ESTIMATES OF BOUNDED RELATIVE ERROR IN PARTICLE COUNTING[1]

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Summary. A statistical problem arising in many fields of activity requires the estimation of the average number of events occurring per unit of a continuous variable, such as area or time. The underlying distribution of events is assumed to be Poisson; the constant to be estimated is the unknown parameter λ of the distribution.

A sampling procedure is proposed in which the continuous variable is observed until a fixed number M of events occurs. Such a procedure enables us to form an estimate l, which with confidence coefficient α does not differ from λ by more than 100 γ per cent of λ. The values of γ and α depend on M but not on λ.

Modifications of this procedure which are sequential in nature and have possible operational advantages are also described.

These procedures are discussed in terms of a chemical problem of particle counting. It is clear, however, that they are generally applicable whenever the basic probability assumptions apply.

1. Introduction. The following problem arising in chemical research is typical of a statistical problem occurring in many fields of activity. A set of inert particles is randomly distributed over a microscope slide of area A. It is assumed that the probability of m particles falling in a subset of area a is

$$P_\lambda(m \mid a) = e^{-\lambda a}(\lambda a)^m / m!$$

and that distributions in disjoint areas are independent. On the basis of particle counts in subareas of A, we want to estimate the unknown parameter λ. With this estimate we can, by performing the indicated multiplication, also estimate, if desired, $N = A\lambda$, the expected number of particles on the slide. In addition, we would like to make a confidence statement about the reliability of our estimate.

In general, in discussing the reliability of an estimate, we would like to bound, at a given confidence level, either the absolute or the relative error of the estimate. That is, for a selected estimate t of an unknown parameter θ, we would like to say with confidence coefficient α, either

$$|t - \theta| \leqq \beta \qquad \text{or} \qquad |t/\theta - 1| \leqq \gamma, \quad \text{i.e., } (1 - \gamma) \leqq t/\theta \leqq (1 + \gamma),$$

where α and β (or γ) are suitably chosen numbers which do not depend on θ.

In the present problem, it seems reasonable to be concerned with the relative

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rather than the absolute error of the estimate, since this error is independent of the units in which the area is measured and is the same whether we are estimating λ, or $A\lambda$, the expected number of particles on the slide. What we would like, therefore, is an estimate l of λ such that we can say with confidence coefficient α that l does not differ from λ by more than 100γ per cent of λ, where neither α nor γ depend on the true value of λ. Such an estimate we call an estimate of *bounded relative error*. Moreover, among all estimates of this type we would like to choose one possessing some optimal properties.

2. Fixed area sampling procedure. In estimating λ in problems of the present kind, it often has been the practice to select n non-overlapping subareas of A, each of size a_0, with n and a_0 determined in advance, and to count the particles in the selected subareas. The usual estimate of λ is then

$$\bar{l} = \frac{1}{na_0} \sum_{i=1}^{n} x_i$$

where x_i is the number of particles observed in the ith subarea. It is easy to see, however, that for a given value of γ,

$$P\{\lambda(1 - \gamma) \leqq \bar{l} \leqq \lambda(1 + \gamma)\}$$

is a function of the unknown parameter λ, so that no confidence statement concerning a bound on the relative error of $\bar{l}$ can be made.

On the other hand, from the observed value of $\bar{l}$, we can determine values for two functions L_μ and L_ν defined by

$$\sum_{j=na_0\bar{l}}^{\infty} \frac{\exp\{-L_\mu(\bar{l})na_0\}(L_\mu(\bar{l})na_0)^j}{j!} = \frac{1-\alpha}{2},$$

$$\sum_{j=0}^{na_0\bar{l}} \frac{\exp\{-L_\nu(\bar{l})na_0\}(L_\nu(\bar{l})na_0)^j}{j!} = \frac{1-\alpha}{2}.$$

Then, we can say with confidence coefficient approximately α that

$$L_\mu(\bar{l}) \leqq \lambda \leqq L_\nu(\bar{l}),$$

$$\lambda[1 - \gamma(\bar{l})] \leqq \hat{l}(\bar{l}) \leqq \lambda[1 + \gamma(\bar{l})],$$

where

$$\hat{l}(\bar{l}) = \frac{2L_\nu)\bar{l})L_\mu(\bar{l})}{L_\nu(\bar{l}) + L_\mu(\bar{l})}, \qquad \gamma(\bar{l}) = \frac{L_\nu(\bar{l}) - L_\mu(\bar{l})}{L_\nu(\bar{l}) + L_\mu(\bar{l})}.$$

It follows that if we estimate λ by $\hat{l}(\bar{l})$, we can say with confidence coefficient approximately α that the relative error of $\hat{l}(\bar{l})$ is bounded by $\gamma(\bar{l})$, but this quantity is a chance variable whose expected value depends on the unknown value of λ.

Since a fixed area sampling procedure does not yield estimates of bounded relative error, we consider the possibility of using another type of sampling procedure. (Some properties of fixed count and fixed time experiments are discussed by Albert and Nelson [1] for the case of counter data for radioactivity.) By a suitable adjustment of the microscope, the aperture can be increased gradually so that the area under observation increases continuously from a point to any desired magnitude, within the limits of the area of the slide.

If the area is expanded only until a fixed number of particles is counted, the magnitude of the observed area becomes a chance variable with an attached probability distribution. In this distribution, the parameter λ appears as a scale parameter. It has been shown [2] that if the single unknown parameter of a distribution is a scale parameter, estimates of bounded relative error exist, and among them one possessing specified optimal properties can be found. By adopting a fixed particle count procedure and applying the general theory, our problem is formally solved.

In the following sections, a fixed particle count procedure is explicitly defined and an estimate of bounded relative error is proposed. For the given value of γ and number of particles counted, it maximizes the value of the confidence coefficient α. Some modifications of this procedure are also discussed.

3. Fixed particle count procedure. Suppose in counting particles under a microscope, the area under observation is expanded until a fixed number of particles, M, is counted. The magnitude of the area so obtained, say a_M, is a chance variable with an attached probability distribution. Since we have assumed that the probability of m particles falling in a subset of fixed area a is

$$P_\lambda(m \mid a) = (\lambda a)^m e^{-\lambda a} / m!,$$

the probability that an area of size x/λ has fewer than M particles is

$$P\{\lambda a_M > x\} = e^{-x} \sum_{j=0}^{M-1} \frac{x^j}{j!} = \int_x^\infty \frac{e^{-t} t^{M-1}}{(M-1)!} dt.$$

It follows that $\lambda a_M = X$ has the density function

$$f_M(x) = e^{-x} x^{M-1} / (M-1)!.$$

That is, λa_M has a gamma distribution with parameter M, or $2\lambda a_M$ has a chi-square distribution with $2M$ degrees of freedom.

Suppose now that an observation a_M is made, and we form the estimate l of λ defined by $l = b/a_M$, where b is a given positive number. If γ is the desired bound on the relative error, the probability, before the observation is made, that

$$(1 - \gamma) \leqq l/\lambda \leqq (1 + \gamma)$$

is given by

$$P\left\{\frac{b}{1+\gamma} \leqq \lambda a_M \leqq \frac{b}{1-\gamma}\right\} = \int_{b/1+\gamma}^{b/1-\gamma} \frac{x^{M-1} e^{-x}}{(M-1)!} dx = \psi_\gamma(b, M), \text{ say.}$$

The quantity $\psi_\gamma(b, M)$ does not depend upon λ, so that for all λ, l is an estimate with relative error bounded by γ at confidence level $\psi_\gamma(b, M)$. Clearly, we would like to find that value of b, say b^*, for which $\psi_\gamma(b, M)$ is a maximum.

As b varies from 0 to ∞, the value of $\psi_\gamma(b, M)$ increases continuously from zero to a maximum value $\psi_\gamma(b^*, M)$, and then decreases again to zero. To determine b^*, we consider

$$\begin{aligned}\frac{\partial}{\partial b}\psi_\gamma(b, M) &= \frac{\partial}{\partial b}\left[\int_0^{b/1-\gamma} \frac{x^{M-1}e^{-x}}{(M-1)!}\,dx - \int_0^{b/1+\gamma} \frac{x^{M-1}e^{-x}}{(M-1)!}\,dx\right] \\ &= \frac{b^{M-1}}{(M-1)!}\left[\frac{e^{-b/(1-\gamma)}}{(1-\gamma)^M} - \frac{e^{-b/(1+\gamma)}}{(1+\gamma)^M}\right].\end{aligned}$$

The maximizing value is the single finite positive value of b for which

$$\partial\psi_\gamma(b, M)/\partial b = 0;$$

that is,

$$b^* = \frac{M(1-\gamma^2)}{2\gamma}\log\frac{1+\gamma}{1-\gamma}. \tag{3.1}$$

For b^*, we have

$$\psi_\gamma(b^*, M) = \int_{\mathfrak{D}} \frac{x^{M-1}e^{-x}}{(M-1)!}\,dx = \psi_\gamma^*(M), \text{ say,}$$

$$\mathfrak{D} = \left[\frac{M(1-\gamma)}{2\gamma}\log\frac{1+\gamma}{1-\gamma},\ \frac{M(1+\gamma)}{2\gamma}\log\frac{1+\gamma}{1-\gamma}\right].$$

For a fixed value of γ, the function ψ_γ^* is a single-valued monotone increasing function. The monotonicity of ψ_γ^* follows from the fact that a_M is a sufficient statistic for $a_1, \cdots, a_M$. It follows, therefore, that if we define M_0 to be the least integer such that $\psi_\gamma^*(M) \geqq \alpha$, then $M_0 = \eta_\gamma(\alpha)$ is a single-valued monotone increasing function of α.

With these considerations in mind, we propose the following fixed particle count procedure. The desired values of γ and α are specified in advance, and from these we determine $M = \eta_\gamma(\alpha)$. The area under observation is expanded until M particles are counted, resulting in an observation a_M. We estimate λ, the average number of particles per unit area, and N, the expected number of particles on the slide with total area A, by

$$l^* = b^*/a_M, \qquad N^* = Al^*,$$

where b^* is defined in (3.1). In either case, the relative error of the estimate is bounded by γ at confidence level at least α, regardless of the true value of λ.

If our estimation problem is formulated as a decision function problem in which our action space is the positive half of the real line, that is,

$$\{l:\ 0 \leqq l < \infty\},$$

and our loss function is

$$L(\lambda, l) = \begin{cases} 0 & \text{if } |(l/\lambda) - 1| \leqq \gamma, \\ 1 & \text{if } |(l/\lambda) - 1| > \gamma, \end{cases}$$

the proposed estimate l^* is the best invariant estimate. Since the loss function is bounded, it is also minimax [3]. The risk when we estimate λ by l^* is $1 - \psi_\gamma^*(M)$.

4. Values of γ, M, and $\psi_\gamma^*(M)$. Table 1 gives values of $\psi_\gamma^*(M)$ for four values of γ and for $M \leqq 40$. For $M > 40$, the distribution of $(\sqrt{4\lambda a_M} - \sqrt{4M - 1})$ is approximately normal with zero mean and unit variance. In this case, therefore,

$$\psi_\gamma^*(M) \sim \Phi\left(\sqrt{\frac{4M(1+\gamma)}{2\gamma}\log\frac{1+\gamma}{1-\gamma}} - \sqrt{4M-1}\right) - \Phi\left(\sqrt{\frac{4M(1-\gamma)}{2\gamma}\log\frac{1+\gamma}{1-\gamma}} - \sqrt{4M-1}\right), \tag{4.1}$$

where $\Phi(u) = \int_{-\infty}^{u} (2\pi)^{-\frac{1}{2}} e^{-t^2/2}\, dt$. Since $0 < \gamma < 1$, we have

$$\frac{1}{2\gamma}\log\frac{1+\gamma}{1-\gamma} = 1 + \frac{\gamma^2}{3} + \frac{\gamma^4}{5} + \frac{\gamma^6}{7} + \cdots$$

$$\psi_\gamma^*(M) \sim \Phi\left[\sqrt{4M}\left(\frac{\gamma^2}{24} + \frac{\gamma}{2} + \frac{7}{48}\gamma^3\right)\right] - \Phi\left[\sqrt{4M}\left(\frac{\gamma^2}{24} - \frac{\gamma}{2} - \frac{7}{48}\gamma^3\right)\right].$$

For values of γ and α which are generally of practical interest, a good approximation for the required value of M is given by the relation

$$\sqrt{4M}\left(\frac{\gamma}{2} + \frac{7}{48}\gamma^3\right) = z_\alpha \tag{4.2}$$

where $z_\alpha = \Phi^{-1}[\frac{1}{2}(1 + \alpha)]$, that is, $\Phi(z_\alpha) - \Phi(-z_\alpha) = \alpha$.

For $\gamma = .10$ and $\alpha = .90$, .95, and .99, respectively, this approximation yields the same values of M as those determined from (4.1), while for $\gamma = .05$, the values differ only slightly:

	α:			
	.90	.95	.99	
$\gamma = .10$	269	382	660	(4.1) or (4.2)
$\gamma = .05$	1081	1535	2651	(4.1)
	1081	1534	2650	(4.2)

5. Modified fixed particle count procedure. Instead of expanding the area continuously until M particles are counted, it may be more convenient to adopt a sampling procedure consisting of k subsamples. In the jth subsample, the area is expanded continuously until a fixed number of particles m_j is counted, with $\sum_1^k m_j = M$, and with the provision that areas observed in the different sub-

TABLE 1

Values of $\psi_\gamma^*(M) = \frac{1}{(M-1)!} \int_{\mathfrak{D}} x^{M-1} e^{-x}\, dx,$

$$\mathfrak{D} = \left[\frac{M(1-\gamma)}{2\gamma} \log \frac{(1+\gamma)}{(1-\gamma)},\ \frac{M(1+\gamma)}{2\gamma} \log \frac{(1+\gamma)}{(1-\gamma)}\right].$$

M	γ			
	.01	.05	.10	.20
2	.0108	.0541	.1083	.2165
4	.0156	.0781	.1558	.3084
6	.0193	.0962	.1915	.3753
8	.0223	.1114	.2211	.4292
10	.0250	.1247	.2469	.4746
12	.0274	.1367	.2700	.5140
14	.0297	.1476	.2909	.5487
16	.0317	.1578	.3102	.5796
18	.0337	.1674	.3281	.6075
20	.0355	.1763	.3449	.6329
22	.0373	.1849	.3607	.6560
24	.0389	.1930	.3757	.6771
26	.0405	.2008	.3898	.6966
28	.0421	.2082	.4033	.7146
30	.0436	.2154	.4162	.7313
32	.0450	.2223	.4286	.7468
34	.0464	.2290	.4404	.7612
36	.0477	.2355	.4518	.7746
38	.0490	.2418	.4628	.7871
40	.0503	.2479	.4733	.7988

samples are nonoverlapping. If $a_{m_j}^{(j)}$ is the area observed in the jth subsample, we now estimate λ by

$$l_k^* = b^* \Big/ \sum_{j=1}^{k} a_{m_j}^{(j)}.$$

Since each of the $a_{m_j}^{(j)}$ is independently distributed in a gamma distribution with parameter m_j, their sum, because of the reproductive property of the gamma (or chi-square) distribution, is again a gamma distribution with parameter $\sum_1^k m_j = M$. The theory now goes through as before.

6. Sequential Procedure. For practical purposes, it may be convenient to modify the sampling procedure in the following manner. The desired bound on the relative error, γ, and the confidence coefficient α are specified, and the corresponding value of M required for the fixed particle count procedure is determined. The area under observation is now expanded until a fixed number of particles, r, is counted; $1 \leqq r < M$.

Let a_r be the size of the area so observed. Subsequently, non-overlapping areas of fixed size ca_r, where c is a given positive number, are examined successively, and the number of particles in each area is counted. Let ν_i be the number of particles counted in the ith area of size ca_r. Sampling stops with the nth such area if

$$N' < M - r \leqq N = M - r + u,$$

say, where

$$N' = \sum_{i=1}^{n-1} \nu_i, \qquad N = \sum_{i=1}^{n} \nu_i.$$

In this procedure, the quantities $a_r, \nu_1, \cdots, \nu_n$, and n are all chance variables. When sampling stops, we estimate λ by

$$\tilde{l}^* = \frac{(M + u)(1 - \gamma^2)}{2\gamma a_r(1 + nc)} \log \frac{1 + \gamma}{1 - \gamma}.$$

That is, we estimate λ in the same manner as before, but replace M and a_n by $(m + u)$ and $a_r(1 + nc)$, respectively. We state with a confidence coefficient at least as large as α that the relative error of $\tilde{l}^*$ is bounded by γ.

This statement is justified as follows: the joint probability density of $\nu_1, \cdots, \nu_n$, n and $X = \lambda a_r$ is given by

$$\phi(\nu_1, \cdots, \nu_n, n, x) = \frac{e^{-x} x^{r-1}}{(r-1)!} \prod_{i=1}^{n} \left(\frac{e^{-cx}(cx)^{\nu_i}}{\nu_i!} \right), \tag{6.1}$$
$$\sum_{i=1}^{n-1} \nu_i < M - r \leqq \sum_{i=1}^{n} \nu_i.$$

(The density is with respect to discrete measure for $\nu_1, \cdots, \nu_n$, n, and to Lebesgue measure for x.) This can be written as

$$\phi(\nu_1, \cdots, \nu_n, n, x) = \frac{e^{-x(1+nc)}(1 + nc)^{r+N} x^{r+N-1}}{(r + N - 1)!} \cdot h(\nu_1, \cdots, \nu_n, n).$$

The probability before any observations are made that

$$1 - \gamma \leqq \frac{\tilde{l}^*}{\lambda} \leqq 1 + \gamma$$

is equal to

$$\sum_{n=1}^{\infty} \sum h(\nu_1, \cdots, \nu_n, n) P \left\{ \left[\frac{1 - \gamma}{2\gamma} \log \frac{1 + \gamma}{1 - \gamma} \leqq \frac{\lambda a_r(1 + nc)}{r + N} \right.\right.$$
$$\left.\left. \leqq \frac{1 + \gamma}{2\gamma} \log \frac{1 + \gamma}{1 - \gamma} \right] \middle| \nu_1, \cdots, \nu_n, n \right\}$$

$$= \sum_{n=1}^{\infty} \sum h(\nu_1, \cdots, \nu_n, n) \cdot \psi_\gamma^*(r + N)$$

$$\geqq \psi_\gamma^*(M) \sum_{n=1}^{\infty} \sum h(\nu_1, \cdots, \nu_n, n)$$

$$= \alpha.$$

All the sums without indices are over all sequences $\nu_1, \cdots, \nu_n$ satisfying (6.1).

7. Probability distribution of n. For any sequential procedure, it is of interest to determine, if possible, the probability distribution of n together with its expected value. For the proposed procedure, this distribution is given by

$$g(n) = \sum h(\nu_1, \cdots, \nu_n, n)$$

$$= \sum \frac{(r + N - 1)!\, c^N}{(r - 1)!\, \nu_1! \cdots \nu_n!\, (1 + nc)^{r+N}}$$

$$= \sum_{t_n = M-r}^{\infty} \sum_{t_{n-1}=0}^{M-r-1} \sum_{t_{n-2}=0}^{t_{n-1}}$$

$$\cdots \sum_{t_1=0}^{t_2} \frac{(r + t_n - 1)!\, c^{t_n}}{(r - 1)!(t_n - t_{n-1})! \cdots (t_2 - t_1)!\, t_1!(1 + nc)^{r+t_n}}$$

$$= \sum_{t_n=M-r}^{\infty} \sum_{t_{n-1}=0}^{M-r-1} \frac{(r + t_n - 1)!\, c^{t_n}(n - 1)^{t_{n-1}}}{(r - 1)!\,(t_n - t_{n-1})!\, t_{n-1}!\,(1 + nc)^{r+t_n}}$$

$$= \sum_{u=0}^{\infty} \sum_{t_{n-1}=0}^{M-r-1} \frac{(M + u - 1)!\, c^{M-r+u}(n - 1)^{t_{n-1}}}{(r - 1)!(M - r + u - t_{n-1})!\, t_{n-1}!\,(1 + nc)^{M+u}}.$$

As before, all sums without indices are over all sequences $\nu_1, \cdots, \nu_n$ satisfying (6.1).

Since

$$\sum_{t_{n-1}=0}^{M-r-1} \frac{(n - 1)^{t_{n-1}}}{t_{n-1}!\,(M - r + u - t_{n-1})!} = n^{M-r+u} \int_{1-1/n}^{1} \frac{t^{M-r-1}(1 - t)^u}{(M - r - 1)!\, u!}\, dt,$$

$$g(n) = \sum_{u=0}^{\infty} \frac{(M + u - 1)!(nc)^{M-r+u}}{(M - r - 1)!(r - 1)!\, u!(1 + nc)^{M+u}} \int_{1-1/n}^{1} t^{M-r-1}(1 - t)^u\, dt$$

$$= \int_{1-1/n}^{1} \frac{(M - 1)!(nc)^{M-r}}{(M - r - 1)!(r - 1)!} \frac{t^{M-r-1}}{(1 + nc)^M}$$

$$\cdot \left[\sum_{u=0}^{\infty} \frac{(M + u - 1)!}{(M - 1)!\, u!} \left(\frac{nc}{1 + nc} \right)^u (1 - t)^u \right] dt$$

$$= \int_{1-1/n}^{1} \frac{(M - 1)!(nc)^{M-r} t^{M-r-1}}{(M - r - 1)!(r - 1)!(1 + nct)^M}\, dt.$$

Writing $\tau = nct \,/\, (1 + nct)$, we have

$$g(n) = \frac{(M-1)!}{(r-1)!\,(M-r-1)!} \int_{\mathfrak{D}} \tau^{M-r-1}(1-\tau)^{r-1}\, d\tau,$$

$$\mathfrak{D} = \left[\frac{c(n-1)}{1+c(n-1)}, \frac{cn}{1+cn}\right].$$

From this we determine the expected value of n as a function of M, r, and c.

$$\begin{aligned} E(n) &= \sum_{n=1}^{\infty} ng(n) = \sum_{n=1}^{\infty} g(n) + \sum_{n=2}^{\infty} g(n) + \cdots \\ &= \frac{(M-1)!}{(r-1)!\,(M-r-1)!} \sum_{n=1}^{\infty} \int_{\mathcal{S}} \tau^{M-r-1}(1-\tau)^{r-1}\, d\tau, \end{aligned}$$

$$\mathcal{S} = \left[\frac{c(n-1)}{1+c(n+1)}, 1\right].$$

8. Modified sequential procedure. Yet another sampling procedure yields an estimate of λ with relative error bounded by γ at confidence level α. This is the procedure in which we first obtain an observation a_r and then count the number of particles falling in a given number n_0 of non-overlapping areas, each of size ca_r, where n_0 is determined by the selected values of γ, α, c, and r. From the observations a_r and $\nu_1, \cdots, \nu_{n_0}$, we estimate λ by

$$\hat{l}^* = \frac{1-\gamma^2}{2\gamma a_r(1+n_0c)}\left[r + \sum_{i=1}^{n_0} \nu_i\right] \log \frac{1+\gamma}{1-\gamma}. \tag{8.1}$$

We state, with confidence coefficient α, that the relative error of $\hat{l}^*$ is bounded by γ. Consider the confidence coefficient with which we make this statement for any arbitrary value of n_0. The joint probability distribution of a_r and $\nu_1, \cdots, \nu_{n_0}$ is defined by the function ϕ given in (6.1), with no restrictions on the values of the ν_i and with $n = n_0$. Hence, the probability, before any observations are made, that

$$1 - \gamma \leqq \hat{\ }^{*} / \lambda \leqq 1 + \gamma$$

is given by

$$\sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \cdots \sum_{\nu_{n_0}=0}^{\infty} \psi_\gamma^*\left(r + \sum_{i=1}^{n_0} \nu_i\right) \cdot h(\nu_1, \cdots, \nu_{n_0}, n_0),$$

and is independent of λ. For fixed values of r and c, this is a monotone increasing function of n_0. By a proper choice of n_0, the value of the confidence coefficient can be made as close to 1 as desired.

9. Estimate of bounded relative error for the variance of a normal distribution. In all of the sampling procedures yielding estimates of bounded relative error for the problem of particle counting, the function ψ_γ^* plays an important part. This function also appears in the problem of estimates of bounded relative error for the variance of a normal distribution. Suppose we have N independent observations $x_1, \cdots, x_N$ on a chance variable X, where now X is normally distributed with unknown mean μ and variance σ^2. Our problem is to estimate σ^2, and we would like our estimate to be of bounded relative error.

If $w^2 = \sum_1^N (x_t - \bar{x})^2$, the distribution of w^2 / σ^2 is a chi-square distribution with $N - 1$ degrees of freedom. In this distribution, σ^2 appears as a scale parameter and is the only unknown parameter in the distribution. With the use of w^2, therefore, we can find an estimate of bounded relative error possessing specified optimal properties. For given values of γ and $n = N - 1$, the estimate $\hat{\sigma}^2$ which maximizes the confidence coefficient with which we state that

$$1 - \gamma \leqq \hat{\sigma}^2 / \sigma^2 \leqq 1 + \gamma$$

is given by

$$\hat{\sigma}^2 = 2\gamma w^2 \bigg/ \left[n \log \frac{1+\gamma}{1-\gamma} \right],$$

$$P\{1 - \gamma \leqq \hat{\sigma}^2/\sigma^2 \leqq 1 + \gamma\} = \frac{1}{\Gamma(n/2)} \int_{\mathfrak{D}} 2^{-n/2} (\chi^2)^{(n-2)/2} e^{-\chi^2/2} \, d(\chi^2)$$

$$= \psi_\gamma^* \left(\frac{n}{2}\right),$$

$$\mathfrak{D} = \left[\frac{n(1+\gamma)}{2\gamma} \log \frac{1+\gamma}{1-\gamma}, \frac{n(1-\gamma)}{2\gamma} \log \frac{1+\gamma}{1-\gamma} \right].$$

Thus, making the substitution $n = 2M$, Table 1 gives us for $n = 4, 8, \cdots, 80$, values of the confidence coefficient with which we assert relation (8.1). As before, we can determine n so that $\psi_\gamma^*(n/2) = \alpha$ with α chosen in advance. For larger values of n, $\sqrt{2\chi_n^2} - \sqrt{2n - 1}$ is approximately normally distributed with zero mean and unit variance, and corresponding approximations can be made.

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