under equivalence $[\mathbf{S}, P]$. A subfield equivalent to a statistic need not itself be a statistic. In an attempt to avoid this difficulty, one may define a *pseudo-statistic* as any subfield equivalent to a statistic. If Lemma 3 remained valid for pseudo-statistics in the sense that a member of C_π is a pseudo-statistic if and only if it is equivalent to $\mathbf{S}_\pi$, this would establish the desired result.

The following example shows that this stronger version of Lemma 3 is not correct. Let $\mathbf{S}_\pi$ be the class of all Lebesgue sets on the real line and $\mathbf{S}_0$ the class of all Lebesgue sets differing only by a set 0 from a set symmetric with respect to the origin. Clearly, $\{x\} \varepsilon \mathbf{S}_0$ for all x so that $\mathbf{S}_0 \varepsilon C_\pi$. Also $\mathbf{S}_0$ is a pseudo-statistic since it is equivalent to the subfield induced by $T(x) = |x|$. But clearly $\mathbf{S}_0$ and $\mathbf{S}_\pi$ are not equivalent.

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A NOTE ON CONFIDENCE SETS FOR RANDOM VARIABLES

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Suppose the chance variables $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ have a joint probability distribution depending on the unknown parameters $\theta_1, \cdots, \theta_k$, but otherwise of known form. We assume that there is a set of sufficient statistics for

$\theta_1, \cdots, \theta_k$, denoted by $T_1(X_1, \cdots, X_m, Y_1, \cdots, Y_n), \cdots,$

$T_r(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$. We shall let X denote the vector $(X_1, \cdots, X_m)$, Y the vector $(Y_1, \cdots, Y_n)$, θ the vector $(\theta_1, \cdots, \theta_k)$, and $T(X, Y)$ the vector $(T_1(X, Y), \cdots, T_r(X, Y))$. $P_\theta(A)$ shall denote the probability of A when the vector of parameters equals θ, and $P_\theta(A \mid B)$ shall denote the conditional probability of A given B when the vector of parameters equals θ.

Given a number α between 0 and 1, if for each vector X we can find a set $S(X)$ in n-dimensional Euclidean space such that $P_\theta(Y \text{ in } S(X)) = \alpha$ identically in θ, then the system of sets $S(X)$ is called a "parameter-free confidence set of level α for the random vector Y."

Since $T(X, Y)$ is a set of sufficient statistics for θ, the joint conditional distribution of Y given that $T(X, Y) = t = (t_1, \cdots, t_r)$ is independent of θ. But then for any given vector t, it is possible to construct a region $S'(t)$ in n-dimensional Euclidean space such that $P_\theta(Y \text{ in } S'(t) \mid T(X, Y) = t) = \alpha$ identically in θ

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(This last sentence is true if the conditional distribution of Y given that $T(X, Y) = t$ is absolutely continuous. In the other cases only approximate equality may be possible.) Then we have $P_\theta[Y \text{ in } S'(T(X, Y))] = \alpha$ identically in θ. Now for any X, we define $S(X)$ as the set of vectors Y such that Y is in $S'(T(X, Y))$. Clearly, $S(X)$ is a parameter-free confidence set of level α.

This construction is the exact analogue of Neyman's construction of confidence sets for a parameter, as described in [1].

As an example, we discuss the problem that motivated this note. We have two solutions, each consisting of a certain type of particles suspended in water. The relative concentration of the two solutions is known, but not the absolute concentration of either. That is, we do not know the average number of particles per unit volume for either solution, but we do know the ratio of the average numbers of particles per unit volume for the two solutions. In taking this ratio, we shall understand that the numerator refers to the second solution, and shall denote the ratio by R. V_1 cubic centimeters of the first solution are drawn and the number of particles in this volume is counted. We denote this number by X. V_2 cubic centimeters from the second solution are to be injected into an experimental animal. We denote the number of particles injected into the animal by Y. Y will never be directly observed. Given a number α between 0 and 1, the problem is to find two functions of X, say $L_1(X)$ and $L_2(X)$, with $L_1(X) < L_2(X)$, such that $P[L_1(X) \leqq Y \leqq L_2(X)] = \alpha$ no matter what the values of the unknown absolute concentrations are. From familiar considerations, it is reasonable to assume that X and Y are independently distributed, each with a Poisson distribution. If θ denotes the expected value of X, then the expected value of Y is $r\theta$, where $r = R(V_2/V_1)$ and is therefore known. θ is unknown. Then we wish to have $P_\theta[L_1(X) \leqq Y \leqq L_2(X)] \equiv \alpha$ identically in θ. It is easily verified that $X + Y$ is a sufficient statistic for θ, and that

$$P(Y = y|X + Y = z) = \binom{z}{y}[r/(1 + r)]^y[1/(1 + r)]^{z-y},$$

that is, the conditional distribution of Y given $X + Y$ is binomial and is independent of θ. For any given value of $X + Y$, we can find two numbers $M_1(X + Y)$ $M_2(X + Y)$ so that $P[M_1(X + Y) \leqq Y \leqq M_2(X + Y) \mid X + Y]$ is approximately equal to α irrespective of the value of θ. If X is large enough to make both $Xr/(1 + r)$ and $X/(1 + r)$ large, then the conditional distribution of Y given $X + Y$ is nearly normal, and we have as a good approximation

$$M_1(X + Y) = (X + Y)r \,/\, (1 + r) - k\sqrt{(X + Y)r/(1 + r)^2},$$

$$M_2(X + Y) = (X + Y)r \,/\, (1 + r) + k\sqrt{(X + Y)r/(1 + r)^2},$$

where k depends on α and is found from the table of the normal distribution. Then the confidence set $S(X)$ is the set of all values of Y such that $M_1(X + Y) \leqq Y \leqq M_2(X + Y)$, and using the approximate values of $M_1(X + Y)$ and $M_2(X + Y)$ from the preceding sentence, we find that $S(X)$ consists of all values of Y between the limits

$$L_1(X) = \tfrac{1}{2}\{2rX + rk^2 - k\sqrt{k^2r^2 + 4rX(1 + r)}\}, \quad \text{and}$$

$$L_2(X) = \tfrac{1}{2}\{2rX + rk^2 + k\sqrt{k^2r^2 + 4rX(1 + r)}\}.$$

Therefore, for these limits, we have that $P[L_1(X) \leqq Y \leqq L_2(X)]$ is approximately equal to α, no matter what the value of θ is.

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AN INCONSISTENCY OF THE METHOD OF MAXIMUM LIKELIHOOD

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An example was given by Neyman and Scott [2] to show that there are situations where the method of maximum likelihood leads to inconsistent estimators. In their example considered, the observations were supposed to be drawn from an infinite sequence of distinct populations involving an infinite sequence of nuisance parameters.

An example is given here to demonstrate that even in simple situations where all the observations are independently and identically distributed and involve only one unknown parameter, the method of maximum likelihood may lead us astray. The example typifies the situations where the correct method of setting up a point estimate should begin with a test of hypothesis between two composite alternatives.

Let A be the set of all rational numbers in the closed interval (0, 1) and B any countable set of irrational numbers in the same interval. Let X be a random variable that takes the two values 0 and 1 with

$$P(X = 1) = \begin{cases} \theta & \text{if } \theta \,\varepsilon\, A, \\ 1 - \theta & \text{if } \theta \,\varepsilon\, B. \end{cases}$$

If r is the total number of 1's in a set of n random observations on X, then from the rationality of r/n it follows at once that the maximum likelihood estimator of θ is r/n. But r/n converges (in probability) to θ or $1 - \theta$ according as $\theta \,\varepsilon\, A$ or $\theta \,\varepsilon\, B$.

Now, since A and B are both countable sets, it follows [1] that there exists a consistent test for the composite hypothesis $\theta \,\varepsilon\, A$ against the composite alterna-

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