We can characterize (2) as saying: Let $\hat{\pi}$ be the linear estimate of minimum variance of π, which is a linear combination of $\pi_1, \cdots, \pi_k$, and let $\hat{v}(\pi)$ be the estimate of the variance of $\hat{\pi}$ based on s^2. Then the confidence interval statements

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2},$$

simultaneously for all π, are correct with probability $1 - \alpha$. This result is contained in [1] and [2].

The quantity $D^2 = \sum \pi_i^2$ is a conventional measure of the "distance" of the null hypothesis that $\pi_1 = \cdots = \pi_k = 0$ from the true state of nature. The power of the analysis of variance test depends only on D^2/σ^2. Hence it would be useful for the experimenter to obtain some information about D.

Making use of the triangle inequality, it follows from (1) that

$$1 - \alpha \leqq \Pr\{(\sum \hat{\pi}_i^2)^{1/2} - (kF_\alpha s^2)^{1/2} \leqq D \leqq (\sum \hat{\pi}_i^2)^{1/2} + (kF_\alpha s^2)^{1/2}\}.$$

The quantity $\sum \hat{\pi}_i^2 = Q_1^2$ is what the experimenter calculates as the "sum of squares due to hypothesis." Hence, instead of just making the statements about the functions π, we can make the simultaneous estimates

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2}, \qquad \text{for all } \pi = \sum a_i \pi_i,$$

$$Q_1 - (kF_\alpha s^2)^{1/2} \leqq D \leqq Q_1 + (kF_\alpha s^2)^{1/2},$$

with the probability of being correct equal to $1 - \alpha$.

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AN INEQUALITY ON POISSON PROBABILITIES

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This note proves an inequality concerning the exponential series or Poisson distribution, however one prefers to view the matter. Specifically, it will be shown that if $[\lambda]$ is the greatest integer not exceeding λ,

$$(1) \qquad \sum_{j=0}^{[\lambda]} \frac{\lambda^j}{j!} > \begin{cases} e^{\lambda - 1} & \text{for all } \lambda \geqq 0; \\ \tfrac{1}{2} e^{\lambda} & \text{for all integral } \lambda > 0. \end{cases}$$

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There are numerous ways of expressing this inequality. It will be demonstrated in the form

$$2) \qquad A_\lambda = \Pr\{X_\lambda \leqq \lambda\} > \begin{cases} e^{-1} & \text{for all } \lambda \geqq 0; \\ \frac{1}{2} & \text{for all integral } \lambda > 0. \end{cases}$$

where X_λ denotes a Poisson chance variable with parameter $\lambda \geqq 0$. The function A_λ is discontinuous at the positive integers but enjoys certain monotonic properties which may be exploited by defining

$$3) \qquad A_{n,\lambda} = \sum_{j=0}^{n} \frac{\lambda^j}{j!} e^{-\lambda} \qquad \text{for all integral } n \geqq 0.$$

Then $A_{n,\lambda} = A_\lambda$ for $n \leqq \lambda < n + 1$. As

$$\frac{d}{d\lambda} A_{n,\lambda} = \frac{-\lambda^n e^{-\lambda}}{n!} < 0,$$

we have

$$4) \qquad A_{n,n} - A_{n,n+1} = \int_n^{n+1} \frac{\lambda^n e^{-\lambda}}{n!} d\lambda = c_n, \text{ (say)}.$$

On the other hand,

$$5) \qquad A_{n+1,n+1} - A_{n,n+1} = \frac{e^{-(n+1)}(n+1)^{n+1}}{(n+1)!} = d_n, \text{ (say)}.$$

Thus to prove that $A_{n,n} > A_{n+1,n+1}$, it suffices to show that $c_n > d_n$.

Now for $0 < y < 1$, the quantity $[1 - y/(n+1)]^n$ decreases monotonically, whence $[1 - y/(n+1)]^n > e^{-y}$ in $(0, 1)$. It follows that

$$6) \qquad \int_0^1 (n + 1 - y)^n e^{-(n+1-y)} \, dy > (n+1)^n e^{-(n+1)}$$

With a change of variable $\lambda = n + 1 - y$, we conclude that $c_n > d_n$, that is $A_{n,n} > A_{n+1,n+1}$.

In an analogous manner, $[1 + x/n]^n < e^x$ for $x > 0$ and from

$$\int_0^1 (1 + x/n)^n e^{-x} \, dx < 1,$$

we have

$$7) \qquad \int_n^{n+1} \frac{\lambda^n}{n!} e^{-\lambda} \, d\lambda < \frac{e^{-n} n^n}{n!}.$$

That is, $A_{n,n} - A_{n,n+1} < A_{n,n} - A_{n-1,n}$. Thus, in summary, we have

$$8) \qquad A_{n,n} > A_{n+1,n+1}, \qquad A_{n-1,n} < A_{n,n+1}.$$

From well known limit theorems,

$$\lim_{\lambda\to\infty} A_\lambda = \lim_{\lambda\to\infty} \Pr\{X_\lambda \leqq \lambda\} = \lim_{\lambda\to\infty} \Pr\left\{\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \leqq 0\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-y^2/2}\, dy = \tfrac{1}{2}. \tag{9}$$

The second portion of inequality (2) follows directly from (8) and (9). The remainder is a consequence of these, of the fact that $A_{n,\lambda}$ decreases monotonically from $A_{n,n}$ to $A_{n,n+1}$ in the interval $[n, n + 1)$, and of

$$A_{n,n+1} > A_{0,1} = e^{-1}, \qquad n = 1, 2, \cdots . \tag{10}$$

If an additional term is included in the sum, we note that

$$b_\lambda = \sum_{j=0}^{[\lambda]+1} \frac{\lambda^j e^{-\lambda}}{j!} \geqq \tfrac{1}{2} \qquad \text{for all } \lambda \geqq 0, \tag{11}$$

since $b_\lambda > A_{n+1,n+1}$ for $\lambda \leqq n + 1$. A final reformulation of part of (2) is

$$\Pr\{X_\lambda \leqq \lambda\} > \Pr\{X_\lambda > \lambda\}, \qquad \text{all integral } \lambda > 0. \tag{12}$$

NOTE ON LINEAR HYPOTHESES WITH PRESCRIBED MATRIX OF NORMAL EQUATIONS

By John E. Maxfield and Robert S. Gardner

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The existence theorem proven in this note relates to the problem of finding an experimental design leading to the analysis determined by the given rational matrix A of the normal equations. The matrix B found by the method used in the proof always has an interpretation as specifying the rational values of some set of regression variables. In the interesting case in which the entries of A are integers, so are the entries of B, but B is not in general interpretable as an analysis of variance. The transpose of a matrix A will be denoted by A^T.

Theorem. *Let A be a symmetric positive semidefinite matrix with rational integral entries. There exist a rational integer a and a matrix B having rational integral entries such that $BB^T = a^2A$.*

Proof. There exists a nonsingular matrix P such that $P^TAP = D$, a diagonal matrix, where P and D have rational entries ([1], p. 56). Then $(P^T)^{-1}$ has rational entries. Let a_1 be the least common denominator of the entries of P^T, a_2 of $(P^T)^{-1}$, and a_3 of D. Let $a^{1/2} = a_1a_2a_3$. Then $a^{1/2}P^Ta^{1/2}AP = aD$. If A is positive semidefinite, then aD has only positive integers or zeros on its diagonal. Let B_1 be a $n \times 4n$ matrix composed of four diagonal $n \times n$ matrices placed side by side,

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