$

The distribution (8) has characteristic function $\Lambda_{s/2+m-1}(a\rho)$. This can be verified by direct calculation, or derived from the facts that a spherical distribution and its projections (in the sense of sec. 3) all have the same characteristic function (proved in [3]), and that a distribution uniform over the surface of a sphere of radius a in $s + 2m$ dimensions obviously has the characteristic function $\Lambda_{s/2+m-1}(a\rho)$. Now

$$\begin{aligned}\Lambda_{s/2+m-1}(a\rho) &= 1 - \frac{a^2\rho^2}{2(s + 2m)} + \frac{a^4\rho^4}{8(s + 2m)(s + 2m + 2)} - \cdots \\ &\sim \exp\left\{-\frac{a^2\rho^2}{2(s + 2m)}\right\}\end{aligned}$$

as s tends to infinity, uniformly in any ρ-interval. Thus the distribution (8) is asymptotically normal with $\mu_2 = a^2s(s + 2m)^{-1}$.

Taking $m = 1$, we obtain the distribution (4) which is therefore asymptotically normal with $\mu_2 = a^2s(s + 2)^{-1}$. The distribution of $\mathbf{r}_1 + \mathbf{r}_2$ is thus asymptotically normal with

$$\mu_2 = 2a^2s(s + 2)^{-1}. \tag{10}$$

Taking $m = 0$, we see that the distribution uniform over the surface of a sphere of radius b is asymptotic to a normal distribution with $\mu_2 = b^2$. Comparing with (10), we see that the distribution of $\mathbf{r}_1 + \mathbf{r}_2$ is asymptotic to a distribution uniform over the surface of a sphere of radius $a(2s)^{1/2}(s + 2)^{-1/2} \simeq a\sqrt{2}(1 + s^{-1})$. This is equivalent to Hammersley's result.

We could avoid the use of characteristic functions in an increasing number of dimensions by projecting onto a diametral subspace of a fixed number of dimensions. Since projection does not alter the characteristic function, the resulting calculation will be the same.

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EXTREME VALUES IN SAMPLES FROM m-DEPENDENT STATIONARY STOCHASTIC PROCESSES

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Summary. The limiting distributions for the order statistics of n successive observations in a sequence of independent and identically distributed random variables are shown to hold also when the sequence is generated by a stationary stochastic process of a certain moving average type.

A sequence of random variables $\{x_i\}$ has been called m-dependent [3] if $|i - j| > m$ implies that x_i and x_j are independent. If the variables in a strictly stationary sequence are m-dependent and have a finite upper bound to their range of variation, the largest in a sample of n successive members tends with probability one to this upper bound. This is a simple extension of Dodd's results [1] for the case of independence.

The following theorem shows that when this upper bound is infinite, the asymptotic distribution of the largest in such a sample is the same as in the case of independence.

Theorem. *Let $\{x_i\}$ be a sequence of random variables, unbounded above and generated by an m-dependent strictly stationary stochastic process with the property that*

$$\lim_{c\to\infty} \frac{1}{P(x_i > c)} \max_{|i-j|\leq m} P[(x_i > c), (x_j > c)] = 0. \tag{1}$$

Then, if $\xi = n\, P[x_i > c_n(\xi)]$, *for* ξ *fixed,*

$$\lim_{n\to\infty} P[x_i \leqq c_n(\xi);\, i = 1, \cdots, n] = e^{-\xi}$$

Proof. Using the formula for the probabilities of the joint occurrence of a set of events in terms of probabilities of occurrence of their contraries (Feller [2],

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