THE RATIO OF VARIANCES IN A VARIANCE COMPONENTS MODEL[1]

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Summary. Our discussion will concern primarily λ, the ratio of two variances which arise in discussing the "mixed" incomplete block model. In Section 1 we find first the class of invariant statistics for a test involving this ratio and second the joint distribution of these statistics. In Section 2 we use these statistics to construct a test (with certain optimum properties) of the hypothesis $\lambda < \lambda_0$ versus $\lambda > \lambda_1$.

1. The general incomplete block variance components model. Suppose that y_{ij} for $i = 1, \cdots, u$, and $j = 1, \cdots, b$ are independent and normal for given $t_1, \cdots, t_u$ with means $E(y_{ij} \mid t) = n_{ij}(t_i + b_j)$ and variance σ^2. Here n_{ij} is 1 or 0 according as the ith treatment does or does not occur in the jth block. The total number of observations is N, that is, $\sum_{i,j} n_{ij} = N$. In addition suppose that the t's are independent and identically normal with mean 0 and variance ϵ^2. If t were an unknown parameter instead of a random variable, we would have the general incomplete block model which appears in analysis of variance (see, for example, Bose [1]).

In the general theory of incomplete block designs we make use of the block totals $B_1, \cdots, B_b$ and of the "adjusted yields" $Q_1, \cdots, Q_u$. It is known from this theory that the latter form the basis of a vector space V_B of dimensionality b, while the former generate a vector space V_Q of dimensionality r, say. (In the case of a connected design, $r = u - 1$.) Further V_B and V_Q are orthogonal to each other and to the error space V' of dimensionality $N - b - r$. We may now choose an orthogonal basis for V', say $Y_1, \cdots, Y_{N-b-r}$.

Again from incomplete block design theory we know that

$$\begin{aligned} E(B_j \mid t) &= k_j b_j + n_{1j}t_1 + n_{2j}t_2 + \cdots + n_{uj}t_u, \\ E(Q_i \mid t) &= c_{i1}t_1 + c_{i2}t_2 + \cdots + c_{iu}t_u, \\ E(Y_i \mid t) &= 0, \end{aligned}$$

and also that the covariance matrix of the B's and Q's and Y's for fixed t is

$$\left[\begin{array}{ccc|c|c} k_1 & & 0 & & \\ & \ddots & & 0 & 0 \\ 0 & & k_b & & \\ \hline & 0 & & C & 0 \\ \hline & 0 & & 0 & I \end{array}\right] \sigma^2,$$

Received January 17, 1955.

[1] Extracted from a special report to the United States Air Force under Contract No. AF 18(600)-83 monitored by the Office of Scientific Research.

where $k_j = \sum_{i=1}^{u} n_{ij}$, and the C matrix, involved in both the expectations and variances of Q, is again from incomplete block design theory.

Now we state several lemmas which were independently developed by Madow [5] and Skibinsky [6] and which we will find useful.

LEMMA 1. $E[E(X \mid Z)] = E(X)$.

LEMMA 2. $\operatorname{Var}(X) = E[\operatorname{Var}(X \mid Z)] + \operatorname{Var}[E(X \mid Z)]$.

LEMMA 3. $\operatorname{Cov}(X, Y) = E[\operatorname{Cov}(X, Y \mid Z)] + \operatorname{Cov}[E(X \mid Z), E(Y \mid Z)]$.

Applying these lemmas, we find the unconditional means and the covariance matrix to be

$$E(B_j) = k_j b_j, \qquad E(Q_i) = E(Y_i) = 0,$$

$\operatorname{Cov}(B, Q, Y) =$

$$\left[\begin{array}{c|c|c} \begin{matrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_b \end{matrix} & 0 & 0 \\ \hline 0 & C & 0 \\ \hline 0 & 0 & I \end{array}\right] \sigma^2 + \left[\begin{array}{c|c|c} * & * & 0 \\ \hline * & C^2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \epsilon^2,$$

where the partitions with the stars in them are known though perhaps complicated constants.

Now in order to simplify the problem as far as possible we will make the transformation $Q_1, \cdots, Q_u \to Z_1, \cdots, Z_u$: $Q = MZ$ where M is an orthogonal matrix such that

$$M'CM = \left[\begin{array}{c|c} \begin{matrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} D_e & 0 \\ \hline 0 & 0 \end{array}\right], \quad .$$

a diagonal matrix with the characteristic roots of C in the diagonal. Note also that

$$M'C^2M = M'C(MM')CM = (M'CM)\,(M'CM)$$

$$= \left[\begin{array}{c|c} D_{e^2} & 0 \\ \hline 0 & 0 \end{array}\right].$$

Thus $Z_1, \cdots, Z_r$ have the covariance matrix $D_e\sigma^2 + D_{e^2}\epsilon^2$ while $Z_{r+1}, \cdots, Z_u$ are zero with probability one.

Because of the orthogonality of the Z's among themselves and the mutual orthogonality of V_B, V_Q, and V', the Y's, B's, and $Z_1, \cdots, Z_r$ are N linearly independent linear functions in the space of the y's and thus form a basis for the y space. We may therefore make a transformation of the y's into the Y's, B's, and $Z_1, \cdots, Z_r$. These last variables are of course multivariate normal. From this and the nature of the covariance matrix of these final variates we see that $B_1, \cdots, B_b, Z_1, \cdots, Z_r$ and $\sum Y_i^2$ are a set of sufficient statistics for the distribution.

Suppose now that we are interested in placing confidence limits on, or testing hypothesis concerning, the ratio $\epsilon^2/\sigma^2 = \lambda$, say. Let us consider a group G of transformations on our set of sufficient statistics. Let G be

$$B_j' = cB_j + k_jc_j, \qquad Z_1' = cZ_1, \cdots, Z_r' = cZ_r, \qquad (\sum Y_i^2)' = c^2(\sum Y_i^2).$$

Since the effect of G is only to change the mean of B_j and multiply the covariance matrix of (B, Z, Y) by an arbitrary constant, c^2, the problem is invariant under G. In this connection, see Lehmann [4]. A maximal invariant under G is

$$Z_1 / \sqrt{\sum Y_i^2}, \quad \cdots, \quad Z_r / \sqrt{\sum Y_i^2}.$$

Thus G induces the group of transformations $\bar{G}$,

$$b_j' = c(b_j + c_j), \qquad \sigma^{2\prime} = c^2\sigma^2, \qquad \epsilon^{2\prime} = c^2\epsilon^2,$$

a maximal invariant for which is $\epsilon^2/\sigma^2 = \lambda$. Thus if we adhere to the principle of invariance, then in making inferences about λ, we may restrict ourselves to functions of

$$Z_1 / \sqrt{\sum Y_i^2}, \quad \cdots, \quad Z_r / \sqrt{\sum Y_i^2}.$$

We now find the joint distribution of the statistics

$$X_1 = Z_1 / \sqrt{e_1X_{r+1}}, \quad \cdots, \quad X_r = Z_r / \sqrt{e_rX_{r+1}},$$

where $X_{r+1} = \sum_1^n Y_i^2$ and $n = N - b - r$.

Let $W_i = Z_i / \sqrt{e_i}$; then W_i is $N(0, \sigma^2 + e_i\epsilon^2)$ and since the W's and X_{r+1} are independent, their joint frequency function is

$$\text{const}(x_{r+1})^{n/2-1} \exp\left[-\frac{1}{2}\left(\frac{w_i^2}{\sigma^2 + e_i\epsilon^2} + \frac{x_{r+1}}{\sigma^2}\right)\right].$$

Making the transformation

$$X_1 = W_1 / \sqrt{X_{r+1}}, \quad \cdots, \quad X_r = W_r / \sqrt{X_{r+1}},$$

we find that the probability element of $X_1, X_2, \cdots, X_r, X_{r+1}$ is

$$\text{const}(x_{r+1})^{(n+r)/2-1} \exp\left[-\frac{x_{r+1}}{2\sigma^2}\left(1 + \sum_{i=1}^{r} \frac{x_i^2}{1 + e_i\lambda}\right)\right].$$

We may now integrate out over X_{r+1}, noting that we have a gamma function in this variable. We find the probability element of $X_1, \cdots, X_r$ to be

$$\text{const}\left(1 + \sum_{i=1}^{r} \frac{x_i^2}{1 + e_i\lambda}\right)^{-(n+r)/2}.$$

2. A one-sided test on λ. Let ω_0: $\lambda \leqq \lambda_0$ and ω_1: $\lambda \geqq \lambda_1$ where $\lambda_0 < \lambda_1$. We now interest ourselves in tests of H_0 : $\lambda \,\varepsilon\, \omega_0$ versus H_1 : $\lambda \,\varepsilon\, \omega_1$. The region between λ_0 and λ_1 is a zone of indifference to be determined by the experimental situation. That is, if $\lambda_0 < \lambda < \lambda_1$, then we do not particularly care whether we accept H_0 or H_1.

Now consider an a priori distribution defined on ω_0 which assigns probability 1 to $\lambda = \lambda_0$, and similarly a distribution on ω_1 which assigns probability 1 to $\lambda = \lambda_1$.

According to a theorem of Lehmann [3], if now we construct a most powerful size α test of λ_0 versus λ_1 which has power β and if we can show that this test has size α for the composite hypothesis and power $\geqq \beta$ for all λ in ω_1, then this test is the one which maximizes the minimum power.

We may use the Neyman-Pearson Lemma to construct a most powerful test of $\lambda = \lambda_0$ versus $\lambda = \lambda_1$. If we let

$$R = \frac{1 + \sum X_i^2 \,/\, (1 + e_i\lambda_0)}{1 + \sum X_i^2 \,/\, (1 + e_i\lambda_1)} = \frac{\sum Y_i^2 + \sum Z_i^2 \,/\, (e_i + e_i^2\lambda_0)}{\sum Y_i^2 + \sum Z_i^2 \,/\, (e_i + e_i^2\lambda_1)},$$

then the above test becomes

$$\text{if } R > c, \text{ accept hypothesis } \lambda = \lambda_1,$$
$$\text{if } R < c, \text{ accept hypothesis } \lambda = \lambda_0.$$

Here c is a constant chosen so that the test has significance level α.

The power function of this test is

$$B(\lambda) = \text{const} \int_{R>c} \exp\left[-\frac{1}{2}\left(\frac{\sum y_i^2}{\sigma^2} + \sum \frac{z_i^2}{e_i\sigma^2 + e_i^2\epsilon^2}\right)\right] dy\, dz$$
$$= \text{const} \int_{R'(\lambda)>c} \exp\left[-\frac{1}{2}\left(\sum f_i^2 + \sum g_i^2\right)\right] df\, dg,$$

where we have made the transformation

$$F_i = Y_i/\sigma, \qquad i = 1, \cdots, n;$$
$$G_i = Z_i/(e_i\sigma^2 + e_i^2\,\epsilon^2)^{1/2}, \qquad i = 1, \cdots, r;$$
$$R'(\lambda) = \frac{\sum F_i^2 + \sum G_i^2(1 + e_i\lambda) \,/\, (1 + e_i\lambda_0)}{\sum F_i^2 + \sum G_i^2(1 + e_i\lambda) \,/\, (1 + e_i\lambda_1)}.$$

We may compute, by straightforward though lengthy algebra, that

$$\frac{\partial R'}{\partial \lambda} = \frac{(\sum F_i^2)(\sum e_i G_i^2 \,/\, (1 + e_i\lambda_0) - \sum e_i G_i^2 \,/\, (1 + e_i\lambda_1))}{(\sum F_i^2 + \sum G_i^2(1 + e_i\lambda) \,/\, (1 + e_i\lambda_1))^2}$$
$$+ \frac{\sum_{j>i} G_i^2 G_j^2((e_j - e_i)^2(\lambda_1 - \lambda_0) \,/\, (1 + e_i\lambda_1)\,(1 + e_j\lambda_0)(1 + e_j\lambda_1)(1 + e_i\lambda_0)}{(\sum F_i^2 + \sum G_i^2(1 + e_i\lambda) \,/\, (1 + e_i\lambda_1))^2}$$

which is greater than 0 except when

$$F_1 = F_2 = \cdots = F_n = G_1 = G_2 = \cdots = G_r = 0.$$

Thus R' is an increasing function of λ. Also if $R_1' \leqq R_2'$, then $c < R_1'$ implies that $c < R_2'$, so that $\int_{R_1'>c} \leqq \int_{R_2'>c}$. Therefore β is an increasing function of R' and thus of λ. Thus

$$\beta(\lambda) \leqq \beta(\lambda_0) = \alpha \text{ for all } \lambda \; \varepsilon \; \omega_0, \qquad \beta(\lambda_1) \leqq \beta(\lambda) \text{ for all } \lambda \; \varepsilon \; \omega_1.$$

Accordingly we have proved the

THEOREM. *The test, accept or reject H_0 according as $R < c$ or $R > c$, is the one which maximizes the minimum power among all invariant tests.*

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