

# THE EFFICIENCY OF TESTS<sup>1</sup>

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**Summary.** The efficiency of a family of tests is defined. Methods for evaluating the efficiency are discussed. The asymptotic efficiency is obtained for certain families of tests under assumptions which imply that the sample size is large.

**1. Introduction.** Let  $w_n^{(1)}$  and  $w_n^{(2)}$  be two tests each of which has the same fixed significance level for testing a hypothesis  $\theta = \theta_1$ , and which are defined for every sample size  $n$ . The relative efficiency of test  $w_n^{(2)}$ , or rather, of the sequence  $\{w_n^{(2)}\}$ , with respect to test  $w_n^{(1)}$  has been defined as the ratio  $n_1/n_2$ , where  $n_2$  is the least sample size required by a test of the sequence  $\{w_n^{(2)}\}$  in order to achieve the same power for a given alternative  $\theta = \theta_2$  as is achieved by the test in  $\{w_n^{(1)}\}$  using a sample of size  $n_1$ . This is essentially the definition used by Pitman (see Noether [6], p. 241).

In Section 2 we extend this definition by replacing sequences of tests by arbitrary families of (nonsequential) tests and the parameters  $\theta_1$  and  $\theta_2$  by two arbitrary classes of distributions. The tests are regarded as general two-decision rules. If  $N(\mathfrak{J})$  is the least sample size used by a test in family  $\mathfrak{J}$  whose probabilities of the two kinds of error (corresponding to the two classes of distributions) do not exceed two given numbers, the ratio  $N(\mathfrak{J}_1)/N(\mathfrak{J}_2)$  is defined as the relative efficiency of family  $\mathfrak{J}_2$  with respect to family  $\mathfrak{J}_1$ .

In Section 3 it is pointed out that the determination of  $N(\mathfrak{J})$  is closely related to finding a test which maximizes the minimum power.

In Section 4 the problem of asymptotic efficiency is considered. In studies of the asymptotic efficiency of tests it is customary to consider a simple hypothesis  $\theta = \theta_1$  (say) and a simple alternative,  $\theta = \theta_2$ , and to assume that as  $n \rightarrow \infty$ ,  $\theta_1$  remains fixed and  $\theta_2$  approaches  $\theta_1$  in a certain way, for instance by setting  $\theta_2 = \theta_1 + kn^{-1/2}$ . Neither the restriction to simple hypotheses nor the assumption that  $\theta_2$  depends on  $n$  seems to be entirely adequate from the point of view of most applications.

In Section 4 of this paper a somewhat different approach is used. As a typical special case, consider a sequence of independent random variables with common cdf (cumulative distribution function)  $F$ . Let  $\theta(F)$  be a real-valued function of  $F$ , and suppose that one or the other decision is undesirable according as  $\theta(F) \leq \theta_1$  or  $\theta(F) \geq \theta_2$ , where  $\theta_1$  and  $\theta_2 > \theta_1$  are fixed numbers. Since  $\theta_1$  and  $\theta_2$  usually will be so chosen that neither decision is strongly preferred when  $\theta_1 < \theta(F) < \theta_2$ , small values of  $\theta_2 - \theta_1$  are frequently of particular interest.

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Let  $N(\mathfrak{J})$  denote the least sample size required by a test in family  $\mathfrak{J}$  for which the probability of a wrong decision does not exceed  $\alpha_1$  if  $\theta(F) \leq \theta_1$ , and does not exceed  $\alpha_2$  if  $\theta(F) \geq \theta_2$ . Thus  $N(\mathfrak{J})$  is a function of  $\theta_1$  and  $\theta_2$ . Under suitable assumptions we derive asymptotic expressions for  $N(\mathfrak{J})$  as  $\delta = \theta_2 - \theta_1$  tends to zero, while  $\alpha_1$  and  $\alpha_2$  remain fixed. Illustrative examples are given in Section 5.

**2. The efficiency of a family of tests.** Let  $X$  be a random variable or a random vector with cdf  $F$ , which is assumed to belong to a class  $\mathcal{C}$  of cdfs. It is desired to decide, on the basis of  $n$  independent observations  $\mathbf{x}_n = (x_1, \dots, x_n)$  on  $X$ , between two alternative courses of action,  $A_1$  and  $A_2$ . Let  $d_i$  denote the decision in favor of  $A_i$ , with  $i = 1, 2$ . Suppose there are given two disjoint subclasses  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of  $\mathcal{C}$  such that decision  $d_i$  is preferred if  $F$  is in  $\mathcal{C}_i$ , for  $i = 1, 2$ . (In most applications one will choose the classes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  in such a way that neither decision is strongly preferred if  $F$  is neither in  $\mathcal{C}_1$  nor in  $\mathcal{C}_2$ .)

Consider a decision rule, briefly referred to as a test, of the following type. Let  $E_n$  denote the space of points  $\mathbf{x}_n$ ; if  $x_j$  is a vector with  $k$  components,  $E_n$  may be taken as the  $nk$ -dimensional Euclidean space. Let  $w_{1n}$  be a subset of  $E_n$ , and denote its complement by  $w_{2n}$ . (All sets of points  $\mathbf{x}_n$  considered in this paper are assumed to be Borel sets, and all functions of  $\mathbf{x}_n$  are understood to be Borel-measurable.) A test determined by the pair of sets  $w_n = (w_{1n}, w_{2n})$  consists in taking  $n$  observations on  $X$ , and making decision  $d_i$  if the observed point  $\mathbf{x}_n$  is in  $w_{in}$ , for  $i = 1, 2$ . This test will be referred to as the test  $w_n$ . A test  $w_n$  with  $w_{in} \subset E_n$  will be called a test based on  $n$  observations.

Let  $\alpha_1$  and  $\alpha_2$  be two positive numbers. We shall say that the test  $w_n$  solves the problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  if

$$(2.1) \quad P(\mathbf{X}_n \varepsilon w_{in} | F) \geq 1 - \alpha_i \quad \text{for all } F \text{ in } \mathcal{C}_i, \quad i = 1, 2,$$

where  $\mathbf{X}_n = (X_1, \dots, X_n)$  is a random vector with values in  $E_n$ , and  $P(R | F)$  denotes the probability of relation  $R$  when  $X_1, \dots, X_n$  are independent with common cdf  $F$ .

Let  $\mathfrak{J}$  be a family of tests. We shall mainly be concerned with families  $\mathfrak{J}$  which, for every positive integer  $n$ , contain at least one test based on  $n$  observations. For example,  $\mathfrak{J}$  may be the family of all tests  $w_n$  with

$$w_{1n} = \{\mathbf{x}_n : t_n(\mathbf{x}_n) < c\}, \quad -\infty < c < \infty, \quad n = 1, 2, \dots$$

where  $\{t_n\}$  is a given sequence of functions, and  $\{\mathbf{x}_n : R\}$  denotes the all set of points  $\mathbf{x}_n$  such that relation  $R$  is satisfied.

Let  $N(\mathfrak{J}) = N(\mathfrak{J}, \mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  be the least integer  $n$  such that the inequalities (2.1) are satisfied for some test in  $\mathfrak{J}$ . Thus  $N(\mathfrak{J})$  is the least number of observations with which problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  can be solved when we restrict ourselves to tests belonging to family  $\mathfrak{J}$ . If no test in  $\mathfrak{J}$  satisfies (2.1), we set  $N(\mathfrak{J}) = \infty$ . The number  $N(\mathfrak{J})$  will be termed the *efficiency index*, or simply the *index*, of family  $\mathfrak{J}$  for problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ . Evidently  $\mathfrak{J}$  may contain more than one test based on  $N(\mathfrak{J})$  observations.

If  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  are two families of tests and at least one of the indices  $N(\mathfrak{J}_i) = N(\mathfrak{J}_i, \mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ , for  $i = 1, 2$ , is finite, the ratio

$$\text{eff}(\mathfrak{J}_1/\mathfrak{J}_2) = N(\mathfrak{J}_2)/N(\mathfrak{J}_1)$$

will be called the *efficiency* of family  $\mathfrak{J}_1$  relative to family  $\mathfrak{J}_2$  for problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ . The relative efficiency can thus equal any nonnegative rational number or infinity. This definition of relative efficiency of two tests is an extension of Pitman's definition of the same term (see Noether [6], p. 241).

Let  $\mathfrak{J}^*$  be the family of all tests  $w_n = (w_{1n}, w_{2n})$ , for all  $n = 1, 2, \dots$ . If  $N(\mathfrak{J}^*) = \infty$ , problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  has no solution (within the family  $\mathfrak{J}^*$ ). If  $N(\mathfrak{J}^*) < \infty$ , and  $\mathfrak{J}$  is any subfamily of  $\mathfrak{J}^*$ , then  $\text{eff}(T/T^*)$  may be called the (absolute) efficiency of family  $\mathfrak{J}$  for problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$ , provided we confine ourselves to tests in  $\mathfrak{J}^*$ .

Clearly,  $\text{eff}(\mathfrak{J}/\mathfrak{J}^*) \leq 1$ , the sign of equality holding if and only if  $\mathfrak{J}$  contains a test which solves problem  $(\mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  and is based on the least possible number of observations with which the problem can be solved by any test in  $\mathfrak{J}^*$ .

All that has been said can immediately be extended to families of randomized tests. A randomized test is determined by a function  $\phi_{1n}(\mathbf{x}_n)$ , where  $0 \leq \phi_{1n}(\mathbf{x}_n) \leq 1$ . Let  $\phi_{2n} = 1 - \phi_{1n}$ . The test consists in taking  $n$  observations  $\mathbf{x}_n$  and performing a random experiment whose two possible outcomes,  $e_1$  and  $e_2$ , have probabilities  $\phi_{1n}(\mathbf{x}_n)$  and  $\phi_{2n}(\mathbf{x}_n)$ , respectively. If event  $e_i$  occurs, decision  $d_i$  is made. A test determined by the pair of functions  $\phi_n = (\phi_{1n}, \phi_{2n})$  will be referred to as the test  $\phi_n$ .

If  $\mathfrak{J}$  is a family of randomized tests, the index  $N(\mathfrak{J}, \mathcal{C}_1, \mathcal{C}_2, \alpha_1, \alpha_2)$  is defined as the least  $n$  such that the relations

$$E(\phi_{in} | F) \geq 1 - \alpha_i \quad \text{for all } F \text{ in } \mathcal{C}_i, \quad i = 1, 2,$$

are satisfied for some  $\phi_n$  in  $\mathfrak{J}$ . Here  $E(\phi_{in} | F)$  denotes the expected value of  $\phi_{in}(\mathbf{X}_n)$  when  $X_1, \dots, X_n$  are independent with the common cdf  $F$ . If randomized tests are admitted which do not use any observations, we could have  $N(\mathfrak{J}) = 0$ . This trivial case can be excluded by assuming that  $\alpha_1 + \alpha_2 < 1$ .

The notion of efficiency could be extended to families of tests such that the choice of the number of observations also depends on a random experiment, the probabilities of whose outcomes may or may not depend on the observations. (In the former case we are dealing with sequential tests.) This paper is confined to families of nonsequential tests based on a nonrandom number of observations.

So far we have assumed, to simplify the exposition, that  $X_1, X_2, \dots$ , is a sequence of independent, identically distributed random variables. Suppose, more generally, that for every  $n$  the random vector  $\mathbf{X}_n = (X_1, \dots, X_n)$  has a cdf  $G_n$  which belongs to a class  $\mathcal{C}_n$ . For every  $n$  there are given two disjoint subclasses,  $\mathcal{C}_{1n}$  and  $\mathcal{C}_{2n}$ , of  $\mathcal{C}_n$ . We say that a test  $\phi_n = (\phi_{1n}, \phi_{2n})$  solves problem  $(\{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$  if

$$E(\phi_{in} | G_n) \geq 1 - \alpha_i \quad \text{for all } G_n \text{ in } \mathcal{C}_{in}, \quad i = 1, 2,$$

where  $E(\phi_{i_n} | G_n)$  denotes the expected value of  $\phi_{i_n}(\mathbf{X}_n)$  when  $\mathbf{X}_n$  has the cdf  $G_n$ . The definitions of  $N(\mathfrak{J}) = N(\mathfrak{J}, \{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$  and  $\text{eff}(\mathfrak{J}_1/\mathfrak{J}_2)$  are obvious extensions of the corresponding definitions in the special case.

**3. The determination of  $N(\mathfrak{J})$ .** Let  $\mathfrak{J}$  be a given family of tests, randomized or not. Since a randomized test  $\phi_n$  such that  $\phi_n(\mathbf{x}_n) = 0$  or 1 for all  $\mathbf{x}_n$  is equivalent to a nonrandomized test  $w_n$ , we may denote the tests in  $\mathfrak{J}$  by  $\phi_n$ . Let  $\mathfrak{J}_n$  be the family of all tests in  $\mathfrak{J}$  which are based on  $n$  observations. Denote by  $\mathfrak{J}_{1n}$  the family of all tests  $\phi_n$  in  $\mathfrak{J}_n$  for which

$$(3.1) \quad E(\phi_{1n} | G_n) \geq 1 - \alpha_1 \quad \text{for all } G_n \text{ in } \mathcal{C}_{1n}.$$

**THEOREM 3.1.** *Let*

$$M(\phi_n) = \sup_{G_n \in \mathcal{C}_{2n}} E(\phi_{1n} | G_n), \quad M_n = \inf_{\phi_n \in \mathfrak{J}_{1n}} M(\phi_n).$$

*If  $N(\mathfrak{J}, \{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$  is finite, it is the least integer  $n$  for which  $M_n \leq \alpha_2$  and either  $M_n < \alpha_2$ , or  $M_n = \alpha_2$  and  $M(\phi_n) = \alpha_2$  for some  $\phi_n$  in  $\mathfrak{J}_{1n}$ .*

The proof is left to the reader.

Due to the symmetry of the problem, the roles of  $\mathcal{C}_{1n}$  and  $\mathcal{C}_{2n}$  obviously can be interchanged.

Adapting a familiar terminology, we may say that a test which satisfies (3.1) has level  $\alpha_1$  with respect to  $\mathcal{C}_{1n}$ , and we may call  $1 - M(\phi_n)$  the *minimum power of test  $\phi_n$  with respect to  $\mathcal{C}_{2n}$* . If there exists a test  $\phi_n$  in  $\mathfrak{J}_{1n}$  such that  $M(\phi_n) = M_n$ , the test is said to *maximize the minimum power with respect to  $\mathcal{C}_{2n}$* . Tests which maximize the minimum power can sometimes be obtained by a method due to Wald and explicitly applied to this problem by Lehmann ([3], Theorem 8.3). A proof of a special case of the theorem and illustrations of its use are given by Lehmann and Stein [4], [5]. Lehmann's theorem immediately applies to cases where  $\mathfrak{J}_n$  is the family of all tests  $\phi_n$ , or of all tests  $\phi_n$  which depend on a given function of  $\mathbf{x}_n$ . It can be extended to arbitrary families of tests.

**4. Asymptotic efficiency.** The rest of this paper considers the asymptotic behavior of the efficiency index  $N(\mathfrak{J})$  for certain families of tests in cases where the "distance" between the classes  $\mathcal{C}_{1n}$  and  $\mathcal{C}_{2n}$  is small. Let  $\theta(G_n)$  be a real-valued function defined for all  $G_n$  in  $\mathcal{C}_n$  and for every (or every sufficiently large) positive integer  $n$ . We assume that the set  $\omega$  of values  $\theta(G_n)$  when  $G_n \in \mathcal{C}_n$  is an interval, finite or infinite, which is independent of  $n$ . Let  $\theta_1$  and  $\theta_2$  be two numbers in  $\omega$ , with  $\theta_1 < \theta_2$ , and let  $\mathcal{C}_{1n}$  and  $\mathcal{C}_{2n}$  be the classes of all  $G_n$  in  $\mathcal{C}_n$  such that  $\theta(G_n) \leq \theta_1$  and  $\theta(G_n) \geq \theta_2$ , respectively.

Let  $\{t_n(\mathbf{x}_n)\}$  for  $n = 1, 2, \dots$ , be a given sequence of functions. Let  $\mathfrak{J}$  be the family of all tests  $\phi_n^{(c)}$  with

$$(4.1) \quad \phi_{1n}^{(c)}(\mathbf{x}_n) = \begin{cases} 1 & \text{if } t_n(\mathbf{x}_n) < c; \\ \text{arbitrary} & \text{if } t_n(\mathbf{x}_n) = c; \\ 0 & \text{if } t_n(\mathbf{x}_n) > c; \end{cases} \quad \begin{matrix} -\infty < c < \infty, \\ n = 1, 2, \dots \end{matrix}$$

For definiteness, we shall assume that  $\phi_{1n}^{(c)} = 1$  if  $t_n = c$ , so that  $\mathfrak{J}$  is the family of nonrandomized tests  $w_n^{(c)}$  with

$$(4.2) \quad w_{1n}^{(c)} = \{\mathbf{x}_n : t_n(\mathbf{x}_n) \leq c\}, \quad -\infty < c < \infty; \quad n = 1, 2, \dots$$

The asymptotic results to be derived for family (4.2) will also hold for any family (4.1).

Nothing will be changed if  $\mathfrak{J}$  consists of all tests (4.1) with  $n$  exceeding a fixed number  $n'$ . In this case the words "for every  $n$ " in the following assumptions should be read "for every  $n > n'$ ."

We shall study the asymptotic behavior of  $N(\mathfrak{J}) = N(\mathfrak{J}, \{\mathcal{C}_{1n}\}, \{\mathcal{C}_{2n}\}, \alpha_1, \alpha_2)$  when  $\alpha_1, \alpha_2$  and  $\theta_1$  are held fixed and  $\delta = \theta_2 - \theta_1$  tends to zero.

ASSUMPTION A. The functions  $t_n(\mathbf{x}_n)$  are independent of  $\delta$ .

Let

$$M_{1n}(x, \theta) = \inf_{\theta(G_n) \leq \theta, G_n \in \mathcal{C}_n} P(t_n \leq x | G_n),$$

$$M_{2n}(x, \theta) = \sup_{\theta(G_n) \geq \theta, G_n \in \mathcal{C}_n} P(t_n \leq x | G_n).$$

The functions  $M_{in}(x, \theta)$  are nonincreasing in  $\theta$ . Hence the conditions

$$P(\mathbf{X}_n \in w_{in} | G_n) \geq 1 - \alpha_i \quad \text{for all } G_n \in \mathcal{C}_{in}, \quad i = 1, 2,$$

are satisfied by test (4.2) if and only if

$$M_{1n}(c, \theta_1) \geq 1 - \alpha_1, \quad M_{2n}(c, \theta_1 + \delta) \leq \alpha_2.$$

ASSUMPTION B. For every  $n$ ,  $\lim_{x \rightarrow \infty} M_{1n}(x, \theta_1) > 1 - \alpha_1$ .

Let  $c_n$  be the least number which satisfies the inequalities

$$(4.3) \quad M_{1n}(c_{n+}, \theta_1) \geq 1 - \alpha_1 \geq M_{1n}(c_{n-}, \theta_1),$$

where  $c_{n+}$  and  $c_{n-}$  are, respectively, the right and left limits. This number exists by Assumption B. In Theorem 3.1, we have  $M_n = M_{2n}(c_{n+}, \theta_1 + \delta)$ . Hence if  $N = N(\mathfrak{J})$  is finite and greater than 1,

$$(4.4) \quad M_{2N}(c_{N+}, \theta_1 + \delta) \leq \alpha_2 \leq M_{2, N-1}(c_{N-1+}, \theta_1 + \delta).$$

ASSUMPTION C.  $\alpha_1 + \alpha_2 < 1$ .

ASSUMPTION D. One of the following conditions is satisfied.

(a) For every  $n$  and every  $x$ ,  $M_{2n}(x, \theta)$  is continuous on the right in  $\theta$  at  $\theta = \theta_1$ .

(b) For every  $n$  and every  $\theta' \in \omega$  there exists a cdf  $G_{\theta', n} \in \mathcal{C}_n$  such that  $\theta(G_{\theta', n}) = \theta'$ , and for every  $x$ ,  $p_n(x, \theta) = P(t_n \leq x | G_{\theta', n})$  is continuous on the right in  $\theta$  at  $\theta = \theta_1$ .

LEMMA 4.1. If Assumptions A, B, C, and D are satisfied,  $N(\mathfrak{J}) \rightarrow \infty$  as  $\delta \rightarrow 0$ .

PROOF. First suppose that part (a) of Assumption D is satisfied. Then for  $n$  fixed and  $\delta \rightarrow 0$ ,

$$M_{2n}(c_{n+}, \theta_1 + \delta) \rightarrow M_{2n}(c_{n+}, \theta_1).$$

Since  $M_{1n}(x, \theta) \leq M_{2n}(x, \theta)$ , we have, by (4.3) and Assumption C,

$$M_{2n}(c_{n+}, \theta_1) \geq M_{1n}(c_{n+}, \theta_1) \geq 1 - \alpha_1 > \alpha_2.$$

Hence  $N(\mathfrak{J})$  will be larger than any fixed  $n$  as  $\delta \rightarrow 0$ .

Now suppose that part (b) of Assumption D is satisfied. Then

$$M_{2n}(c_{n+}, \theta_1 + \delta) \geq p_n(c_n, \theta_1 + \delta), \quad p_n(c_n, \theta_1) \geq M_{1n}(c_{n+}, \theta_1) \geq 1 - \alpha_1 > \alpha_2.$$

Since  $p_n(c_n, \theta_1 + \delta) \rightarrow p_n(c_n, \theta_1)$  as  $\delta \rightarrow 0$ , we arrive at the same conclusion.

ASSUMPTION E. There exist a sequence  $\{h_n(x)\}$  of nondecreasing, continuous functions, a positive number  $r$ , and functions  $H_1(x)$  and  $H_2(x, d)$ , defined for every  $d > 0$ , such that

$$(4.5) \quad M_{1n}(h_n(x), \theta_1) \rightarrow H_1(x),$$

$$(4.6) \quad M_{2n}(h_n(x), \theta_1 + dn^{-r}) \rightarrow H_2(x, d)$$

as  $n \rightarrow \infty$ , for every  $x$  which is a point of continuity of  $H_1(x)$  and  $H_2(x, d)$ , respectively, and every  $d > 0$ .

Note that  $H_2(x, d)$  is necessarily nonincreasing in  $d$ , while  $H_1(x)$  and  $H_2(x, d)$  are nondecreasing in  $x$ .

ASSUMPTION F. (i) The equation  $H_1(x) = 1 - \alpha_1$  has a unique root  $x = a$ , at which  $H_1(x)$  is continuous.

(ii) The equation  $H_2(a, d) = \alpha_2$  has a unique positive root  $d = D(\alpha_1, \alpha_2)$ . The function  $H_2(x, d)$  is continuous at  $x = a$  for all  $d$  in a neighborhood of  $D(\alpha_1, \alpha_2)$ .

Assumptions E and F have the following meaning. The function  $M_{in}(x, \theta)$  is a bound of cdfs of  $t_n$ . The function  $M_{in}(h_n(x), \theta)$  is the corresponding bound of the cdfs of  $t'_n = f_n(t_n)$ , where  $f_n = h_n^{-1}$ , the inverse of  $h_n$ . The function  $f_n$  is strictly increasing. The tests with  $w_{1n}$  determined by  $t'_n \leq c$  are equivalent to the tests (4.2). Thus Assumptions E and F require, roughly speaking, that a suitable function of  $t_n$  have a limiting distribution which satisfies certain regularity conditions.

THEOREM 4.1. Let  $\mathfrak{J}$  be a family of tests of the form (4.1), or the subfamily with  $c = c'_n$  fixed in such a way that  $h_n^{-1}(c'_n) \rightarrow a$  as  $n \rightarrow \infty$ . If Assumptions A through F are satisfied, then asymptotically as  $\delta \rightarrow 0$ ,

$$(4.7) \quad N(\mathfrak{J}) \sim \{\delta^{-1}D(\alpha_1, \alpha_2)\}^{1/r}.$$

PROOF. We first assume that family  $\mathfrak{J}$  is defined by (4.2). By the remark following Assumption F we may assume that  $h_n(x) = x$ . Then relations (4.5) and (4.6) are replaced by

$$(4.8) \quad M_{1n}(x, \theta_1) \rightarrow H_1(x),$$

$$(4.9) \quad M_{2n}(x, \theta_1 + dn^{-r}) \rightarrow H_2(x, d).$$

From (4.3), (4.8) and Assumption F(i) it follows that  $c_n \rightarrow a$  as  $n \rightarrow \infty$ .

We now show that  $N = N(\mathfrak{J})$  is finite for every  $\delta > 0$ , that is,  $M_{2n}(c_{n+}, \theta_1 + \delta) \leq \alpha_2$  for some  $n$ . By Assumption F we can choose  $d > D =$

$D(\alpha_1, \alpha_2)$  so that  $H_2(x, d)$  is continuous at  $x = a$ . For every  $\delta > 0$  and  $n$  so large that  $\delta > dn^{-\tau}$  we have

$$M_{2n}(c_{n+}, \theta_1 + \delta) \leq M_{2n}(c_{n+}, \theta_1 + dn^{-\tau}).$$

By Assumptions E and F, the right side tends to  $H_2(a, d)$  as  $n \rightarrow \infty$ , and  $H_2(a, d) < \alpha_2$  by Assumption F. Hence  $N$  is finite.

We have to show that

$$(4.10) \quad \delta N^\tau \rightarrow D \quad \text{as} \quad \delta \rightarrow 0.$$

Suppose this is not true. Then there exists a sequence  $\{\delta_k\}$  of positive numbers such that  $\delta_k \rightarrow 0$  and  $\delta_k N_k^\tau \rightarrow D' \neq D$  as  $k \rightarrow \infty$ , where  $0 \leq D' \leq +\infty$  and  $N_k$  is the value of  $N$  for  $\delta = \delta_k$ . By (4.4) we have for every  $k$

$$(4.11) \quad M_{2N_k}(c_{N_k+}, \theta_1 + \delta_k) \leq \alpha_2 \leq M_{2, N_k-1}(c_{N_k-1+}, \theta_1 + \delta_k).$$

First assume  $D' < D$ . By Assumption F(ii) there exists a number  $D''$  such that  $D' < D'' < D$  and  $H_2(x, D'')$  is continuous at  $x = a$ . For  $k$  sufficiently large we have  $\delta_k N_k^\tau < D''$  and hence

$$(4.12) \quad M_{2N_k}(c_{N_k+}, \theta_1 + \delta_k) \geq M_{2N_k}(c_{N_k+}, \theta_1 + D'' N_k^{-\tau}).$$

As  $k \rightarrow \infty$ , we have (by Lemma 4.1)  $N_k \rightarrow \infty$  and  $c_{N_k} \rightarrow a$ . Hence

$$(4.13) \quad M_{2N_k}(c_{N_k+}, \theta_1 + D'' N_k^{-\tau}) \rightarrow H_2(a, D'') > \alpha_2.$$

But relations (4.12) and (4.13) contradict (4.11). Hence  $D' \geq D$ .

If we assume  $D' > D$ , a similar argument starting with the right member of (4.11) leads again to a contradiction. This completes the proof for family (4.2).

An inspection of the proof shows that (4.7) also holds for any family (4.1) and the subfamily with  $c = c'_n$  fixed, as stated in the theorem.

We observe that the argument will not be essentially affected if the factor  $n^{-\tau}$  in (4.6) is replaced by an arbitrary decreasing function  $k(n)$  which tends to zero as  $n \rightarrow \infty$ . Then  $N(\mathfrak{J}) \sim k^{-1}(\delta/D)$ .

Let  $\{t_{1n}\}$  and  $\{t_{2n}\}$  be two sequences of statistics, and denote by  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  the corresponding families of tests of the form (4.1). Suppose that Assumptions A through F are satisfied by both families. If  $r_i$  and  $D_i(\alpha_1, \alpha_2)$ , for  $i = 1, 2$ , denote the values of  $r$  and  $D(\alpha_1, \alpha_2)$  for the two families, an application of Theorem 4.1 gives immediately

**THEOREM 4.2.** *Let  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  be two families of tests of the form (4.1) which both satisfy Assumptions A through F. Then as  $\delta \rightarrow 0$ ,*

$$\text{eff}(\mathfrak{J}_1/\mathfrak{J}_2) = \frac{N(\mathfrak{J}_2)}{N(\mathfrak{J}_1)} \sim \frac{D_2(\alpha_1, \alpha_2)^{1/r_2}}{D_1(\alpha_1, \alpha_2)^{1/r_1}} \delta^{(r_2-r_1)/r_1 r_2}.$$

Thus if  $r_1 < r_2$ , the efficiency of family  $\mathfrak{J}_1$  relative to family  $\mathfrak{J}_2$  tends to zero. If  $r_1 = r_2 = r$ ,

$$\lim_{\delta \rightarrow 0} \text{eff}(\mathfrak{J}_1/\mathfrak{J}_2) = (D_2(\alpha_1, \alpha_2) / D_1(\alpha_1, \alpha_2))^{1/r}.$$

Consider now the particular case where the distribution of  $t_n$  depends on  $G_n$  only through  $\theta(G_n)$ . We shall write  $P(t_n \leq x | \theta)$  for  $P(t_n(\mathbf{X}_n) \leq x | G_n)$  when  $\theta(G_n) = \theta$ . In addition we shall assume that the power  $P(t_n > c | \theta)$  does not decrease as  $\theta$  increases; a "reasonable" test can be expected to have this property.

ASSUMPTION B'. The distribution of  $t_n(\mathbf{X}_n)$  depends on  $G_n$  only through  $\theta(G_n)$ , and  $P(t_n \leq x | \theta)$  is a nonincreasing function of  $\theta$ .

If Assumption B' is satisfied, we have  $M_{1n}(x, \theta) = M_{2n}(x, \theta) = P(t_n \leq x | \theta)$ . Assumption B is then satisfied for every  $\alpha_1 > 0$ . Assumption D is implied by Assumption D'. For every  $x$ ,  $P(t_n \leq x | \theta)$  is continuous on the right in  $\theta$  at  $\theta = \theta_1$ .

In Assumptions E and F we have  $H_1(x) = H_2(x, 0)$ . If we let  $H(x, d) = H_2(x, d)$ , then  $D(\alpha_1, \alpha_2)$  is the unique positive root of the equation  $H^{-1}(\alpha_2, D) = H^{-1}(1 - \alpha_1, 0)$ . Theorems 4.1 and 4.2 hold with Assumptions B and D replaced by B' and D'.

In many applications the statistic  $t_n$  or a function of  $t_n$  will be asymptotically normally distributed. Let

$$(4.14) \quad \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-y^2/2} dy.$$

ASSUMPTION E'. There exist a sequence  $\{g_n(t)\}$  of everywhere increasing functions, a positive number  $r$ , and two functions  $\mu(\theta)$  and  $\sigma(\theta)$  defined for  $\theta \in \omega$ , such that

$$(4.15) \quad P \left\{ n^r \frac{g_n(t_n) - \mu(\theta_1 + dn^{-r})}{\sigma(\theta_1 + dn^{-r})} \leq x \mid \theta_1 + dn^{-r} \right\} = \Phi(x)$$

as  $n \rightarrow \infty$ , for every  $x$  and every  $d \geq 0$ .

ASSUMPTION F'. The function  $\mu(\theta)$  has at  $\theta = \theta_1$  a right derivative  $\mu'(\theta_1) > 0$ . The function  $\sigma(\theta)$  is positive and continuous on the right at  $\theta = \theta_1$ .

By Assumption F' we can write

$$\mu(\theta_1 + dn^{-r}) = \mu(\theta_1) + dn^{-r} \mu'(\theta_1)(1 + \epsilon_n), \quad \sigma(\theta_1 + dn^{-r}) = \sigma(\theta_1)(1 + \epsilon'_n),$$

where  $\epsilon_n \rightarrow 0$  and  $\epsilon'_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence

$$n^r \frac{g_n(t_n) - \mu(\theta_1 + dn^{-r})}{\sigma(\theta_1 + dn^{-r})} = n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)(1 + \epsilon'_n)} - d \frac{\mu'(\theta_1)(1 + \epsilon_n)}{\sigma(\theta_1)(1 + \epsilon'_n)}$$

has the same limiting distribution as

$$n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)} - d \frac{\mu'(\theta_1)}{\sigma(\theta_1)}.$$

Thus if we replace  $x$  in (4.15) by  $x - d\mu'(\theta_1) / \sigma(\theta_1)$ , we obtain

$$P \left\{ n^r \frac{g_n(t_n) - \mu(\theta_1)}{\sigma(\theta_1)} \leq x \mid \theta_1 + dn^{-r} \right\} \rightarrow \Phi \left( x - d \frac{\mu'(\theta_1)}{\sigma(\theta_1)} \right).$$

Assumption E is now satisfied with  $H_1(x) = \Phi(x)$  and  $H_2(x, d) = \Phi(x - d\mu'(\theta_1) / \sigma(\theta_1))$ .



Let  $\lambda(u)$  be defined by

$$(4.16) \quad u = 1 - \Phi(\lambda(u)) = \Phi(-\lambda(u)).$$

Then Assumption F is satisfied with

$$D(\alpha_1, \alpha_2) = \frac{\sigma(\theta_1)}{\mu'(\theta_1)} \{\lambda(\alpha_1) + \lambda(\alpha_2)\}.$$

Hence we can state

**THEOREM 4.3.** *Let  $\mathfrak{J}$  be a family of tests of the form (4.1) or the subfamily with  $c = c'_n$  fixed in such a way that  $n^r \sigma(\theta_1)^{-1} [g_n(c'_n) - \mu(\theta_1)] \rightarrow \lambda(\alpha_1)$ . If Assumptions A, B', C, D', E', F' are satisfied, then asymptotically as  $\delta \rightarrow 0$ ,*

$$(4.17) \quad N(\mathfrak{J}) \sim \left( \frac{\sigma(\theta_1)}{\mu'(\theta_1)} \right)^{1/r} \left( \frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta} \right)^{1/r}.$$

If  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  are two families of tests which both satisfy the conditions of Theorem 4.3 with the same number  $r$ , we have

$$(4.18) \quad \lim_{\delta \rightarrow 0} \text{eff} (\mathfrak{J}_1/\mathfrak{J}_2) = \left( \frac{\mu'_1(\theta_1)}{\sigma_1(\theta_1)} \right)^{1/r} / \left( \frac{\mu'_2(\theta_1)}{\sigma_2(\theta_1)} \right)^{1/r},$$

where  $\mu'_i(\theta_1)$  and  $\sigma_i(\theta_1)$  are the values of  $\mu'(\theta_1)$  and  $\sigma(\theta_1)$  for family  $\mathfrak{J}_i$ .

Thus in this case the asymptotic relative efficiency is independent of  $\alpha_1$  and  $\alpha_2$ .

Relation (4.18) is essentially due to Pitman, who obtained an analogous result (for  $r = \frac{1}{2}$ ) under somewhat different assumptions. Pitman's result was extended by Noether [7].

Assumption A ( $t_n$  independent of  $\delta$ ) is somewhat restrictive. Thus if  $\mathcal{C}_n$  is a class of distributions depending only on the parameter  $\theta$ , the most powerful test for testing the hypothesis  $\theta = \theta_1$  against the alternative  $\theta = \theta_1 + \delta$  will in general depend on  $\delta$ . If Assumption A is dropped,  $M_{in}(x, \theta)$  and  $c_n$  will depend on  $\delta$ . Theorem 4.1 can be extended to this case by suitably modifying the assumptions. We shall state a corresponding theorem for the special case where the distribution of  $t_n$  depends only on  $\theta(G_n)$  (and on  $\delta$  through  $t_n$ ), and a function of  $t_n$  has a normal limiting distribution.

To emphasize the dependence on  $\delta$  we shall write  $t_{\delta n}$  for  $t_n$  and  $c_n(\delta)$  for  $c_n$ . Let Assumption B' be satisfied.

**ASSUMPTION D''.** For every  $n$ ,

$$(4.19) \quad P(t_{\delta n} \leq x \mid \theta_1 + \delta) - P(t_{\delta n} \leq x \mid \theta_1) \rightarrow 0$$

as  $\delta \rightarrow 0$ , uniformly in  $x$ .

**LEMMA 4.2.** *If Assumptions B', C, and D'' are satisfied,  $N(\mathfrak{J}) \rightarrow \infty$  as  $\delta \rightarrow 0$ .*

**PROOF.** After substituting  $c_n(\delta)$  for  $x$  in (4.19), the proof parallels that of Lemma 4.1.

**ASSUMPTION E''.** There exist functions  $g_n(t, \delta)$ , defined for  $\delta > 0$  and  $n = 1, 2, \dots$ , which are strictly increasing in  $t$ ; two positive numbers  $r$  and  $\delta_1$ ; and two functions  $\mu(\theta, \delta)$  and  $\sigma(\theta, \delta)$  defined for  $\theta \in \omega$  and  $\delta > 0$ , all such that

$$P \left\{ n^r \frac{g_n(t_{\delta n}, \delta) - \mu(\theta, \delta)}{\sigma(\theta, \delta)} \leq x \mid \theta \right\} \rightarrow \Phi(x)$$

as  $n \rightarrow \infty$  for every  $x$ , uniformly for  $\theta_1 \leq \theta < \theta_1 + \delta_1$  and  $0 < \delta < \delta_1$ .

ASSUMPTION F". The limit

$$\mu'(\theta_1, 0) = \lim_{0 < \delta \rightarrow 0} \frac{\mu(\theta_1 + \delta, \delta) - \mu(\theta_1, \delta)}{\delta}$$

exists and is positive. Also,  $\sigma(\theta, \delta)$  is positive and continuous on the right at  $(\theta_1, 0)$ , that is,  $\sigma(\theta, \delta) \rightarrow \sigma(\theta_1, 0) > 0$  as  $\theta \rightarrow \theta_1$  and  $\delta \rightarrow 0$ , with  $\theta \geq \theta_1$  and  $\delta > 0$ .

THEOREM 4.4. Let  $\mathfrak{J}$  be a family of tests of the form (4.1) or the subfamily with  $c = c_n(\delta)$ . If Assumptions B', C, D'', E'', F'' are satisfied, then asymptotically as  $\delta \rightarrow 0$ ,

$$N(\mathfrak{J}) \sim \left( \frac{\sigma(\theta_1, 0)}{\mu'(\theta_1, 0)} \right)^{1/r} \left( \frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta} \right)^{1/r}.$$

PROOF. Assumption E'' implies

$$n^r \frac{g_n(c_n(\delta), \delta) - \mu(\theta_1, \delta)}{\sigma(\theta_1, \delta)} \rightarrow \lambda(\alpha_1)$$

as  $n \rightarrow \infty$ , uniformly for  $0 < \delta < \delta_1$ . The rest of the proof is similar to the proofs of Theorems 4.1 and 4.3.

**5. Illustrations.** Three examples are offered.

EXAMPLE 1. A test for regression. Let  $G_n$  be the cdf of  $n$  independent, normally distributed random variables with common variance  $\sigma^2$  and means  $EX_j = j\xi$  for  $j = 1, \dots, n$ . Let  $\theta = \theta(G_n) = \xi/\sigma$ , with  $\theta_1 = 0$  and  $\theta_2 = \delta$ . Let

$$t_n(\mathbf{x}_n) = \frac{\sum jx_j}{\sqrt{\sum j^2 \sum x_j^2 - (\sum jx_j)^2}} = \frac{y_n}{\sqrt{\sum x_j^2 - y_n^2}},$$

where  $y_n = \sum jx_j / \sqrt{\sum j^2}$  and all summations are from 1 to  $n$ . Here  $y_n$  is normally distributed with mean  $\xi \sqrt{\sum j^2}$  and variance  $\sigma^2$ . Also,  $\sqrt{(\sum x_j^2 - y_n^2)/n}$  tends to  $\sigma$  in probability. Hence  $\sqrt{n}t_n$  is asymptotically normal  $(\theta \sqrt{\sum j^2}, 1)$ . Observing that  $\sum j^2 \sim n^3/3$ , it is easy to verify that the conditions of Theorem 4.3 are satisfied with

$$g_n(t) = \sqrt{3} t/n, \quad r = 3/2, \quad \mu(\theta) = \theta, \quad \sigma(\theta) = \sqrt{3}.$$

Thus

$$N \sim 3^{1/3} \left( \frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta} \right)^{2/3}.$$

EXAMPLE 2. A one-sided test of fit. Let  $\theta(F) = \sup_{-\infty < x < \infty} \{F_0(x) - F(x)\}$ , where  $F_0(x)$  is a fixed continuous cdf. Let  $F(x)$  belong to the class  $\mathcal{C}$  consisting of  $F_0$  and all continuous cdfs  $F$  with  $\theta(F) > 0$ . Let  $F_n(x)$  be the empirical cdf of a random sample of  $n$  observations from  $F$ , and let  $t_n = \sqrt{n}\theta(F_n)$ . For deciding

whether  $F = F_0$  or  $\theta(F) > 0$ , consider a test which accepts the former alternative when  $t_n < c$ . Smirnov has shown (see, e.g., Feller [2]) that as  $n \rightarrow \infty$ ,

$$P(t_n < x \mid F_0) \rightarrow 1 - e^{-2x^2}, \quad x > 0.$$

Birnbaum [1] obtained the best upper and lower bounds for the power of the test in the class of continuous cdfs with  $\theta(F)$  fixed. From Birnbaum's Theorem 1 we have for  $0 < x < \delta \sqrt{n}$ ,

$$\sup_{\theta(F)=\delta} P(t_n < x \mid F) = \sup_{\delta' \leq v \leq 1} Q_n(n^{-1/2}x, \delta, v),$$

where

$$Q_n(\epsilon, \delta, v) = \sum_{i=j+1}^n \binom{n}{i} (v - \delta)^i (1 - v + \delta)^{n-i}, \quad j = [n(v - \epsilon)].$$

For  $x \geq \delta \sqrt{n}$ ,  $\sup_{\theta(F)=\delta} P(t_n < x \mid F) = 1$ .

The function  $Q_n(\epsilon, \delta, v)$  is decreasing in  $\delta$ . Hence if  $\delta < \delta'$ ,

$$\sup_{\delta' \leq v \leq 1} Q_n(\epsilon, \delta, v) \geq \sup_{\delta' \leq v \leq 1} Q_n(\epsilon, \delta, v) \geq \sup_{\delta' \leq v \leq 1} Q_n(\epsilon, \delta', v).$$

Thus for  $0 < x < \delta \sqrt{n}$ ,

$$M_{2n}(x, \delta) = \sup_{\theta(F) \geq \delta} P(t_n < x \mid F) = \sup_{\delta' \leq v \leq 1} Q_n(n^{-1/2}x, \delta, v).$$

For  $x \leq 0$ ,  $M_{2n}(x, \delta) = 0$ . For  $x \geq \delta \sqrt{n}$ ,  $M_{2n}(x, \delta) = 1$ .

If  $v$  is fixed,  $0 < v < 1$ , we have for  $n \rightarrow \infty$ ,

$$Q_n(n^{-1/2}x, n^{-1/2}d, v) \rightarrow \Phi\left(\frac{x - d}{\sqrt{v(1 - v)}}\right), \quad 0 < x < d.$$

Using this result it can be shown that as  $n \rightarrow \infty$ ,

$$M_{2n}(x, n^{-1/2}d) \rightarrow \Phi(2x - 2d), \quad 0 < x < d.$$

It can now be verified that the conditions of Theorem 4.1 are satisfied with  $h_n(x) = x$  and

$$H_1(x) = \begin{cases} 0, & x \leq 0, \\ 1 - e^{-2x^2}, & x > 0; \end{cases} \quad H_2(x, d) = \begin{cases} 0, & x \leq 0, \\ \Phi(2x - 2d), & 0 < x < d, \\ 1, & x \geq d. \end{cases}$$

That Assumption D(b) is satisfied can be seen from Birnbaum's Theorem 1 and the fact that  $Q_n(\epsilon, \theta, v)$  is continuous in  $\theta$ . Thus if  $\alpha_1 + \alpha_2 < 1$  and  $\alpha_2 < \frac{1}{2}$ ,

$$N(\beta) \sim \frac{1}{\delta^2} \left\{ \sqrt{\frac{1}{2} \log \frac{1}{\alpha_1} + \frac{1}{2} \lambda(\alpha_2)} \right\}^2.$$

**EXAMPLE 3.** *The double-exponential distribution and the sign test.* Let  $\mathbf{x}_n$  have the probability density  $2^{-n} \exp(-\sum_{j=1}^n |x_j - \theta|)$ . The most powerful test for testing  $\theta = 0$  against  $\theta = \delta > 0$  accepts  $\theta = 0$  for small values of  $t_{\delta n} = \sum_{j=1}^n a(x_j, \delta)$ , where  $a(x, \delta) = (|x| - |x - \delta|) / \delta$ . Since  $a(x, \delta)$  is a non-

decreasing function of  $x$ , it is easily seen that  $P(t_{3n} \leq c | \theta)$  is a nonincreasing function of  $\theta$ .

As  $n \rightarrow \infty$ ,  $t_{3n}$  is asymptotically normal  $(n\mu(\theta, \delta), n\sigma^2(\theta, \delta))$ , where

$$\mu(\theta, \delta) = \begin{cases} -1 + (e^\theta - e^{\theta-\delta})/\delta, & \theta \leq 0, \\ (2\theta - \delta)/\delta + (e^{-\theta} - e^{\theta-\delta})/\delta, & 0 \leq \theta \leq \delta, \\ 1 + (e^{-\theta} - e^{-\theta+\delta})/\delta, & \delta \leq \theta; \end{cases}$$

$$\sigma^2(\theta, \delta) = \mu_2(\theta, \delta) - \mu(\theta, \delta)^2;$$

$$\mu_2(\theta, \delta) = \begin{cases} 1 + \{(4 - 2\delta)e^\theta - (4 + 2\delta)e^{\theta-\delta}\}/\delta^2, & \theta \leq 0, \\ \{8 + 4(\theta - \frac{1}{2}\delta)^2\}/\delta^2 - (4 + 2\delta)(e^{-\theta} + e^{\theta-\delta})/\delta^2, & 0 \leq \theta \leq \delta, \\ 1 + \{(4 - 2\delta)e^{-\theta+\delta} - (4 + 2\delta)e^{-\theta}\}/\delta^2, & \delta \leq \theta. \end{cases}$$

It can be verified that the conditions of Theorem 4.4 are satisfied with  $g_n(t, \delta) = t/n$ ,  $r = \frac{1}{2}$ ,  $\mu'(0, 0) = 1$ , and  $\sigma(0, 0) = 1$ . That the uniformity condition in Assumption E'' is satisfied can be seen from the fact that  $E\{|a(X, \delta) - \mu(\theta, \delta)|^3 | \theta\} / \sigma(\theta, \delta)^3$  is bounded (since  $a(x, \delta)$  is bounded); this implies uniform convergence by Liapunov's or Berry's bounds for the remainder term in the central limit theorem.

Thus if  $\mathfrak{J}_\delta$  denotes the family of the most powerful tests based on  $t_{3n}$ , we have  $N(\mathfrak{J}_\delta) \sim \delta^{-2}[\lambda(\alpha_1) + \lambda(\alpha_2)]^2$ .

As  $\delta \rightarrow 0$ ,  $t_{3n}$  tends to  $t_{0n} = 2m - n$ , where  $m$  is the number of positive values  $x_j$ , for  $j = 1, \dots, n$ . The family  $\mathfrak{J}_0$  of the tests based on  $t_{0n}$  (sign tests) has asymptotically the same efficiency index (up to order  $\delta^{-2}$ ), so that the sign test has asymptotic efficiency 1 for the present problem. This is not surprising in the light of a recent result of Ruist [8], who showed that the sign test is most powerful for discriminating between two symmetrical continuous distributions which can be regarded as approximations of double-exponential distributions.

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