A NOTE ON SIMULTANEOUS CONFIDENCE INTERVALS

By Meyer Dwass

Northwestern University

Scheffé [1] and Bose and Roy [2] have recently given a method for making confidence interval estimates useful in the analysis of variance. Here we show that we can append to such a set of confidence interval estimates an additional one on D, where D^2/σ^2 is the "distance function" which determines the power of the analysis of variance test. For completeness, we first determine the same set of simultaneous confidence intervals as in [1] and [2] in a simple way. This may be of at least pedagogic interest.

Lemma. *Let $x_1, \cdots, x_k$ stand for a point in k-dimensional Euclidean space. The equations of the two planes normal to a vector $a_1, \cdots, a_k$ and tangent to a sphere of radius R centered at $c_1, \cdots, c_k$ are given by*

$$\sum a_i x_i = \sum a_i c_i + R(\sum a_i^2)^{1/2}, \qquad \sum a_i x_i = \sum a_i c_i - R(\sum a_i^2)^{1/2}.$$

This is an elementary geometric fact which we will not prove.

The analysis of variance problem can be formulated in terms of

$$y = (y_1, \cdots, y_n),$$

a set of n independent, normal random variables, each with variance σ^2, and of $\hat{\pi}_1(y), \cdots, \hat{\pi}_k(y)$, which are k independent random variables, each a linear function of $y_1, \cdots, y_n$. Each $\hat{\pi}_i$, for $i = 1, \cdots, k$, is the linear estimate of minimum variance of $E\hat{\pi}_i = \pi_i$; the variance of $\hat{\pi}_i$ is σ^2. Let ms^2 be independent of

$$\hat{\pi}_1, \cdots, \hat{\pi}_k$$

and distributed as the sum of squares of m independent $N(0, \sigma)$ random variables. The problem is to estimate or test hypotheses about $\pi_1, \cdots, \pi_k$ or linear combinations of them.

Let F_α be a number such that $(0, F_\alpha)$ includes $1 - \alpha$ of the probability of the F distribution with (k, m) d.f. Let C be a k-sphere centered at $\hat{\pi}_1, \cdots, \hat{\pi}_k$ with radius $(kF_\alpha s^2)^{1/2}$. For a vector $a = a_1, \cdots, a_k$, let $P_1(a)$ and $P_2(a)$ be the two planes which are normal to a and tangent to C. Then

$$(1) \qquad 1 - \alpha = \Pr\left\{\sum_{i=1}^{k} \frac{(\hat{\pi}_i - \pi_i)^2}{ks^2} \leqq F_\alpha\right\},$$

which is the probability that C covers the point $\pi_1, \cdots, \pi_k$. This in turn is the probability that $\pi_1, \cdots, \pi_k$ is located between $P_1(a)$ and $P_2(a)$ for all vectors a. Thus

$$(2) \quad 1 - \alpha = \Pr\{\sum a_i\hat{\pi}_i - (kF_\alpha \sum a_i^2 s^2)^{1/2} \leqq \sum a_i \pi_i \leqq \sum a_i \hat{\pi}_i + (kF_\alpha \sum a_i^2 s^2)^{1/2} \text{ for all vectors } a\}.$$

The last step follows from the lemma.

Received April 24, 1954, revised July 21, 1954.

We can characterize (2) as saying: Let $\hat{\pi}$ be the linear estimate of minimum variance of π, which is a linear combination of $\pi_1, \cdots, \pi_k$, and let $\hat{v}(\pi)$ be the estimate of the variance of $\hat{\pi}$ based on s^2. Then the confidence interval statements

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2},$$

simultaneously for all π, are correct with probability $1 - \alpha$. This result is contained in [1] and [2].

The quantity $D^2 = \sum \pi_i^2$ is a conventional measure of the "distance" of the null hypothesis that $\pi_1 = \cdots = \pi_k = 0$ from the true state of nature. The power of the analysis of variance test depends only on D^2/σ^2. Hence it would be useful for the experimenter to obtain some information about D.

Making use of the triangle inequality, it follows from (1) that

$$1 - \alpha \leqq \Pr\{(\textstyle\sum \hat{\pi}_i^2)^{1/2} - (kF_\alpha s^2)^{1/2} \leqq D \leqq (\textstyle\sum \hat{\pi}_i^2)^{1/2} + (kF_\alpha s^2)^{1/2}\}.$$

The quantity $\sum \hat{\pi}_i^2 = Q_1^2$ is what the experimenter calculates as the "sum of squares due to hypothesis." Hence, instead of just making the statements about the functions π, we can make the simultaneous estimates

$$\hat{\pi} - [kF_\alpha \hat{v}(\pi)]^{1/2} \leqq \pi \leqq \hat{\pi} + [kF_\alpha \hat{v}(\pi)]^{1/2}, \qquad \text{for all } \pi = \textstyle\sum a_i \pi_i,$$

$$Q_1 - (kF_\alpha s^2)^{1/2} \leqq D \leqq Q_1 + (kF_\alpha s^2)^{1/2},$$

with the probability of being correct equal to $1 - \alpha$.

REFERENCES

[1] H. Scheffé, "A method for judging all contrasts in the analysis of variance," *Biometrika*, Vol. 40 (1953), pp. 87–104.

[2] S. N. Roy and R. C. Bose, "Simultaneous confidence interval estimation," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 513–536.

AN INEQUALITY ON POISSON PROBABILITIES

By Henry Teicher

Purdue University

This note proves an inequality concerning the exponential series or Poisson distribution, however one prefers to view the matter. Specifically, it will be shown that if $[\lambda]$ is the greatest integer not exceeding λ,

$$(1) \qquad \sum_{j=0}^{[\lambda]} \frac{\lambda^j}{j!} > \begin{cases} e^{\lambda - 1} & \text{for all } \lambda \geqq 0; \\ \frac{1}{2} e^{\lambda} & \text{for all integral } \lambda > 0. \end{cases}$$

Received February 16, 1954.