A CONFIDENCE INTERVAL FOR VARIANCE COMPONENTS

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1. Introduction.

Summary. In this paper an approximate confidence interval is found for the expected value of the difference between two quantities which are independently distributed proportionally to χ^2 variates. Three methods are used. The first is based on the work of Welch [13], [14] and Aspin [1], [2] on the generalized "Student's" problem, and involves neglecting successively higher powers of the reciprocal of one of the degrees of freedom. This method is used to check the other two solutions, both of which involve neglecting successive increasing and decreasing powers, respectively, of a nuisance parameter. Finally a solution is formed using those resulting from the second and third methods, and is more accurate than those solutions. The order of accuracy, and the use of the final solution, are discussed.

The paper does not present a method of computing confidence intervals in a form suitable for immediate *practical* application. Series developments of a certain hypothetical function are given; more remains to be said about the relation between the series and the function, and the problem of computing tables. A computational exploration of the solution is at present in hand.

Applications. In what has sometimes been termed a Model II multiple classification, each observation is the sum of a constant and of contributions due to the different factors which feature in the classification, the interaction effects, and an error term. These contributions are taken to be normally and independently distributed with zero means, and variances independent of the particular levels of the appropriate factors. These variances are called variance components, since each gives that portion of the total variance of each observation appropriate to a particular source. In a balanced layout, each of these variance components, except that due to the error term, can be written as a known constant multiplied by the difference of the expected values of two mean squares, which are independently distributed proportionally to χ^2 variates. Thus the results of this paper may be applied to these variance components.

In the other main model of multiple classifications, the so-called Model I, the factors make constant contributions to the observations at the different levels. Here all the mean squares except the residual are proportional to noncentral χ^2 variates, so that the results of this paper cannot be applied. However, in a "Mixed Model", where some factors are as in Model I and some as in Model II, some mean squares will be suitable.

The general balanced Model II classification will be exemplified by considering the two-way layout. Let y_{ijk} be the kth observation in the ith row and the jth

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column, and take $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$, where μ is a constant, and α_i, β_j, γ_{ij}, and ϵ_{ijk} are independent normal deviates with variances σ_α^2, σ_β^2, σ_γ^2 and σ_ϵ^2, respectively. The appropriate table is thus

Source	D. F.	Mean square	$\mathcal{E}$ (mean square)
Between rows	$a - 1$	M_α	$\sigma_\epsilon^2 + n\sigma_\gamma^2 + nb\sigma_\alpha^2$
Between cols.	$b - 1$	M_β	$\sigma_\epsilon^2 + n\sigma_\gamma^2 + na\sigma_\beta^2$
Interaction	$(a - 1)(b - 1)$	M_γ	$\sigma_\epsilon^2 + n\sigma_\gamma^2$
Error	$ab(n - 1)$	M_ϵ	σ_ϵ^2

Consider the variance component σ_α^2, for example. Now

$$\sigma_\alpha^2 = (nb)^{-1}\mathcal{E}(M_\alpha - M_\gamma),$$

and M_α and M_γ are independently distributed as

$$(a - 1)^{-1}(\sigma_\epsilon^2 + n\sigma_\gamma^2 + nb\sigma_\alpha^2)\chi^2, \qquad (b - 1)^{-1}(\sigma_\epsilon^2 + n\sigma_\gamma^2)\chi^2,$$

so that the results of this paper may be applied to obtain confidence limits for σ_α^2.

It is well-known that a confidence interval or confidence limit can be used to provide a test of a hypothesis which postulates a particular value for the parameter concerned; for if the hypothesis be accepted when the hypothetical value of the parameter lies inside the interval or on the appropriate side of a single confidence limit, then the probability of rejecting a true hypothesis is fixed at some chosen level. This is true for the limits found for K, the variance component for which an interval estimate is obtained. An examination of the power of this test may require the tabulation of the function derived in Section 7. However it seems reasonable to expect that, for a sufficiently large difference between the true and hypothetical values of K, the power of the test will be a monotonically increasing function of this difference, since the interval continues to cover the true value in the fixed proportion of cases.

Crump [5] states three main fields of application of work on variance components.

(i) *The interpretation of significance tests.* Here variance component estimates are used to locate the sources of undesirable variation, so that this variation can be partially or completely eliminated. Tippett [12] discusses significance tests in the analysis of variance in terms of these components, giving a numerical example of the quality control of spectacle glass. Daniels [8] gives an example from the woollen industry.

(ii) *The selection of efficient sampling designs.* This is the most important use of variance components. Usually interest is focussed on a particular function of the observations, such as the grand mean. The reciprocal of the variance of this statistic is then regarded as a measure of the efficiency of the sampling design. Both the cost and the efficiency are functions of the sample sizes and variance components. The usual procedure for choosing a good design is to estimate the variance components from a preliminary experiment, and then, using these estimates instead of the true values, to calculate the sample sizes which either minimise the cost for fixed efficiency or maximise the efficiency for fixed cost. Alternatively, Yates [15] suggests the general principle that an experiment should be so designed that the sum of the cost and the expected losses due to errors in the results should be minimised. Examples are given by Marcuse [10], and Nordskog and Crump [6].

(iii) *Various problems in genetics.* An example is given by Robinson and Comstock [4].

In all of these fields, point estimates of variance components are now used. They seem to be more appropriate than interval estimates for many of the examples met in practice. However, a confidence interval is useful for assessing the accuracy of an estimate. If the confidence interval is wide, then little trust can be placed in a point estimate; if it is narrow, then the estimate can reasonably be regarded as trustworthy. Estimates do exist for the variances of the variance component estimates, but these, being estimates, are less reliable than confidence intervals for assessing the accuracy of the variance component estimates. Also, they are less informative, since the usual type of variance component estimate has a complicated distribution, involving a nuisance parameter (see K. Pearson [11]).

When variance components are used qualitatively to assess the amount of variation present, a confidence interval may be a more reliable guide to the judgement.

Previous work. A full discussion of previous work would require too much space and anything brief is scarcely illuminating. Attention is directed to papers by Fisher [9], Bross [3], and a comprehensive survey by Crump [7]. In this paper we do not follow Fisher's method of computing fiducial limits.

2. The problem. The previous problems may be subsumed under the following canonical form. Two statistics M_1 and M_2 are given, which are independently distributed as $\sigma_1^2\chi^2/r_1$ and $\sigma_2^2\chi^2/r_2$, with r_1 and r_2 degrees of freedom, respectively. Confidence limits are required for $\sigma_1^2 - \sigma_2^2$, both σ_1^2 and σ_2^2 being unknown. For the present, it will be assumed that $\sigma_1^2 > \sigma_2^2$, but this restriction will be withdrawn later, as discussed in Section 9.

We define

$$\mathrm{K} = \sigma_1^2 - \sigma_2^2, \qquad \rho = \frac{\sigma_2^2}{\mathrm{K}} = \frac{1}{(\sigma_1/\sigma_2)^2 - 1}, \qquad y = \frac{M_1}{\mathrm{K}}, \qquad x = \frac{M_2}{\mathrm{K}}.$$

Thus a function f is sought such that

$$\Pr\,[y \leqq f(x)] = \alpha, \qquad 0 < \alpha < 1, \tag{2.1}$$

where α is given and f must be independent of the nuisance parameter. Later we shall require to find K such that $M_1/\mathrm{K} = f(M_2/\mathrm{K})$. The problem was put into this form originally so that the method of approach due to Welch (later referred to as Method I) might be exploited.

Now r_2x/ρ and $r_1y/(1 + \rho)$ are independently distributed as χ^2 on r_1 and r_2

degrees of freedom, respectively. With $y_1 = r_1y/(1 + \rho)$, requirement (2.1) becomes

$$\int_0^\infty \frac{e^{-r_2x/2\rho}}{\Gamma(\frac{1}{2}r_2)}\left(\frac{r_2x}{2\rho}\right)^{r_2/2-1}\left\{\int_0^{r_1f(x)/(1+\rho)}\left(\frac{y_1}{2}\right)^{r_1/2-1}\frac{e^{-y_1/2}}{2\Gamma(\frac{1}{2}r_1)}\,dy_1\right\}\frac{r_2\,dx}{2\rho} = \alpha. \tag{2.2}$$

Since x and y are nonnegative, the discussion is confined entirely to the first quadrant of the plane of x and y. Thus it is essential that $f(x) \geqq 0$ (see Section 9). Put

$$I_{r_1}(z) = \int_0^z \{(\tfrac{1}{2}y_1)^{r_1/2-1}e^{-y_1/2}/2\Gamma(\tfrac{1}{2}r_1)\}\,dy_1, \qquad g(x) = I_{r_1}\{r_1f(x)\,/\,(1+\rho)\}.$$

Further, let ξ be such that $I_{r_1}(\xi) = \alpha$. Thus an f is required such that

$$\int_0^\infty \frac{e^{-r_2x/2\rho}}{\Gamma(\frac{1}{2}r_2)}\left(\frac{r_2x}{2\rho}\right)^{r_2/2-1}\frac{r_2g(x)}{2\rho}\,dx = \alpha \doteq I_{r_1}(\xi), \tag{2.3}$$

independently of ρ. We do not know whether there exists a function $f(x)$ which satisfies the above conditions nor, if it exists, whether it is regular. In this paper we derive a function $f_{IV}(x)$ which is such that (2.2) is approximately satisfied when $f_{IV}(x)$ is inserted in place of $f(x)$. How good this approximation is can be determined only by computational means.

3. Method I. In equation (2.3) we expand $g(x)$ in a Taylor series about $x = \rho$. That is, we confine the investigation to the finding of a solution for which this is permissible, if one exists. Now $g(x) = e^{(x-\rho)\partial}g(w)$, where $\partial^r = [\partial^r/\partial w^r]_{w=\rho}$. Thus (2.3) becomes

$$\int_0^\infty \frac{e^{-r_2x/2\rho}}{\Gamma(\frac{1}{2}r_2)}\left(\frac{r_2x}{2\rho}\right)^{r_2/2-1} e^{(x-\rho)\partial}\,\frac{r_2\,dx}{2\rho}\,g(w) = \alpha.$$

With $\Theta = (1 - 2\rho\partial/r_2)^{-r_2/2}e^{-\rho\partial}$, this becomes $\Theta I_{r_1}\{r_1f(w)\,/\,(1+\rho)\} = \alpha$. Expansion of $I_{r_1}\{r_1f(w)\,/\,(1+\rho)\}$ about $\{r_1f(w)\,/\,(1+\rho)\} = \xi$ yields

$$I_{r_1}\left\{\frac{r_1f(w)}{1+\rho}\right\} = \exp\left[\left\{\frac{r_1f(w)}{1+\rho} - \xi\right\}D\right]I_{r_1}(z), \qquad D^r = \left[\frac{d^r}{dz^r}\right]_{z=\xi}.$$

Hence the equation to be solved becomes

$$\Theta \exp\,(\{[r_1f(w)\,/\,(1+\rho)] - \xi\}D)I_{r_1}(z) = \alpha. \tag{3.1}$$

Equation (2.2) is very similar to the one solved approximately by Welch [13], [14] and Aspin [1], [2] in their work on the problem of comparing two means; the method used here is the same as theirs. The different functional form of the inner integrand's upper limit, and the different type of the inner integral, prevent deriving our solution from Welch's, although a comparison means of checking for $r_1 = 1$ can be used. It is evident that Θ is essentially the same in both cases.

Continuing, we put $f = f_0 + f_1 + f_2 + \cdots$, where f_s is of order $-s$ in r_2, and the expansion may be finite or infinite. The quantity f_0 is, as in Welch's work, the large sample approximation, here $\xi(1 + x)/r_1$. Expanding,

$$\begin{aligned}\Theta &= \exp\{-\rho\partial - \tfrac{1}{2}r_2 \log(1 - 2\rho\partial/r_2)\}\\ &= \exp\{\rho^2\partial^2/r_2 + 4\rho^3\partial^3/3r_2^2 + \cdots\} \qquad (3.2)\\ &= 1 + \rho^2\partial^2/r_2 + \{4\rho^3\partial^3/3r_2^2 + \rho^4\partial^4/2r_2^2\} + \cdots .\end{aligned}$$

Neglecting terms of order r_2^{-3} in (3.1), we have

$$\Theta \exp\left\{\xi D\left[\frac{1+w}{1+\rho} - 1\right]\right\}\left\{1 + \frac{r_1 f_1(w)D}{1+\rho} + \left[\frac{r_1 f_2(w)D}{1+\rho} + \frac{r_1^2 f_1^2(w)D^2}{2(1+\rho)^2}\right] + \cdots\right\} \cdot I_{r_1}(z) = I_{r_1}(\xi).$$

Substituting for Θ, and grouping separately terms of order r_2^{-1} and r_2^{-2}, we obtain

$$\left[\frac{r_1 f_2(\rho)}{1+\rho} D + \frac{r_1 f_1^2(\rho)D^2}{2(1+\rho)^2} + \frac{\rho^2\partial^2}{r_2}\exp\left\{\xi D\left(\frac{1+w}{1+\rho} - 1\right)\right\}\frac{r_1 f_1(w)D}{1+\rho} + \left\{\frac{4\rho^3\partial^3}{3r_2^2} + \frac{\rho^4\partial^4}{2r_2^2}\right\}\exp\left\{\xi D\left(\frac{1+w}{1+\rho} - 1\right)\right\}\right] I_{r_1}(z)$$
$$+ \left[\frac{r_1 f_1(\rho)}{1+\rho} D + \frac{\rho^2\partial^2}{r_2}\exp\left\{\xi D\left(\frac{1+w}{1+\rho} - 1\right)\right\}\right] I_{r_1}(z) = 0.$$

Equating to zero the first order term yields

$$[r_1 f_1(\rho) / (1+\rho)]I'_{r_1}(\xi) + [\rho^2\xi^2 / r_2(1+\rho)^2]I''_{r_1}(\xi) = 0.$$

Therefore $f_1(\rho) = -\rho^2\xi^2 I''_{r_1}(\xi) / r_1 r_2(1+\rho)I'_{r_1}(\xi)$. We put $R_s = I^{(s)}_{r_1}(\xi) / I'_{r_1}(\xi)$, so that

$$f_1(x) = -x^2\xi^2 R_2 / r_1 r_2(1+x).$$

Equating to zero the second order term yields

$$f_2(x) = \frac{-x^2\xi}{6r_1 r_2^2(1+x)^3}\{x^2[3R_2(\xi^3R_2^2 - 2\xi^3R_3 - 4\xi^2R_2) + 8\xi^2R_3 + 3\xi^3R_4] + 8x[\xi^2R_3 - 3\xi^2R_2^2] - 12\xi R_2\}.$$

It is required to express R_s in terms of ξ and r_1. Now

$$I_{r_1}(\xi) = \int_0^\xi (\tfrac{1}{2}y)^{r_1/2-1}[e^{-y/2}/2\Gamma(\tfrac{1}{2}r_1)]\,dy,$$

$$I'_{r_1}(\xi) = (\tfrac{1}{2}\xi)^{r_1/2-1}[e^{-\xi/2}/2\Gamma(\tfrac{1}{2}r_1)].$$

Using Leibniz's formula,

$$I^{(s)}_{r_1}(\xi) = 2^{-s+1}\frac{(\tfrac{1}{2}r_1 - 1)!}{(\tfrac{1}{2}r_1 - s)!}\left(\frac{\xi}{2}\right)^{r_1/2-s}\frac{e^{-\xi/2}}{2\Gamma(\tfrac{1}{2}r_1)}$$

$$- \binom{s-1}{1} \frac{(\frac{1}{2}r_1 - 1)!}{(\frac{1}{2}r_1 - s + 1)!} 2^{-s+1} \left(\frac{\xi}{2}\right)^{r_1/2-s+1} \frac{e^{-\xi/2}}{2\Gamma(\frac{1}{2}r_1)}$$

$$+ \cdots + (-2)^{-s+1} \left(\frac{\xi}{2}\right)^{r_1/2-1} \frac{e^{-\xi/2}}{2\Gamma(\frac{1}{2}r_1)}$$

$$= 2^{-s} \left(\frac{\xi}{2}\right)^{r_1/2-s} \frac{e^{-\xi/2}}{\Gamma(\frac{1}{2}r_1)} \sum_{i=0}^{s-1} \left(\frac{-\xi}{2}\right)^{i} \frac{(\frac{1}{2}r_1 - 1)!}{(\frac{1}{2}r_1 - s + i)!} \binom{s-1}{i}.$$

Therefore

$$R_s = (2\xi)^{-s+1} \sum_{i=0}^{s-1} 2^{1+s-i}(-\xi)^{i} \frac{(\frac{1}{2}r_1 - 1)!}{(\frac{1}{2}r_1 - s + i)!} \binom{s-1}{i}$$

$$= (2\xi)^{-s+1} \Big\{ [r_1 - 2][r_1 - 4] \cdots [r_1 - 2(s-1)] - \binom{s-1}{1}$$

$$\cdot \xi([r_1 - 2] \cdots [r_1 - 2(s-2)]) + \cdots + (-\xi)^{s-1} \Big\}.$$

Substituting for R_s in the f's, we obtain

$$f_0(x) = (1 + x)\frac{\xi}{r_1}, \qquad f_1(x) = \frac{x^2}{1+x} \frac{\xi(\xi - r_1 + 2)}{2r_1 r_2}$$

$$f_2(x) = \frac{x^2 \xi}{24 r_1 r_2^2 (1+x)^3} \{x^2[4\xi^2 - 11\xi(r_1 - 2) + (r_1 - 2)(7r_1 - 10)]$$

$$+ 16x[\xi^2 - 2\xi(r_1 - 2) + (r_1 - 1)(r_1 - 2)] + 24(r_1 - 2 - \xi)\}.$$

For further terms operate on

$$I_{r_1}\left\{\frac{f_0(w) + \cdots + f_r(w)}{1 + \rho} r_1\right\}$$

by $-\Theta$ and arrange the result as a power series in $1/r_2$, say $\sum a_{r+1}(\rho)/r_2^{r+1}$. Then

$$r_1 f_{r+1}(\rho) I_{r_1}(\xi)/(1 + \rho) = a_{r+1}(\rho)/r_2^{r+1}$$

whence $f_{r+1}(\rho)$. Now this expansion is in descending powers of r_2, but though this may be large compared with r_1 it may not be large compared with certain powers of r_1 which may occur in the numerators of the f's, or compared with ξ. This matter is considered in Section 8.

Only the terms shown above have been worked out by this method, as the calculations become laborious and this solution is used only as a check on the solutions obtained by other methods.

Another point regarding this solution is that in replacing ρ by x it has been assumed that a solution $f(x)$ exists which is independent of ρ, whereas the function $f(w)$ which is operated upon by ∂ may be actually of the form $f(w, \rho)$. However, if such a solution exists, then this method will give it. Moreover, $f = f_0 + f_1 + f_2$ does satisfy (2.2) to the order r_2^{-2}, whether or not an exact solution exists.

A check has been made in the case $r_1 = 1$, when it is possible to deduce the appropriate series from Welch's solution of the two means problem.

4. Graphical representation. At this stage, a picture helps one to visualize Methods II and III, described in the subsequent sections. For simplicity, put

$$v = \tfrac{1}{2}r_2x, \qquad u = \tfrac{1}{2}r_1y, \qquad a = \tfrac{1}{2}r_1 - 1, \qquad b = \tfrac{1}{2}r_2 - 1.$$

The joint probability density function of u and v is

$$\frac{1}{a!b!}\left(\frac{u}{1+\rho}\right)^a\left(\frac{v}{\rho}\right)^b\frac{1}{\rho(1+\rho)}\exp\left\{-\left[\frac{u}{1+\rho}+\frac{v}{\rho}\right]\right\},$$

where $a! = \Gamma(a+1)$ whether a is an integer or not. It is required to find $g(u, v)$ such that the integral of this density function over the region $g(u, v) \leqq 0$ shall equal α, that is, such that

$$(1/a!b!)\iint_{\mathfrak{D}} u^a v^b e^{-(u+v)}\,du\,dv = \alpha, \qquad \mathfrak{D} = \{u, v : g[u(1+\rho), v\rho] \leqq 0\}.$$

This equation shows that g is required such that when the curve $g(u, v) = 0$ is scaled down by dividing the u-value of every point by $(1+\rho)$, and the v-value by ρ, then the integral of $(1/a!b!)\,u^a v^b e^{-(u+v)}$ over the region on one side of the resulting curve is α, independently of ρ. Also, when $g(u, v) = 0$, only one value of v corresponds to one of u.

When $\rho \to 0$, the slope becomes very small and the curve flattens out to the form $v =$ constant. If the integral under the curve is α, then the constant value of v will be $\tfrac{1}{2}\xi$, where ξ is such that $I_{r_1}(\xi) = \alpha$.

When $\rho \to \infty$, the curve cannot lie completely in the range $u \leqq U$ for any finite U. If it did, the scaled curve would lie completely in the range $u \leqq U/(1+\rho)$, which becomes arbitrarily small as $\rho \to \infty$. Thus the integral on one side of the curve can be made arbitrarily small, or close to 1, as the case may be. Similarly the curve cannot lie wholly in the region $v \leqq V$, for any finite V. Thus the curve extends to infinity in both variables.

Further, looking for a solution whose slope tends to a definite limit (not necessarily finite) for large u and v (if such a solution exists), then for ρ large the shape of the scaled curve will be roughtly that of a straight line through the origin, with slope equal to the slope at infinity. Now the integral on one side of this line, say below it, must be α, that is, Pr $(y/x \leqq m/r_1) = \alpha$. Since y/x is distributed as F_{r_1, r_2}, (i.e. the F variate with degrees of freedom r_1, r_2),

$$m = r_1F_{r_1, r_2}(\alpha), \qquad \text{where Pr } [(F_{r_1, r_2} \leqq F_{r_1, r_2}(\alpha)] = \alpha.$$

From this crude picture of the approximations for ρ very large and very small, the first stages of the approximations derived in Methods II and III can be obtained. A greater understanding of these methods is also provided.

The same conclusions are reached by the following intuitive reasoning:

When $\rho \to 0$, then $\rho = \sigma_2^2/\mathrm{K}$ and $\mathrm{K} = \sigma_1^2 - \sigma_2^2$. Let $\sigma_2^2 \to 0$, then $\mathrm{K} = \sigma_1^2$, so that M_1 tends to become an unbiased estimate of K, distributed as $r_1^{-1}K\chi^2$ on r_1 degrees of freedom. Hence r_1y is distributed as χ^2 on r_1 degrees of freedom. An approximate $f(x)$ such that Pr $[r_1y \leqq r_1f(x)] = \alpha$ is $f(x) = \xi/r_1$, which is thus the limiting solution as $\rho \to 0$.

When $\rho \to \infty$, let $K \to 0$ and $\sigma_2^2 \to \sigma_1^2$, so that M_1/M_2 tends to become an F_{r_1,r_2} variate. Thus in the limit y/x is an F_{r_1,r_2} variate, and the appropriate $f(x)$ such that Pr. $[y \leqq f(x)] = \alpha$ is mx/r_1 which is thus the limiting solution as $\rho \to \infty$.

5. Method II. As $\rho \to 0$, M_1 tends to become an unbiased estimate of K, so that for ρ small, $f(x) = \xi/r_1$ would approximately satisfy (2.2), as just pointed out in the preceding section. This solution neglects terms of order ρ^2, so that the accuracy could be improved by neglecting higher orders of ρ instead.

We take $b = \frac{1}{2}r_2 - 1$ as before, and change (2.2), by the transformations of Section 4, to the form

$$\int_0^\infty \frac{e^{-u}u^b}{b!} I_{r_1}\left\{\frac{r_1 f(2\rho u/r_2)}{1+\rho}\right\} du = \alpha. \tag{5.1}$$

It can be seen easily that it is appropriate to seek an $f(x)$ of the form

$$r_1 f(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots.$$

We now expand the function I_{r_1} in (5.1) about the point where its argument equals ξ. When $\rho = 0$, we find that $b_0 = \xi$. The expansion gives

$$I_{r_1}\left\{\frac{b_0 + b_1(2\rho u/r_2) + b_2(2\rho u/r_2)^2 + \cdots}{1+\rho}\right\} = \alpha + \frac{p}{1!}I'_{r_1}(\xi) + \frac{p^2}{2!}I''_{r_1}(\xi) + \cdots,$$

where

$$\begin{aligned} p &= \{[b_0 + b_1(2\rho u/r_2) + b_2(2\rho u/r_2)^2 + \cdots] / [1+\rho]\} - \xi \\ &= [\rho(2b_1u/r_2 - \xi) \quad + b_2(2\rho u/r_2)^2 + \cdots] / [1+\rho]. \end{aligned}$$

Substitution in (5.1) gives

$$\int_0^\infty [e^{-u}u^b/b!][pI'_{r_1}(\xi) + (p^2/2!)I''_{r_1}(\xi) + \cdots]\, du = 0.$$

Dividing through by $I'_{r_1}(\xi)/(1+\rho)$ gives

$$\begin{aligned} \int_0^\infty \frac{e^{-u}u^b}{b!} [\{\rho(2b_1u/r_2 - \xi) + b_2(2\rho u/r_2)^2 + \cdots\} + \{R_2/2(1+\rho)\} \\ \{\rho(2b_1u/r_2 - \xi) + b_2(2\rho u/r_2)^2 + \cdots\}^2 + \{R_3/3!(1+\rho)^2\} \\ \{\rho(2b_1u/r_2 - \xi) + \cdots\}^3 + \cdots]\, du = 0. \end{aligned}$$

Equating to zero the coefficient of ρ gives $2b_1(b+1) = r_2\xi$, or $b_1 = \xi$. Similarly equating to zero the coefficient of ρ^2 gives

$$(2/r_2)^2b_2(b+2)(b+1) = -\tfrac{1}{2}R_2\{(2b_1/r_2)^2(b+2)(b+1) + \xi^2 - (4b_1/r_2)(b+1)\xi\}$$

Therefore $b_2(b+2)/(b+1) = -\frac{1}{2}R_2\xi^2\{(b+2)/(b+1)+1-2\}$, and consequently $b_2 = -\frac{1}{2}R_2\xi^2/(b+2)$.

Similar consideration of the coefficients of ρ^3 and ρ^4 gives, in turn,

$$b_3 = \frac{(b+1)}{(b+2)(b+3)}\left\{\frac{R_2^2\xi^3}{b+1} + \frac{R_2\xi^2}{2} - \frac{R_3\xi^3}{3(b+1)}\right\},$$

$$b_4 = \frac{(b+1)^2}{(b+2)(b+3)(b+4)}\left\{\frac{(b+11)R_2R_3\xi^4}{4(b+1)^2} - \frac{5R_2^2\xi^3}{2(b+1)} + \frac{2R_3\xi^3}{3(b+1)} - \frac{R_2\xi^2}{2} - \frac{(b^2+31b+60)R_2^3\xi^4}{8(b+1)^2(b+2)} - \frac{(b+3)R_4\xi^4}{8(b+1)^2}\right\}.$$

This term is as far as this solution was taken, since the work involved increases very rapidly. One would expect this solution to be unreliable for $\rho \geqq 1$, but it will not be used by itself. No simplification seems likely from replacing the R's by their expressions in terms of ξ and r_1, in this case.

6. Method III. This is rather similar to the last method, involving the neglect of successive descending powers of ρ. Thus it is suitable for $\rho > 1$. In Method II a Taylor expansion about a constant was used; in this method the corresponding expansion is about a function of x. We look for a solution of (2.3) of the form

$$r_1 f(x) = mx + m_0 + m_1x^{-1} + m_2x^{-2} + \cdots,$$

for which it is legitimate to write

$$I_{r_1}\{r_1 f(2\rho u/r_2) \,/\, (1+\rho)\} = I_{r_1}\{2mu/r_2\} + [r_1 f(2\rho u/r_2) \,/\, (1+\rho) - 2mu/r_2]I'_{r_1}\{2mu/r_2\} + \cdots + R_n,$$

for some $n > 3$, with R_n being of order ρ^{-n}. Now, for $N \geqq n$,

$$f(2\rho u/r_2) \,/\, (1+\rho) - 2mu/r_2) = [m_0 - 2mu/r_2]\rho^{-1} + [m_1r_2/2u - (m_0 - 2mu/r_2)]\rho^{-2} + [m_2(r_2/2u)^2 - \{m_1r_2/2u - (m_0 - 2mu/r_2)\}]\rho^{-3} + \cdots + T_N.$$

Thus (2.3) becomes

$$\int_0^\infty (e^{-u}u^b/b!)(\{[m_0 - 2mu/r_2]\rho^{-1} + [m_1 r_2/2u - (m_0 - 2mu/r_2)]\rho^{-2} + \cdots\}I'_{r_1}(2mu/r_2) + \{(1/2!)[m_0 - 2mu/r_2]\rho^{-1} + \cdots\}^2 \cdot I''_{r_1}(2mu/r_2) + \cdots)\, du = 0.$$

Equating to zero the coefficient of ρ^{-1} gives

$$\int_0^\infty (e^{-u}u^b/b!)(m_0 - 2mu/r_2)I'_{r_1}(2mu/r_2)\, du = 0,$$

that is, $(m/r_2)^a(1/a!b!)\int_0^\infty e^{-u(1+m/r_2)}u^{a+b}(m_0 - 2mu/r_2)\, du = 0.$

Therefore, $m_0 = 2m(a + b + 1) / (r_2 + m) = m(r_1 + r_2 - 2) / (r_2 + m)$.

Similarly, by considering the coefficients of ρ^{-2} and ρ^{-3}, we find successively

$$m_1 = \tfrac{1}{2} m(r_1 + r_2 - 2)(r_2 + m)^{-3}\{r_2(m - r_1 + 2) - 2m\},$$

$$m_2 = \tfrac{1}{6} m(r_1 + r_2 - 2)(r_2 + m)^{-5}\{2m^2(r_2 - 2)(r_2 - 4) - r_2m(3r_2^2 + 7r_1r_2 - 32r_2 - 26r_1 + 76) + r_2^2(r_1 - 2)(5r_1 + 3r_2 - 14)\}.$$

Again, this is as far as this solution was carried, due to the heavy work involved in proceeding further.

7. Final approximate solution. A *type A function* will be defined to be of the form $\phi(x) = (1 + x)^{-r}(a_{r+1}x^{r+1} + a_rx^r + a_{r-1}x^{r-1} + \cdots + a_1x + a_0)$. Now it is evident that this type of function can be put into the form of a type II or III solution, so as to agree with the first $(r + 2)$ terms in either expansion. Further solutions of the form I, II, or III (save that a finite number of terms only are considered, so that $(r + 2)$ of the calculated constants are involved) can be put into the form of type A. In this way Solution I was used to check Solutions II and III.

For a final solution a type A function is formed using Solutions II and III. Since four constants of Solution III and five of Solution II have been calculated, then $r + 2 = 4 + 5$, so that $r = 7$. The coefficients a_0, a_1, a_2, a_3, and a_4 are calculated from b_0, b_1, b_2, b_3, and b_4, and the coefficients a_8, a_7, a_6, and a_5 from m, m_0, m_1, and m_2. We put

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4, \qquad a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5$$

respectively equal to the corresponding terms in

$$(1 + x)^7(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4), \qquad (1 + x)^7(mx + m_0 + m_1x^{-1} + m_2x^{-2}).$$

Hence

$$a_0 = b_0, \quad a_1 = b_1 + 7b_0, \quad a_2 = b_2 + 7b_1 + 21b_0,$$

$$a_3 = b_3 + 7b_2 + 21b_1 + 35b_0, \quad a_4 = b_4 + 7b_3 + 21b_2 + 35b_1 + 35b_0,$$

$$a_5 = m_2 + \tbinom{7}{1}m_1 + \tbinom{7}{2}m_0 + \tbinom{7}{3}m = m_2 + 7m_1 + 21m_0 + 35m,$$

$$a_6 = m_1 + \tbinom{7}{1}m_0 + \tbinom{7}{2}m = m_1 + 7m_0 + 21m,$$

$$a_7 = m_0 + \tbinom{7}{1}m = m_0 + 7\,m, \qquad a_8 = m.$$

Using these values, we take

$$r_1f(x) = (1 + x)^{-7}(a_8x^8 + a_7x^7 + \cdots + a_1x + a_0).$$

From Solutions II and III we obtain

$$m = r_1F_{r_1,r_2}(2), \qquad m_0 = m(r_2 + m)^{-1}(r_1 + r_2 - 2),$$

$$m_1 = \tfrac{1}{2}m(r_2 + m)^{-3}(r_1 + r_2 - 2)[m(r_2 - 2) - r_2(r_1 - 2)]$$
$$= \tfrac{1}{2}m_0(r_2 + m)^{-2}[m(r_2 - 2) - r_2(r_1 - 2)],$$

$$m_2 = \tfrac{1}{6}m(r_2 + m)^{-5}(r_1 + r_2 - 2)[2m^2(r_2 - 2)(r_2 - 4)$$
$$-mr_2(3r^2 + 7r_1 r_2 - 32r_2 - 26r_1 + 76) + r_2^2(r_1 - 2)(5r_1 + 3r_2 - 14)];$$

$$b_0 = \xi; \quad b_1 = \xi, \quad b_2 = -\tfrac{1}{2}R_2/(b + 2),$$

$$b_3 = \frac{b + 1}{(b + 2)(b + 3)}\left\{\frac{R_2^2\xi^2}{b + 1} + \frac{R_2\xi^2}{2} - \frac{R_3\xi^3}{3(b + 1)}\right\},$$

$$b_4 = \frac{(b + 1)^2}{(b + 2)(b + 3)(b + 4)}\left\{\frac{(b + 11)^4 R_2 R_3 \xi^4}{4(b + 1)^2} - \frac{5R_2^2\xi^3}{2(b + 1)}\right.$$
$$\left. + \frac{2R_3\xi^3}{3(b + 1)} - \frac{R_2\xi^2}{2} - \frac{(b^2 + 31b + 60)R_2^3\xi^4}{8(b + 1)^2(b + 2)} - \frac{(b + 3)R_4\xi^4}{8(b + 1)^2}\right\}.$$

The solution thus derived will be called Solution IV.

8. Accuracy of solution. It can be shown that for c large compared with a,

$$\int_0^c (y^a e^{-y}/a!)\, dy = 1 - o(1).$$

Thus $r_1 f(x)/(1 + \rho)$ cannot be large compared with r_1; if it were, the left side of (2.2) would be $1 - o(1)$ and the equation would not be satisfied. So, $f(x)$ must be $O(1)$; similarly, ξ must be $O(r_1)$.

Now for Solution I to exist for large r_1 there must be at least a finite k such that $f_r(x)$ is $O(r_1^{rk})$. Since ξ is $O(r_1)$, f_0, f_1, and f_2 are of orders r_1^0, $r_1^{\frac{1}{2}}$, and r_1^1, respectively, which suggests $k = \frac{1}{2}$. That this k will suffice, or even that there exists a suitable $k \leqq 1$, has not been proved and may be only conjectured. Fortunately, it is not necessary to make any such assumption about the value.

For quick (or even any) convergence of Solution I it is necessary that $O(r_1^k) \leqq O(r_2)$ for a suitable choice of k. In practical cases r_2 is usually greater than r_1 and is often large compared with it for K positive.

It can be shown that the $f_r(x)$ of Solution I is of type A_{2r-1}, where "type A_i" will mean "of the form $(1 + x)^{-i}(a_{i+1}x^{i+1} + a_i x^i + \cdots + a_1 x + a_0)$." If Solution I were developed as far as $f_4(x)$, it would yield a type A_7 function, which could be compared with the A_7 function Solution IV. The former of these two type A_7 functions is correct to the order r_2^{-4}. Thus when it is expanded in ascending or descending powers of x, the resulting coefficients are correct to the order r_2^{-4}, and thus differ from the exact ones obtained from Methods II and III, respectively, by terms of the order r_2^{-5}.

Consequently it is readily seen that the type A_7 form of Solution I and that of Solution IV differ only by terms of the order r_2^{-5}, so that Solution IV is correct to the order r_2^{-4}. Using the consequences of the above discussion, we see that Solution IV is correct to the order $(r_2/r_1^k)^{-4}$.

Further, if Solution IV is put in place of $f(x)$ in (2.2), the error involved for ρ small will be of the order ρ^5. Similarly the error involved in using Solution IV for ρ large is of the order ρ^{-4}. These statements apply whether or not there exists an $f(x)$ exactly satisfying this equation.

For convenience of calculation one could, of course, use fewer leading terms of Solutions II and III to form a less accurate Solution IV. The error involved in substituting this Solution IV into (2.2) may be rather less than the error in the approximate $f(x)$, particularly if the upper limit of the inner integrand is sufficiently large or small, when the rate of change of the inner integral with respect to the upper limit will be negligible.

However, when tables have been prepared using the solution given in this paper, there will be no need to use a less accurate approximation to save labour.

The solution given in Section 7, say $f_{IV}(x)$, has been calculated for $r_1 = 8$, $r_2 = 50$, $\alpha = .975$, and a series of x values. The left side of (2.2) was then calculated for $\rho = 1$. One would not expect this to give a particularly accurate value to a function correct to the orders ρ^5 for ρ small and ρ^{-4} for ρ large. Further $r_2/r_1 = 6.25$, which is not very large. Thus one would expect the majority of practical cases to be more favourable than that chosen. By numerical integration, the value of the left side of (2.2) was found correct to five significant figures as .97492, a satisfactory approximation to α.

9. Obtaining and using the confidence limit. If f_{IV} is of suitable form, a suitable approximate confidence limit for K will be given by solving

$$M_1/\mathrm{K} = f_{IV}(M_2/\mathrm{K}),$$

where f_{IV} is the function given by Solution IV. In view of the complicated form of $f_{IV}(x)$, a numerical method of solution evidently will be necessary. Since $x = M_2/\mathrm{K}$ and $y = M_1/\mathrm{K}$, the ratio y/x is M_1/M_2, which has an observed value. Thus the confidence limit, K_α, is given by the intersection (x_0, y_0) of the curve $y = f_{IV}(x)$ with the line $y = (M_1/M_2)x$, since

$$\mathrm{K}_\alpha = M_1/y_0 = M_2/x_0 .$$

This gives a lower limit such that $\Pr(\mathrm{K}_\alpha \leqq \mathrm{K}) = \alpha$.

Certain questions arise immediately:

(*i*). How can one be sure that there will be only one point of intersection?

(*ii*). How can one be sure that there will be any point of intersection?

(*iii*). The previous results depend on the assumption that $\mathrm{K} > 0$. What modifications are required for the case where the sign of K is not known?

These matters will be considered in turn.

(*i*). The complicated expression for f_{IV} makes uniqueness of intersection difficult to prove. A sufficient condition for having no more than one point of intersection (since the asymptote to the curve and curve itself intersect the y-axis in positive values of y) is that the slope of the curve $y = f_{IV}(x)$ should be a monotonically increasing or decreasing function of x. The asymptote to the curve is given by the first two terms of Solution III, the second of which (that independent of x) is positive. This condition is satisfied in the particular example mentioned at the end of the previous section, in which the slope is monotonically increasing. Further, the condition is satisfied when r_2 is sufficiently large compared with r_1; also, the slope of the curve is increasing when α is chosen to correspond to the lower limit, and decreasing in the upper limit case, provided the two appropriate choices of α are both reasonably different from 50 per cent. Hence, at least for r_2 sufficiently large compared with r_1, if not more generally (as the author conjectures), there is no more than one confidence limit corresponding to one value of α.

(*ii*). When the slope is monotonically increasing or decreasing, evidently there will be no intersection of the line with the curve, unless M_1/M_2 is greater than or equal to the asymptotic slope of the curve, which equals $F_{r_1,r_2}(\alpha)$. Now M_1/M_2 is distributed as

$$(\sigma_1^2/\sigma_2^2)F_{r_1,r_2} = [(1 + \rho)/\rho] F_{r_1,r_2} .$$

Thus the probability of nonintersection is the probability of an F_{r_1,r_2} variate not exceeding $\rho F_{r_1,r_2}(\alpha)/(1 + \rho)$. This probability is α when $\rho = \infty$, and decreases with decreasing ρ to zero at $\rho = 0$. Since M_1 and M_2 are positive, it is evident that an intersection in the first quadrant leads to a positive K_α. Further, it can easily be shown that, as $M_1/M_2 \to F_{r_1,r_2}(\alpha)$ from above, $\mathrm{K}_\alpha \to 0$ from above.

At this stage it is convenient to consider (*iii*) along with (*ii*). Now all the previous investigation of a solution providing a confidence limit for K could have been treated in exactly the same way with M_1 and M_2 and all related quantities interchanged. In this way an approximation to a function h, such that

$$\Pr\{M_2/(-\mathrm{K}) \leqq h\,[M_1/(-K)]\} = \alpha$$

independently of a nuisance parameter

$$\rho' = -\mathrm{K}/\sigma_1^2 = -\mathrm{K}/(\mathrm{K} + \sigma_2^2) = -1/(1 + \rho^{-1}) = -\rho/(1 + \rho)$$

would have been obtained. An approximate h_{IV} would have been derived equal to f_{IV} with interchanged r_1 and r_2. A variation of ρ' between 0 and ∞ corresponds to one of ρ between 0 and -1, and is appropriate for K negative. The interchanged ranges are appropriate for K positive. The accuracy of h_{IV} with respect to ρ' would be the same as that of f_{IV} with respect to ρ as far as neglected orders are concerned. However, if r_2 is large compared with r_1, favouring the accuracy of f_{IV}, then the accuracy of h_{IV} would not be so favoured; the reverse is true for r_1 large compared with r_2.

It will be seen to be satisfactory to use the curve $y = f_{IV}(x)$ in the first quadrant together with a second curve, $-x = h_{IV}(-y)$, in the third to provide a confidence limit. However, if the coefficient α is chosen for the first curve, then $1 - \alpha$ will be taken for the second. The reasons for this are explained in the following paragraphs.

Since $F_{r_1,r_2}(\alpha)\, F_{r_2,r_1}(1 - \alpha) = 1$, the asymptote to the first curve in the first quadrant will be parallel to the asymptote to the second curve in the third. Thus a straight line through the origin will intercept the first curve if its slope is less than $F_{r_1,r_2}(\alpha)$, while if its slope is exactly equal to this value, it intersects both curves at infinity.

The set of points S, consisting of those points of the first quadrant lying below the first curve and those of the third quadrant lying above the second curve, will be used to obtain a confidence limit for K. If $\mathrm{K} > 0$, the probability density function is zero outside the first quadrant and the probability of M_1 and M_2 being such that (x, y) lies beneath the first curve is α. If $\mathrm{K} < 0$, the probability density function is zero outside the third quadrant, and the probability of (x, y) lying above the second curve is α.

Thus, whether K is greater or less than zero, the probability is α that M_1 and M_2 are such that (x, y) lies in S. Just as an intersection of the first curve with the line $y = (M_1/M_2)x$ gives a positive value of K_α, an intersection of the second curve with this line gives a negative value. Further, it can be shown easily that $\mathrm{K}_\alpha \to 0$ from below as $M_1/M_2 \to F_{r_1,r_2}(\alpha)$ from below, or as $M_2/M_1 \to F_{r_2,r_1}(1 - \alpha)$ from above.

Thus the two curves together provide a lower confidence limit which falls below K with probability α. Evidently they provide equivalently an upper limit with coefficient $1 - \alpha$. Accordingly, two suitable values of α are selected, one for each limit, and an interval is obtained. The values .025 and .975, giving an interval coefficient of .95, are frequently used in practice. The complicated form of K_α does not lend itself to an examination of which pair of values of α having a given difference (confidence coefficient) yield the shortest interval.

Incidentally, the curves obtained by imaging radially the two curves through the origin into the opposite quadrants can be shown easily to form the curved parts of the boundary of an alternative set of points which yields a confidence limit. However, it is usual to have $\mathrm{K} > 0$ and $r_2 > r_1$, and one would prefer the positive confidence limit to be more accurate. To ensure this, the two curves should be used as discussed above.

Under (*iii*), it remains to be decided whether or not the confidence coefficient is affected by using only that part of the confidence interval which has the same sign as K, if this sign is known. Consider the lower limit with coefficient α_1, when K is known to be positive

$$\begin{aligned}\Pr\{\max(0, \mathrm{K}_{\alpha_1}) \leqq \mathrm{K} \mid \mathrm{K} > 0\} &= 1 - \Pr\{\max(0, \mathrm{K}_{\alpha_1}) \geqq \mathrm{K} \mid \mathrm{K} > 0\} \\ &= 1 - \Pr\{\mathrm{K}_{\alpha_1} \geqq \mathrm{K} \mid \mathrm{K} > 0\} \\ &= 1 - (1 - \alpha_1) = \alpha_1 = \Pr\{\mathrm{K}_{\alpha_1} \leqq \mathrm{K}\}.\end{aligned}$$

A similar discussion applies when K is known to be negative, also for the upper limit when K has known sign. Hence the natural procedure does not distort the confidence coefficient.

In the Introduction, we discussed the use of this confidence interval, or a single limit, for testing a hypothetical value of K. When K is known to be greater than or equal to zero and the hypothesis to be tested is $\mathrm{K} = 0$, the use of the upper limit alone is more appropriate. In this case the hypothesis is rejected if $M_1/M_2 > F_{r_1,r_2}(\alpha)$, which is the usual test of the analysis of variance.

However, throughout this paper there is one possibility which has not been discussed—nor is it obvious how it could be, considering the technique that has been used. This is the case of K = 0, when the above derivation of a confidence interval would be completely invalid. That is not to say that the interval does not apply in this case—(the author conjectures that it does), but the point is just not proved either way, although it is true that $\Pr[y/x \leqq f(x)/x] = \alpha$ in the limit as $K \to 0$. Thus the confidence interval carries with it the perhaps unnecessary proviso that K is not zero.

10. Tabulation. For the practical use of the two curves, discussed in the preceding section, to obtain a confidence limit, the following procedure seems the most satisfactory. For each selected value of α, the values of $f_{IV}(x)/x$ are tabulated for different values of r_1, r_2, and x. It might then be advisable to retabulate, so that for each set of values of r_1, r_2, and $f_{IV}(x)/x$ a value of x, or of $f_{IV}(x)$, is tabulated; otherwise use of the table would require inverse interpolation. To use the hypothetical table,

If $M_1/M_2 \geqq F_{r_1,r_2}(\alpha)$, it is set equal to $f_{IV}(x)/x$ and, by direct interpolation, the appropriate value of $x = M_2/K_\alpha$ is obtained and since M_2 is known, K_α can then be derived;

if $M_1/M_2 = F_{r_1,r_2}(\alpha)$, then $K_\alpha = 0$;

if $M_1/M_2 < F_{r_1,r_2}(\alpha)$, then r_2, r_1, and $(1 - \alpha)$ are used as new values of r_1, r_2, and α, respectively, in the first procedure.

The expression for $f_{IV}(x)$ is very complicated, and the tabulation discussed above, for a suitable selection of values of r_1, r_2, and x, would require tens of thousands of cells. Accordingly the table has not been constructed for inclusion in this paper, and that task remains.

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