A CHARACTERIZATION OF THE GAMMA DISTRIBUTION

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1. Introduction. The sum and the difference of two independently and identically distributed normal variates are uncorrelated and therefore also independent. Conversely, we can conclude from the independence of the sum and the difference of two identically and independently distributed random variables that both these variables are normally distributed. In this manner one obtains a characterization of the normal distribution.

We denote in the following by

$$F(x; \alpha, \lambda) = \begin{cases} 0 & x < 0, \\ \dfrac{\alpha^\lambda}{\Gamma(\lambda)} \displaystyle\int_0^x t^{\lambda-1} e^{-\alpha t}\, dt & x > 0, \end{cases} \tag{1.1}$$

the distribution function of the gamma distribution. The corresponding characteristic function is

$$f(t; \alpha, \lambda) = (1 - it/\alpha)^{-\lambda}. \tag{1.2}$$

Here α and $\lambda > 0$ are two parameters. It is seen easily that α is a scale parameter.

Let X and Y be two identically and independently distributed random variables each having the distribution (1.1). It is known [2] that in this case $U = X + Y$ and $V = X/Y$ are also two independent random variables.

In the present paper, we use this fact to derive a characterization of the gamma distribution which is similar to the characterization of the normal distribution mentioned above. Our result can be formulated in the following manner.

Theorem. *Let X and Y be two nondegenerate and positive random variables, and suppose that they are independently distributed. The random variables $U = X + Y$ and $V = X/Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter.*

In the following it will be convenient to introduce the random variable

$$W = 1/(1 + V) = Y/(X + Y). \tag{1.3}$$

The random variables U and W are both nonnegative. Moreover, W is a bounded random variable and

$$0 \leqq W \leqq 1. \tag{1.4}$$

2. Analytic properties of the characteristic functions of X, Y, U, and W. In this section we consider the characteristic functions of these random variables and investigate in particular whether the integrals defining these functions exist

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also for some complex values of the argument. In addition, we discuss some properties of these functions which we need in the subsequent sections.

We denote in the following by $F(x)$, $G(y)$, $H_1(u)$, and $H_2(w)$ the distribution functions of the random variables X, Y, U, and W respectively. Since X and Y are nonnegative, the characteristic functions of these random variables are

$$f(v) = \int_0^\infty e^{ivx}\, dF(x), \qquad g(v) = \int_0^\infty e^{ivy}\, dG(y). \tag{2.1}$$

Due to the nonnegativity of X and Y, the range of integration extends only from 0 to ∞. Therefore these integrals exist not only for real v but also for complex values $v = s + it$, where s and t are real, for which $t = \operatorname{Im}(v) \geqq 0$. It follows then from well-known properties of the Laplace integral that the functions (2.1) are analytic for $t = \operatorname{Im}(v) > 0$ and that

$$\begin{cases} f'(v) = i \int_0^\infty x e^{ivx}\, dF(x), & f''(v) = -\int_0^\infty x^2 e^{ivx}\, dF(x), \\ g'(v) = i \int_0^\infty y e^{ivy}\, dG(y), & g''(v) = -\int_0^\infty y^2 e^{ivy}\, dG(y). \end{cases} \tag{2.2}$$

We need also the following continuity property of the integrals (2.1),

$$\lim_{t \downarrow 0} f(s + it) = f(s); \qquad \lim_{t \downarrow 0} g(s + it) = g(s) \tag{2.3}$$

where $t \downarrow 0$ means that t approaches zero from above.

We consider next $\mathbf{E}\{e^{izW}\}$, the characteristic function of the random variable W. It is seen from (1.4) that

$$\mathbf{E}\{e^{izW}\} = \int_{-\infty}^{+\infty} e^{izw}\, dH_2(w) = \int_0^1 e^{izw}\, dH_2(w).$$

This integral exists for all values of z and is an entire function of z. The random variable U is nonnegative. Its characteristic function is

$$\mathbf{E}\{e^{ivU}\} = \int_{-\infty}^{\infty} e^{ivu}\, dH_1(u) = \int_0^\infty e^{ivu}\, dH_1(u).$$

Therefore this integral exists for all values of v for which $t = \operatorname{Im}(v) \geqq 0$ and is analytic for all v for which $t > 0$. Similarily the expectation $\mathbf{E}\{e^{ivU+izW}\}$ exists for all z and all v such that $t = \operatorname{Im}(v) \geqq 0$ and is analytic for all z and all v with $t = \operatorname{Im}(v) > 0$.

We conclude this section with another remark concerning W. The random variable W is bounded; therefore all its moments exist. We denote in the following by

$$\theta_1 = \mathbf{E}\left(\frac{Y}{X+Y}\right), \qquad \theta_2 = \mathbf{E}\left[\left(\frac{Y}{X+Y}\right)^2\right]. \tag{2.4}$$

It is then easy to see that

$$0 < \theta_1^2 \leqq \theta_2 \leqq \theta_1 < 1. \tag{2.5}$$

3. The independence of U and V. The random variables U and V are assumed to be independent. It follows then from (1.3) that this is equivalent to the independence of U and W. We denote by $H(u, w)$ the joint distribution of the random variables U and W and have then

$$H(u, w) = H_1(u)\, H_2(w). \tag{3.1}$$

We conclude from (3.1) that $\mathbf{E}[\exp(ivU + izW)] = \mathbf{E}[\exp(ivU)]\, \mathbf{E}[\exp(izW)]$, or

$$\begin{aligned} &\mathbf{E}\left\{\exp\left[iv(X+Y) + \frac{izY}{X+Y}\right]\right\} \\ &\qquad = \mathbf{E}\left\{\exp\left[iv(X+Y)\right]\right\}\, \mathbf{E}\left\{\exp\left[\frac{izY}{X+Y}\right]\right\}. \end{aligned} \tag{3.2}$$

We have shown in the preceding section that the expectations in (3.2) are analytic for all z and for all v such that $t = \operatorname{Im}(v) > 0$. We now write (3.2) in a more explicit form and obtain

$$\begin{aligned} &\int_0^\infty\int_0^\infty \exp\left[iv(x+y) + \frac{izy}{x+y}\right] dF(x)\, dG(y) \\ &\qquad = \int_0^\infty\int_0^\infty \exp\left[iv(x+y)\right] dF(x)\, dG(y) \int_0^\infty\int_0^\infty \exp\left[\frac{izy}{x+y}\right] dF(x)\, dG(y). \end{aligned} \tag{3.3}$$

The integrals in (3.3) converge for all complex z and **all** $v = s + it$ such that $t \geqq 0$. Therefore they are analytic in z and also in v, provided that $t = \operatorname{Im}(v) > 0$.

Equation (3.3) is the starting point for our investigation. We shall derive from it two relations for the unknown characteristic functions.

4. The relation between the characteristic functions. We assume for the time being that $t > 0$. Therefore we can differentiate (3.3), first with respect to v and then with respect to z. In this manner we obtain

$$\begin{aligned} &\int_0^\infty\int_0^\infty y \exp\left[iv(x+y) + \frac{izy}{x+y}\right] dF(x)\, dG(y) \\ &\qquad = \int_0^\infty\int_0^\infty (x+y) \exp\left[iv(x+y)\right] dF(x)\, dG(y) \\ &\qquad\qquad \cdot \int_0^\infty\int_0^\infty \frac{y}{x+y} \exp\left[\frac{izy}{x+y}\right] dF(x)\, dG(y). \end{aligned} \tag{4.1}$$

If we put here $z = 0$ and use the notation (2.4), we see that, if $t = \operatorname{Im}(v) > 0$,

$$\begin{aligned} &\int_0^\infty\int_0^\infty y \exp\left[iv(x+y)\right] dF(x)\, dG(y) \\ &\qquad = \theta_1 \int_0^\infty\int_0^\infty (x+y) \exp\left[iv(x+y)\right] dF(x)\, dG(y). \end{aligned} \tag{4.2}$$

We use (2.1) to express the integrals in (4.2) in terms of the functions $f(v)$ and $g(v)$ and obtain

$$(4.3)\qquad (1 - \theta_1)\, g'(v)\, f(v) = \theta_1 f'(v)\, g(v), \qquad t = \mathrm{Im}(v) > 0.$$

From (2.3) and from

$$(4.4)\qquad f(0) = g(0) = 1$$

we conclude that there exists a neighborhood of the origin such that

$$(4.5)\qquad f(v) \neq 0, \qquad g(v) \neq 0,$$

for all values of v belonging to this neighborhood for which $t = \mathrm{Im}(v) \geqq 0$. (This neighborhood could, of course, be the half-plane $t \geqq 0$.) Moreover, it follows from (2.1) that $f(it)$ and $g(it)$ are positive for real $t > 0$. From the continuity of $f(v)$ and $g(v)$ we see that every point of the segment $0 < \mathrm{Im}(v) \leqq 1$ has a neighborhood in which (4.5) holds. We conclude then from the Heine-Borel covering theorem that there exists a simply connected domain $\mathfrak{D}$ containing the interval $0 < \mathrm{Im}(v) \leqq 1$ in which $f(v)$ and $g(v)$ do not vanish. In the following we restrict ourselves to this domain. We may then divide (4.3) by $f(v)g(v)$ and obtain

$$(4.6)\qquad (1 - \theta_1)(g'(v)/g(v)) = \theta_1(f'(v)/f(v)).$$

In the domain $\mathfrak{D}$ we may introduce the logarithms of $f(v)$ and $g(v)$ and integrate (4.6) using the initial conditions (4.4). We obtain finally

$$(4.7)\qquad [g(v)]^{1-\theta_1} = [f(v)]^{\theta_1}.$$

This relation is certainly valid for values of v for which the relations (4.5) hold together with

$$(4.8)\qquad t = \mathrm{Im}\, v > 0.$$

5. The differential equation. We derived relation (4.7) which connects the two unknown functions $f(v)$ and $g(v)$. To determine these function we must find a second relation. This may be accomplished by repeating the procedure which led from (4.1) to (4.6). We still restrict our considerations to values of v from $\mathfrak{D}$ for which (4.5) and (4.8) hold. We may therefore differentiate (4.1) twice, first with respect to v then with respect to z. We put finally $z = 0$ and see, using again the notations of (2.4), that

$$\int_0^\infty \int_0^\infty y^2 \exp[i(x + y)v]\, dF(x)\, dG(y)$$
$$= \theta_2 \int_0^\infty \int_0^\infty (x^2 + 2xy + y^2) \exp[iv(x + y)]\, dF(x)\, dG(y).$$

If we substitute for these integrals the expressions (2.1) and (2.2), we obtain a second relation between the functions $f(v)$ and $g(v)$,

$$g''(v)\, f(v) = \theta_2[f''(v)\, g(v) + 2f'(v)\, g'(v) + g''(v)\, f(v)].$$

On account of (4.5) we may write this in the form

$$\frac{g''(v)}{g(v)} = \theta_2 \left[\frac{f''(v)}{f(v)} + 2\frac{f'(v)}{f(v)}\frac{g'(v)}{g(v)} + \frac{g''(v)}{g(v)}\right]. \tag{5.1}$$

It is now convenient to introduce the logarithms

$$\phi(v) = \log f(v), \qquad \psi(v) = \log g(v). \tag{5.2}$$

Then

$$\begin{cases} \dfrac{f'(v)}{f(v)} = \phi'(v), & \dfrac{g'(v)}{g(v)} = \psi'(v) \\ \dfrac{f''(v)}{f(v)} = \phi''(v) + [\phi'(v)]^2, & \dfrac{g''(v)}{g(v)} = \psi''(v) + [\psi'(v)]^2. \end{cases} \tag{5.3}$$

From (4.6) and (5.2) we see that

$$(1 - \theta_1)\psi'(v) = \theta_1 \phi'(v), \qquad (1 - \theta_1)\psi''(v) = \theta_1 \phi''(v). \tag{5.4}$$

After some elementary computations we obtain from (5.1), (5.3), and (5.4) the differential equation

$$(1 - \theta_1)(\theta_1 - \theta_2)\phi''(v) = (\theta_2 - \theta_1^2)[\phi'(v)]^2. \tag{5.5}$$

6. Solution of the differential equation. We first leave aside the cases where $\theta_1 = \theta_2$ or $\theta_2 = \theta_1^2$ and consider the case where $0 < \theta_1^2 < \theta_2 < \theta_1 < 1$. Then

$$\rho = \frac{(1 - \theta_1)(\theta_1 - \theta_2)}{\theta_2 - \theta_1^2} > 0 \tag{6.1}$$

and we may write (5.5) as

$$\phi''(v)/[\phi'(v)]^2 = 1/\rho \tag{6.2}$$

for values of v satisfying (4.5) and (4.8). We denote by

$$k_1 = \mathbf{E}(e^{-X}), \qquad k_2 = \mathbf{E}(Xe^{-X}), \qquad \alpha = (k_1\rho - k_2)/k_2 \tag{6.3}$$

and integrate (6.2) with the initial condition $\phi'(i) = ik_2/k_1$. We obtain easily

$$\frac{1}{\phi'(v)} = -\frac{v}{\rho} + \frac{k_1\rho - k_2}{i\rho k_2} = -\frac{v}{\rho} + \frac{\alpha}{i\rho}$$

so that $\phi'(v) = (i\rho/\alpha)/(1 - iv/\alpha)$. We integrate this again and obtain, considering (5.2),

$$f(v) = c_2(1 - iv/\alpha)^{-\rho} \tag{6.4}$$

where c_2 is a constant of integration. It follows from (2.3) that $\lim_{t\downarrow 0} f(it) = f(0) = 1$, therefore $c_2 = 1$. It is then seen from (6.4) and (4.7) that

$$f(v) = (1 - iv/\alpha)^{-\rho}, \qquad g(v) = (1 - iv/\alpha)^{-\theta_1\rho/(1-\theta_1)}. \tag{6.5}$$

The equations (6.5) were derived for all points v of the domain $\mathfrak{D}$. This restriction on v can now be removed. The functions $f(v) = \int_0^\infty e^{itv}\, dF(x)$ and $(1 - iv/\alpha)^{-\rho}$

are both analytic in the upper half-plane $t = \text{Im}(v) > 0$, and agree in the domain $\mathfrak{D}$. Hence they agree in the half-plane $\text{Im}(v) > 0$. The same argument applies to $g(v)$. Finally it follows from (2.3) that (6.5) holds also for real values of v. The characteristic functions of X and Y are therefore given by (6.5) so that X and Y have, indeed, gamma distributions with the same scale parameter. We still have to consider the cases $\theta_1 = \theta_2$ and $\theta_1^2 = \theta_2$. If $\theta_1 = \theta_2$ then we see from (2.5) that $\theta_2 > \theta_1^2$, therefore (5.5) reduces to $\phi'(v) = 0$. If on the other hand $\theta_1^2 = \theta_2$, then we obtain from (5.5) the equation $\phi''(v) = 0$. By a reasoning similar to the one we employed earlier, it is seen that in both these cases the random variables X and Y have degenerate distributions.

Therefore, we have established that the independence of U and V implies that X and Y have gamma distributions with the same scale parameter. We still have to prove the converse. This does not follow from Pitman's result [2] quoted in the introduction, since we did not assume that X and Y have the same distribution.

Suppose, therefore, that X and Y are independent random variables and that their distributions are $F(x; \alpha, \lambda_1)$ and $F(x; \alpha, \lambda_2)$, respectively. We see, then, from (1.1) that the characteristic function of the joint distribution of U and V is given by

$$\mathbf{E}\left\{\exp\left[it(X+Y) + \frac{izX}{Y}\right]\right\}$$
$$= \frac{\alpha^{\lambda_1+\lambda_2}}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty\int_0^\infty \exp\left[-(\alpha - it)(x+y) + \frac{izx}{y}\right] x^{\lambda_1-1}y^{\lambda_2-1}\,dx\,dy.$$

If we introduce new variables under the integral sign by setting

$$\zeta = (\alpha - it)x, \qquad \eta = (\alpha - it)y,$$

we see easily that the integral can be written as a product of two functions, one depending on t, the other on z. Therefore, the random variables $U = X + Y$ and $V = X/Y$ are independent and the theorem is fully established.

Our theorem is closely related to a recent result of R. G. Laha [1], who characterized the gamma distribution essentially by the independence of the ratio of certain quadratic forms from the mean. Laha assumed that the random variables are identically distributed and that their second moment exists; both these assumptions were avoided in the present paper.

REFERENCES

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