SOME CLASSES OF PARTIALLY BALANCED DESIGNS[1]

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1. Summary. Incomplete block designs with a few replications are of practical interest to experimenters. Partially balanced incomplete block (PBIB) designs with two associate classes, and $k > r = 2$, were studied by one of the authors [1]. The present paper extends this investigation to the case $k > r$ with $\lambda_1 = 1$ and $\lambda_2 = 0$. It is shown that the parameters of all PBIB designs in this case are given by (4.27) and thus depend upon three integral parameters k, r, and t, with the additional restrictions that

(i) $1 \leqq t \leqq r$,

(ii) $rk(r - 1)(k - 1)/t(k + r - t - 1)$ is a positive integer.

For the particular case $r = 3$ it is shown that all designs with $t = 2$ or 3 necessarily exist, but if $t = 1$, then the only possible value of $k > r$ is 5. However designs with parameters (4.27) with $r = 3$, $t = 1$, and $k = 2$ or 3 are also combinatorially possible though they do not belong to the class $k > r$.

Interesting by-products of this study are a lemma and five corollaries which give an insight into the structure of PBIB designs with $\lambda_1 = 1$ and $\lambda_2 = 0$, and $p_{11}^1 = k - 2$, no special assumptions being made regarding r and k.

2. Introduction. PBIB designs with m associate classes ($m \geqq 1$) were introduced by Bose and Nair [2]. Balanced incomplete block designs and square lattices were included as special cases. Nair and Rao [5] broadened the definition so as to further include cubic and other higher dimensional lattices. Bose and Shimamoto [3] have rephrased the definition so as to stress the fact that the relations between the treatments are determined only by the parameters n_i and p_{jk}^i with $i, j, k = 1, 2, \cdots, m$. For the special case of two associate classes ($m = 2$) the Bose and Shimamoto definition is substantially as follows.

A PBIB block design with two associate classes is an arrangement of v treatments (or varieties) in b blocks such that:

(i) Each of the v treatments is replicated r times in b blocks each of size k, and no treatment appears more than once in any block.

(ii) There exists a relationship of association between every pair of the v treatments satisfying the following conditions:

(a) Any two treatments are either first or second associates.

(b) Each treatment has n_1 first and n_2 second associates.

(c) Given any two treatments which are ith associates, the number p_{jk}^i of treatments common to the jth associates of the first and the kth associates

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of the second is independent of the pair of treatments with which we start. Furthermore, $p^i_{jk} = p^i_{kj}$ for $i, j, k = 1, 2$.

(iii) Any pair of treatments which are ith associates occur together in exactly λ_i blocks for $i = 1, 2$.

If either n_1 or n_2 assumes the value zero, one associate class does not exist. Hence, we shall require that both n_1 and n_2 be positive integers.

$$vr = bk, \tag{2.1}$$

$$v = n_1 + n_2 + 1, \tag{2.2}$$

$$\lambda_1 n_1 + \lambda_2 n_2 = r(k - 1), \tag{2.3}$$

$$p^1_{11} + p^1_{12} + 1 = p^2_{11} + p^2_{12} = n_1, \tag{2.4}$$

$$p^1_{21} + p^1_{22} = p^2_{21} + p^2_{22} + 1 = n_2, \tag{2.5}$$

$$n_1 p^1_{12} = n_2 p^2_{11}, \qquad n_1 p^1_{22} = n_2 p^2_{12}. \tag{2.6}$$

Furthermore, it was proved that if values are assigned to the parameters of the first kind (v, b, r, k, λ_1, λ_2, n_1, and n_2) satisfying (2.1), (2.2), and (2.3), then there is one independent parameter of the second kind (p^i_{jk} for $i, j, k = 1, 2$).

The parameters of the second kind will be exhibited as elements of two symmetric matrices

$$P_1 = \begin{pmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix}. \tag{2.7}$$

Nair [6] established a necessary condition for PBIB designs having $k > r$. This condition was used by Bose [1] in exhausting the subclass of PBIB designs with two associate classes and $\lambda_1 = 1$ and $\lambda_2 = 0$, with $r = 2$. For the special case of PBIB designs with two associate classes with $\lambda_1 = 1$ and $\lambda_2 = 0$, and $k > r$, Nair's condition simplifies to

$$r p^1_{12} - (r - 1) p^2_{12} = r(r - 1), \tag{2.8}$$

a useful tool in the present investigation.

3. A less demanding definition for PBIB designs with two associate classes. We shall now show that for PBIB designs with two associate classes the Bose and Shimamoto definition is more demanding than it need be. To this end we establish two theorems.

THEOREM 3.1. *Let there exist a relationship of association between every pair among the v treatments satisfying the conditions:*

(a) *Any two treatments are either first or second associates.*

(b) *Each treatment has n_1 first and n_2 second associates.*

(c) *For any pair of treatments which are first associates, the number p^1_{11} of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.*

Then, for every pair of first associates among the v treatments the numbers p_{12}^1, p_{21}^1, *and* p_{22}^1 *are constants, and* $p_{12}^1 = p_{21}^1$.

PROOF. Let θ and ϕ be an arbitrarily chosen pair of first associates from among the v treatments. Let $p_{jk}^1(\theta, \phi)$ denote the number of treatments common to the jth associates of θ and to the kth associates of ϕ, for $j, k = 1, 2$. The n_1 first associates of θ are made up of ϕ, the $p_{11}^1(\theta, \phi)$ treatments which are first associates of θ as well as ϕ, and the $p_{12}^1(\theta, \phi)$ treatments which are first associates of θ but second associates of ϕ. Hence

$$1 + p_{11}^1(\theta, \phi) + p_{12}^1(\theta, \phi) = n_1. \tag{3.1}$$

Likewise, classifying the first associates of ϕ, we have

$$1 + p_{11}^1(\theta, \phi) + p_{21}^1(\theta, \phi) = n_1. \tag{3.2}$$

Similarly, the n_2 second associates of θ are made up of the $p_{21}^1(\theta, \phi)$ treatments which are second associates of θ and first associates of ϕ, and the $p_{22}^1(\theta, \phi)$ treatments which are second associates of both θ and ϕ. Hence

$$p_{21}^1(\theta, \phi) + p_{22}^1(\theta, \phi) = n_2. \tag{3.3}$$

But by hypothesis, $p_{11}^1(\theta, \phi)$ is independent of the pair of treatments θ and ϕ and is p_{11}^1. Hence, from (3.1), (3.2), and (3.3) we obtain

$$p_{12}^1(\theta, \phi) = p_{21}^1(\theta, \phi) = n_1 - p_{11}^1 - 1, \tag{3.4}$$

$$p_{22}^1(\theta, \phi) = n_2 - n_1 + p_{11}^1 + 1. \tag{3.5}$$

Since θ and ϕ are an arbitrarily chosen pair of first associates, the relations (3.4) and (3.5) amount to a proof of the theorem.

THEOREM 3.2. *Let there exist a relationship of association between every pair among the v treatments satisfying the conditions:*

(a) *Any two treatments are either first or second associates.*

(b) *Each treatment has* n_1 *first and* n_2 *second associates.*

(c) *For any pair of treatments which are second associates, the number* p_{11}^2 *of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.*

Then, for every pair of second associates among the v treatments the numbers p_{12}^2, p_{21}^2, *and* p_{22}^2 *are constants, and* $p_{12}^2 = p_{21}^2$.

Proof is similar to that of Theorem 3.1. In fact, it can be shown that

$$p_{12}^2(\theta, \phi) = p_{21}^2(\theta, \phi) = n_1 - p_{11}^2, \tag{3.6}$$

$$p_{22}^2(\theta, \phi) = n_2 - n_1 + p_{11}^2 - 1, \tag{3.7}$$

where θ and ϕ are an arbitrarily chosen pair of second associates.

The question naturally arises whether one of the preceding theorems implies the other. The answer is no. Consider the following design with $v = 7$ treatments in $b = 14$ blocks, each of size $k = 2$, and with each treatment replicated $r = 4$ times:

$$(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 0),$$
$$(0, 3), (1, 4), (2, 5), (3, 6), (4, 0), (5, 1), (6, 2).$$

Each treatment of this design has $n_1 = 4$ first associates and $n_2 = 2$ second associates with $\lambda_1 = 1$ and $\lambda_2 = 0$. For any pair of treatments θ and ϕ which are second associates, $p_{11}^2(\theta, \phi) = 3$. Hence, this design satisfies Theorem 3.2 with $p_{12}^2 = p_{21}^2 = 1$ and $p_{22}^2 = 0$. However, for any two treatments α and β which are first associates, $p_{11}^1(\alpha, \beta) = 1$ or 2, and Theorem 3.1 is not satisfied.

Insofar as PBIB designs with two associate classes are concerned, the consequence of Theorems 3.1 and 3.2 is that the definition of a PBIB design given by Bose and Shimamoto demands more than is needed. For PBIB designs with two associate classes, a less demanding definition could be formed by replacing condition (c) of (ii) in Section 2 by

(c′) For any pair of the v treatments which are ith associates, the number, p_{11}^i for $i = 1, 2$, of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.

This new definition and Theorems 3.1 and 3.2 are then equivalent to the Bose and Shimamoto definition. Under the Bose and Shimamoto definition, to prove that an arrangement of objects is a PBIB design, it is necessary, insofar as (c) of (ii) is concerned, to show the constancy of all eight parameters p_{jk}^i for $i, j, k = 1, 2$, and also to show that the equalities $p_{jk}^i = p_{kj}^i$ hold for $j \neq k$ and $i, k = 1, 2$. By use of Theorems 3.1 and 3.2, with regard to the parameters of the second kind, it is necessary only to show the constancy of p_{11}^1 and p_{11}^2.

4. Complete enumeration of PBIB designs with two associate classes and $k > r \geqq 2$, with $\lambda_1 = 1$ and $\lambda_2 = 0$.

We first establish the following useful lemma.

LEMMA 4.1. *For any* PBIB *design with two associate classes and* $k > r \geqq 2$, with $\lambda_1 = 1$ *and* $\lambda_2 = 0$,

$$k - 2 \leqq p_{11}^1 \leqq (k - 2) + (r - 1)^2. \tag{4.1}$$

PROOF. The left portion of (4.1) is obvious; consider the right portion. Let the treatments θ and ϕ be first associates. Then they occur together in exactly one block which we shall denote by $B(\theta, \phi)$. There are $k - 2$ other treatments in this block which are first associates of θ as well as ϕ. Since $\lambda_1 = 1$, both θ and ϕ cannot occur together in any other block. Denote the $r - 1$ blocks in which θ but not ϕ occurs by $B_i(\theta)$, for $i = 1, 2, \cdots, r - 1$, and similarly the $r - 1$ blocks in which ϕ but not θ occurs by $B_j(\phi)$, for $j = 1, 2, \cdots, r - 1$. If a treatment does not occur in $B(\theta, \phi)$ but is a first associate of θ as well as ϕ, it must occur exactly once in the blocks $B_i(\theta)$, and exactly once in the blocks $B_j(\phi)$. But the block $B_j(\phi)$ cannot have more than one treatment in common with any of the blocks $B_1(\theta)$, $B_2(\theta)$, $\cdots$, $B_{r-1}(\theta)$. Hence $B_j(\phi)$ cannot contain more than $r - 1$ first associates of θ. This holds for $j = 1, 2, \cdots, r - 1$. Therefore there cannot exist more than $(r - 1)^2$ treatments which occur once among the blocks $B_i(\theta)$ and once among the blocks $B_j(\phi)$. Thus p_{11}^1 cannot exceed $(k - 2) + (r - 1)^2$, which proves the lemma.

We shall now obtain the combinatorial parameters of all designs belonging to the class under consideration. From (2.4) and (2.6) we obtain

$$n_1 p_{12}^1 + n_2 p_{12}^2 = n_1 n_2 . \tag{4.2}$$

Solving (2.8) and (4.2) simultaneously, we have

$$p_{12}^1 = n_2(r - 1)(r + n_1) / [n_1(r - 1) + n_2 r] \tag{4.3}$$

$$p_{12}^2 = r n_1(n_2 - r + 1) / [n_1(r - 1) + n_2 r]. \tag{4.4}$$

From (2.2) and (2.3) we get

$$n_1 = r(k - 1) \tag{4.5}$$

$$v = n_2 + 1 + r(k - 1). \tag{4.6}$$

From (2.1) and (4.6) it follows that

$$b = r^2 + (r/k)(n_2 - r + 1). \tag{4.7}$$

Since both b and r^2 must be integral, we set

$$r(n_2 - r + 1) = sk \tag{4.8}$$

where s is an integer. Then, from (4.8), (4.7), (2.1), (4.3), (4.4), and (4.5) we get

$$n_2 = sk/r + r - 1, \tag{4.9}$$

$$b = r^2 + s, \tag{4.10}$$

$$v = k(r^2 + s)/r, \tag{4.11}$$

$$p_{12}^1 = (r - 1)k - r(r - 1)^2(k - 1) / [s + r(r - 1)], \tag{4.12}$$

$$p_{12}^2 = r(k - 1) - r^2(r - 1)(k - 1) / [s + r(r - 1)]. \tag{4.13}$$

From (4.12) and (4.13) it is seen that $r(r - 1)^2(k - 1)$ and $r^2(r - 1)(k - 1)$ must both be integral multiples of $s + r(r - 1)$. Hence, their difference must also be divisible by $s + r(r - 1)$. Therefore, we introduce an auxiliary integral parameter t defined by

$$t = r(r - 1)(k - 1) / [s + r(r - 1)]. \tag{4.14}$$

Then

$$s = r(r - 1)(k - t - 1)/t, \qquad t \neq 0. \tag{4.15}$$

From (4.9) through (4.13) and (4.15) it follows that

$$v = k[(r-1)(k-1)+t]/t, \tag{4.16}$$

$$b = r[(r-1)(k-1)+t]/t, \tag{4.17}$$

$$n_2 = (r-1)(k-1)(k-t)/t, \tag{4.18}$$

$$p_{12}^1 = (r-1)(k-t), \tag{4.19}$$

$$p_{12}^2 = r(k-t-1). \tag{4.20}$$

From (4.19), (4.20), (2.4), and (2.5) we get

$$p_{11}^1 = t(r-1)+k-r-1, \tag{4.21}$$

$$p_{22}^1 = (r-1)(k-t)(k-t-1)/t, \tag{4.22}$$

$$p_{11}^2 = rt, \tag{4.23}$$

$$p_{22}^2 = [(r-1)(k-1)(k-2t)+t(rt-k)]/t. \tag{4.24}$$

Applying Lemma 4.1 to (4.21), we obtain

$$1 \leqq t \leqq r, \tag{4.25}$$

a most useful set of bounds on the integral parameter t. It is now seen that the divisor, $s + r(r-1)$, in (4.12) produces no difficulty because

$$(r-1)(k-1) \leqq s + r(r-1) \leqq r(r-1)(k-1), \qquad k > r \geqq 2. \tag{4.26}$$

In summary, all PBIB designs with two associate classes belonging to the class characterized by $k > r \geqq 2$, with $\lambda_1 = 1$ and $\lambda_2 = 0$, are obtainable from

$$\left\{\begin{array}{l} v = k[(r-1)(k-1)+t]/t, \quad r = r, \quad \lambda_1 = 1, \\ b = r[(r-1)(k-1)+t]/t, \quad k = k, \quad \lambda_2 = 0, \\ n_1 = r(k-1), \quad n_2 = (r-1)(k-1)(k-t)/t, \\ P_1 = \begin{pmatrix} (t-1)(r-1)+k-2 & (r-1)(k-t) \\ (r-1)(k-t) & (r-1)(k-t)(k-t-1)/t \end{pmatrix}, \\ P_2 = \begin{pmatrix} rt & r(k-t-1) \\ r(k-t-1) & [(r-1)(k-1)(k-2t)+t(rt-k)]/t \end{pmatrix}, \end{array}\right. \tag{4.27}$$

where $1 \leqq t \leqq r$.

Connor and Clatworthy [4] have shown (Theorem 5.1) that for a PBIB design with two associate classes it is necessary (but not sufficient) that the quantities

$$\alpha_1 = [(v-1)(-\gamma+\sqrt{\Delta}+1) - 2n_1]/2\sqrt{\Delta} \tag{4.28}$$

$$\alpha_2 = [(v-1)(\gamma+\sqrt{\Delta}+1) - 2n_2]/2\sqrt{\Delta} \tag{4.29}$$

be positive integers, where

$$\gamma = p_{12}^2 - p_{12}^1, \qquad \beta = p_{12}^1 + p_{12}^2, \qquad \Delta = \gamma^2 + 2\beta + 1. \tag{4.30}$$

Thus for any design of (4.27) it is necessary that

$$\alpha_1 = rk(r-1)(k-1) \,/\, t(k+r-t-1). \tag{4.31}$$

In the case of designs of three replications, there are only three series, one corresponding to each of the three permissible values of t. Setting $t = 1$, 2, and 3 in (4.27) we obtain

$$(4.32)\begin{cases} v = k(2k-1), & r = 3, \quad \lambda_1 = 1, \quad n_1 = 3(k-1), \\ b = 3(2k-1), & k = k, \quad \lambda_2 = 0, \quad n_2 = 2(k-1)^2, \\ P_1 = \begin{pmatrix} k-2 & 2(k-1) \\ 2(k-1) & 2(k-1)(k-2) \end{pmatrix}, & P_2 = \begin{pmatrix} 3 & 3(k-2) \\ 3(k-2) & 2k^2-7k+7 \end{pmatrix}; \end{cases}$$

$$(4.33)\begin{cases} v = k^2, & r = 3, \quad \lambda_1 = 1, \quad n_1 = 3(k-1), \\ b = 3k, & k = k, \quad \lambda_2 = 0, \quad n_2 = (k-1)(k-2), \\ P_1 = \begin{pmatrix} k & 2(k-2) \\ 2(k-2) & (k-2)(k-3) \end{pmatrix}, & P_2 = \begin{pmatrix} 6 & 3(k-3) \\ 3(k-3) & k^2-6k+10 \end{pmatrix}; \end{cases}$$

$$(4.34)\begin{cases} v = k(2k+1)/3, & r = 3, \quad \lambda_1 = 1, \quad n_1 = 3(k-1), \\ b = 2k+1, & k = k, \quad \lambda_2 = 0, \quad n_2 = 2(k-1)(k-3)/3, \\ P_1 = \begin{pmatrix} k+2 & 2(k-3) \\ 2(k-3) & 2(k-3)(k-4)/3 \end{pmatrix}, & \\ P_2 = \begin{pmatrix} 9 & 3(k-4) \\ 3(k-4) & (2k^2-17k+39)/3 \end{pmatrix}. & \end{cases}$$

In (4.34), k must be of the form $3p$ or $3p + 1$, p being an integer.

The designs given by (4.33) are the well known lattice designs of three replications. For all integral values of $k \geqq 3$ solutions exist.

The results of Reiss [7], together with those of Shrikhande [8], are sufficient to guarantee the existence of solutions for the designs of (4.34) as duals of the corresponding balanced incomplete block designs given by

$$v^* = 2k+1, \qquad b^* = k(2k+1)/3, \qquad k^* = 3, \qquad r^* = k, \qquad \lambda^* = 1, \tag{4.35}$$

where k is of the form $3p$ or $3p + 1$, p being a positive integer.

It follows from (4.31), that for designs of (4.32), $12/(k + 1)$ must be integral. The only permissible values of k are therefore 2, 3, 5, and 11.

The designs corresponding to $k = 2$ and 3 are known, the design corresponding to $k = 5$ is new, and the case $k = 11$ is impossible. We shall defer the discussion of these designs to Section 7.

5. The block structure of PBIB designs with two associate classes having $\lambda_1 = 1$ and $\lambda_2 = 0$, with $p_{11}^1 = k - 2$. First, we shall introduce the notation and terminology used throughout the remainder of this paper. We shall then establish a lemma and five corollaries which provide necessary conditions for the existence of the designs we consider.

Consider any PBIB design with two associate classes, and having $\lambda_1 = 1$ and $\lambda_2 = 0$, with $p_{11}^1 = k - 2$. Let the Greek letter θ represent any treatment of the design. The $n_1 = r(k - 1)$ first associates of θ will be denoted by r Latin letters each bearing subscripts $1, 2, \cdots, k - 1$. The $k - 1$ first associates of θ appearing together with θ in a block will be denoted by the same letter, and first associates of θ appearing in different blocks with θ will be denoted by different letters. The n_2 second associates of θ will be denoted by the integers $1, 2, \cdots, n_2$. The n_1 first associates of θ will sometimes be referred to as *lettered treatments* and the n_2 second associates of θ as *numbered treatments*. The r blocks containing treatment θ will be referred to as the *θ-blocks*.

Likewise, the blocks containing the lettered treatments x_j, where $x = a$ or b or c, etc., and $j = 1, 2, \cdots, k - 1$, but not θ will be referred to as the *x-blocks*, and the r blocks containing the numbered treatment i, with $1 \leqq i \leqq n_2$, will be called the *i-blocks*. Blocks containing only first associates of θ or only second associates of θ will be called *pure blocks* while blocks containing both first and second associates of θ will be called *mixed blocks*.

In the special case of designs having $\lambda_1 = 1$ and $\lambda_2 = 0$, with $p_{11}^1 = k - 2$, it will be shown that there are exactly $r(r - 1)(k - 1)$ mixed blocks each of which contains one lettered treatment and $k - 1$ numbered treatments. Those $(r - 1)(k - 1)$ mixed blocks containing first associates of θ denoted by the same letter (and subscripts $1, 2, \cdots, k - 1$) will be called a *group of blocks*. There are r such groups of blocks. The group of mixed blocks containing the lettered treatment x with any subscript will be referred to as the *x-group* of blocks ($x = a$, b, c, etc.) Within the x-group of blocks there are $r - 1$ blocks containing the lettered treatment x_j. These $r - 1$ blocks will be referred to as the *x_j-set* of blocks.

LEMMA 5.1. *For a* PBIB *design with two associate classes having* $\lambda_1 = 1$ *and* $\lambda_2 = 0$, *with* $p_{11}^1 = k - 2$, *any two treatments appearing in different blocks containing a common treatment must be second associates of each other.*

PROOF. There is no loss in generality if the treatment common to the two blocks is taken as treatment θ. Let x_i and y_j, with $x \neq y$ and $i, j = 1, 2, \cdots, k - 1$, be any two treatments appearing in different blocks containing θ. Since $\lambda_1 = 1$ and $\lambda_2 = 0$, two treatments appearing in the same block are first associates of each other while any pair of treatments which do not occur together in any block are second associates of each other. The $k - 2$ other treatments appearing in the same block with the pair (θ, x_i) are first associates of both θ and x_i. Since $p_{11}^1 = k - 2$, all other first associates of θ (including y_j) must be second associates of x_i. This proves the lemma.

COROLLARY 5.1. *Under the conditions of Lemma* 5.1, *each mixed block contains only one lettered treatment, and the* $r(r - 1)(k - 1)$ *mixed blocks can be divided into* r *distinct groups, each group containing* $k - 1$ *sets and each set containing* $r - 1$ *blocks.*

PROOF. Let the r θ-blocks of the design be

$$\begin{array}{ccccc} \theta & a_1 & a_2 & \cdots & a_{k-1} \\ \theta & b_1 & b_2 & \cdots & b_{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta & l_1 & l_2 & \cdots & l_{k-1}. \end{array}$$

Since $\lambda_1 = 1$ and $\lambda_2 = 0$, no pair of treatments can occur together in more than one block. By Lemma 5.1 any pair of lettered treatments x_i and y_j, with $x \neq y$, must be second associates of each other. Hence, no two lettered treatments can appear together in a block free from θ. Therefore the design must contain $r(r-1) \cdot (k-1)$ mixed blocks, each of which contains one lettered and $k-1$ numbered treatments. Since each lettered treatment must appear in $r-1$ blocks free from θ, there are $(r-1)(k-1)$ mixed blocks containing treatments $x_1, x_2, \cdots, x_{k-1}$. These blocks constitute the x-group of blocks, where $x = a, b, \cdots$, or l, and there are r distinct groups of blocks. Within the x-group there are $r-1$ blocks containing the lettered treatment x_i, and these we have called the x_i-set of blocks for $i = 1, 2, \cdots, k-1$. Each group of blocks obviously contains $k-1$ distinct sets of blocks.

COROLLARY 5.2. *Under the conditions of Lemma* 5.1, *the numbered treatments appearing in a group of blocks must all be distinct. Hence* $n_2 \geqq (r-1)(k-1)^2$. If $n_2 = (r-1)(k-1)^2$, *each group must contain precisely one complete replication of the numbered treatments.*

PROOF. Since $\lambda_1 = 1$ and $\lambda_2 = 0$, no numbered treatment can appear in two blocks belonging to the same set. Suppose the same numbered treatment j, for $j = 1, 2, \cdots, n_2$, appears in blocks belonging to different sets of the same group. Let the lettered treatment in one of the sets containing j be x_m and the lettered treatment in the other set containing j be x_n, with $m \neq n$ and $m, n = 1, 2, \cdots, k-1$. Now x_m and x_n are first associates of each other since they appear together in the same θ-block. But by Lemma 5.1, x_n and x_m must be second associates of each other since they appear in different blocks containing the common treatment j, a contradiction. Therefore no numbered treatment can appear in two or more blocks belonging to the same group. Hence the numbered treatments appearing in the same group of blocks must all be distinct. Since each mixed block contains $k-1$ numbered treatments, $(r-1)(k-1)^2$ distinct numbered treatments appear in a group of blocks. Since numbered treatments are second associates of θ, $n_2 \geqq (r-1)(k-1)^2$. If $n_2 = (r-1)(k-1)^2$, each group must obviously contain exactly one complete replication of the numbered treatments.

COROLLARY 5.3. *Under the conditions of Lemma* 5.1, *the mixed blocks contain* p_{11}^2 *complete replications of the numbered treatments. The pure blocks must contain* $r - p_{11}^2$ *complete replications of the numbered treatments.*

PROOF. Consider treatment θ and an arbitrarily chosen numbered treatment i, for $i = 1, 2, \cdots, n_2$. Since θ and i are second associates of each other, and since θ has only lettered treatments as first associates, treatment i must have exactly p_{11}^2 lettered first associates. Since lettered treatments appear only in θ-blocks and mixed blocks, and since no numbered treatments can appear in a θ-block, it follows by Corollary 5.1 that treatment i must appear in exactly p_{11}^2 mixed blocks. This is true for $i = 1, 2, \cdots, n_2$; therefore the mixed blocks contain exactly p_{11}^2 complete replications of the numbered treatments. Now, there are, in all, r complete replications of the numbered treatments. Therefore the pure blocks consist of $r - p_{11}^2$ complete replications of the numbered treatments.

Corollary 5.4. *Under the conditions of Lemma* 5.1, *two sets of blocks belonging to different groups must intersect in* $p_{11}^2 - 1$ *numbered treatments; that is, they must have* $p_{11}^2 - 1$ *numbered treatments in common.*

Proof. By Lemma 5.1 and Corollary 5.1, the pair of lettered treatments x_i and y_j, where $x \neq y$, appearing in two sets belonging to different groups are second associates of each other. Since x_i and y_j have θ common to their first associates but no common lettered first associates, they must have $p_{11}^2 - 1$ common numbered first associates. Since x_i and y_j appear only in the θ-blocks and in the sets under consideration, these two sets must contain the $p_{11}^2 - 1$ common numbered first associates of x_i and y_j. This proves the corollary.

Corollary 5.5. *Under the conditions of Lemma* 5.1 *a mixed block cannot intersect a set belonging to another group in more than one treatment. If* $n_2 = (r - 1) \cdot (k - 1)^2$, *then each mixed block must intersect each set of another group in exactly one numbered treatment.*

Proof. Suppose a mixed block intersects a set of another group in two or more treatments. Then consider a pair of numbered treatments common to the given mixed block and the intersected set of blocks belonging to another group. These two numbered treatments are first associates of each other since they occur together in a block. Their common first associates will include the $k - 2$ other treatments in the block containing them both and also the lettered treatment occurring in all blocks of the set intersected. Hence $p_{11}^1 \geqq k - 1$, in contradiction to the hypothesis. This proves the first part of the corollary.

Each group must contain a complete replication of numbered treatments when $n_2 = (r - 1)(k - 1)^2$, and we have proved that a mixed block cannot intersect a set of another group in more than one treatment. Therefore, when $n_2 = (r - 1)(k - 1)^2$, the $k - 1$ numbered treatments in a mixed block must be distributed one to each set in the other groups. This proves the second part of the corollary.

Definition. Two number-pairs may be defined to be *set compatible* if it is possible for them to occur in the same set without violating Lemma 5.1 and its corollaries.

6. The relationship of duality between members of a certain two-parameter series of PBIB designs. Let D be a design with a known solution. To form a design D^*, let the treatments of D be the blocks of D^*, and the blocks of D

be the treatments of D^*. The design D^* has been called the *dual* of design D. The process described is sometimes called *inverting* or *dualizing the* design D.

In general, the dual of a PBIB design is not itself a PBIB design. Inversion of a partially balanced design D can result in a dual design D^* having a different number of associate classes than design D. Inversion of a PBIB design D may yield a balanced incomplete block design.

Consider the two-parameter series of designs having

$$(6.1)\quad \begin{cases} v = k[(r-1)(k-1)+1], \quad r = r, \quad \lambda_1 = 1, \quad n_1 = r(k-1), \\ b = r[(r-1)(k-1)+1], \quad k = k, \quad \lambda_2 = 0, \quad n_2 = (r-1)(k-1)^2, \\ P_1 = \begin{pmatrix} k-2 & (r-1)(k-1) \\ (r-1)(k-1) & (r-1)(k-1)(k-2) \end{pmatrix}, \\ P_2 = \begin{pmatrix} r & r(k-2) \\ r(k-2) & k^2(r-1)-(3r-2)(k-1) \end{pmatrix}, \end{cases}$$

obtained by setting $t = 1$ in (4.27). Note that (6.1) satisfies the conditions of Lemma 5.1 and its five corollaries, with equality holding in Corollary 5.2.

Assuming the existence of a solution of some design belonging to (6.1), let us examine its dual. Let the parameters of the dual be distinguished by the asterisk as superscript. Clearly

$$(6.2)\quad \begin{aligned} v^* &= r[(r-1)(k-1)+1], \qquad r^* = k, \\ b^* &= k[(r-1)(k-1)+1], \qquad k^* = r. \end{aligned}$$

The r blocks of (6.1) containing treatment θ go over into r treatments appearing in a single block of the dual design. Since no pair of θ-blocks of (6.1) contain another common treatment, in the dual no treatment-pair corresponding to a pair of blocks containing θ can occur together in a block (excepting the one block corresponding to treatment θ). Hence

$$(6.3)\quad \lambda_1^* = 1, \qquad \lambda_2^* = 0.$$

Arbitrarily choose a block of the original design and a treatment within this block. There are $r - 1$ other blocks containing this treatment. The arbitrarily chosen treatment may be called θ and the r blocks containing θ are then the θ-blocks. The arbitrarily chosen θ-block intersects each of the other $r - 1$ θ-blocks in exactly one treatment. It also intersects each of the $(r - 1)(k - 1)$ blocks of one group in a single treatment, but it intersects no other blocks. Hence in the dual

$$(6.4)\quad n_1^* = k(r-1).$$

These n_1^* blocks in the original design (treatments in the dual) we shall call the first associates of the arbitrarily chosen block. Since the arbitrarily chosen block fails to intersect any of the blocks in the other $r - 1$ groups, in the dual

$$n_2^* = (k - 1)(r - 1)^2. \tag{6.5}$$

We shall call these n_2^* blocks the second associates of the arbitrarily chosen block.

Now consider any pair of blocks which are first associates. There is no loss in generality in assuming that they are the ith and jth blocks containing treatment θ for $i \neq j$; with $1 \leqq i$ and $j \leqq r$. The first associates of the ith block containing θ consist of all θ-blocks except the ith, and of all the blocks in the ith group (the ith group being the one whose lettered treatments occur in the ith θ-block). The first associates of the jth block containing θ consist of all θ-blocks except the jth, and of all blocks in the jth group. The blocks which are common to the first associates of the ith and jth θ-blocks are the other $r - 2$ θ-blocks. Hence,

$$p_{11}^{1*} = r - 2, \tag{6.6}$$

a constant for any particular design. Since the conditions of Theorem 3.1 are satisfied, it follows that p_{12}^{1*}, p_{21}^{1*}, and p_{22}^{1*} are constants, and from (3.4) and (3.5) we obtain

$$p_{12}^{1*} = p_{21}^{1*} = (r - 1)(k - 1), \tag{6.7}$$

$$p_{22}^{1*} = (r - 1)(r - 2)(k - 1). \tag{6.8}$$

Next consider a pair of blocks which are second associates of each other, that is, a pair of blocks containing no common treatment. There is no loss in generality if we take one of them to be the first θ-block and the other to be a block in the jth group, where $j \neq 1$. The first associates of the first θ-block are the $r - 1$ other θ-blocks and all the blocks in the first group. The first associates of a block in the jth group consist of the $r - 2$ other blocks in the same set of the jth group, one block (not the first) among the θ-blocks, and $k - 1$ blocks in each of the r groups excepting the jth. Hence

$$p_{11}^{2*} = k, \tag{6.9}$$

a constant for a particular design. Thus it is seen that the conditions of Theorem 3.2 are satisfied, and hence, p_{12}^{2*}, p_{21}^{2*}, and p_{22}^{2*} are constants. Furthermore, from (3.6) and (3.7) we obtain

$$p_{12}^{2*} = p_{21}^{2*} = k(r - 2), \tag{6.10}$$

$$p_{22}^{2*} = r^2(k - 1) - (3k - 2)(r - 1). \tag{6.11}$$

In the original design, any pair of blocks which are first associates interesect in a single treatment, so $\lambda_1^* = 1$, while any pair of blocks which are second associates fail to intersect at all, so $\lambda_2^* = 0$.

We have shown that the conditions (i), (ii), and (iii) stated in the definition of a PBIB design with two associate classes are satisfied. Hence, the dual of design (6.1) is a PBIB design with two associate classes having parameters

$$(6.12)\begin{cases} v^* = r[(r-1)(k-1)+1], \quad r^* = k, \quad \lambda_1^* = 1, \quad n_1^* = k(r-1), \\ b^* = k[(r-1)(k-1)+1], \quad k^* = r, \quad \lambda_2^* = 0, \quad n_2^* = (k-1)(r-1)^2, \\ P_1^* = \begin{pmatrix} r-2 & (r-1)(k-1) \\ (r-1)(k-1) & (r-1)(r-2)(k-1) \end{pmatrix}, \\ P_2^* = \begin{pmatrix} k & k(r-2) \\ k(r-2) & r^2(k-1)-(3k-2)(r-1) \end{pmatrix}. \end{cases}$$

If in (6.12) we replace k with r^* and r with k^*, it is seen that the dual has the same form as the original design (6.1) and hence belongs to the same series. We have proved that the existence of the original design (6.1) implies the existence of (6.12) as the dual of (6.1). If we now assume the existence of (6.12), it can be shown by exactly the same type of argument that its dual exists and is given by (6.1). We have thus proved the following theorem.

THEOREM 6.1. *Between corresponding designs of* (6.1) *and* (6.12) *there exists a relationship of duality. They exist or fail to exist simultaneously.*

The theorem of duality implies that if a design belonging to one of the series (6.1) or (6.12) is impossible, then the corresponding design of the other series is also impossible. Suppose that there exists a solution of a design of one of the series, but that the corresponding design of the other series is impossible. Then by Theorem 6.1, upon inverting the design whose solution is known, we obtain the solution of the corresponding design of the other series. But this contradicts our assumption. Consequently, the first statement of this paragraph is true.

Actually, the series (6.1) and (6.12) are the same, since we may obtain (6.1) from (6.12) by interchanging r and k. However, the correspondence between designs of (6.1) and (6.12) gives a means of pairing off those designs which are duals of each other.

7. Constructions and impossibility proofs. Putting $k = 3$ in the two-parameter series of designs (6.1), we obtain the single-parameter series, with $r \geqq 2$,

$$(7.1)\begin{cases} v = 3(2r-1), \quad r = r, \quad \lambda_1 = 1, \quad n_1 = 2r, \\ b = r(2r-1), \quad k = 3, \quad \lambda_2 = 0, \quad n_2 = 4(r-1), \\ P_1 = \begin{pmatrix} 1 & 2(r-1) \\ 2(r-1) & 2(r-1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} r & r \\ r & 3r-5 \end{pmatrix}. \end{cases}$$

Putting $k = 3$ in the two-parameter series of designs (6.12), we obtain the single-parameter series

$$(7.2)\quad \begin{cases} v^* = r(2r-1), & r^* = 3, \quad \lambda_1^* = 1, \quad n_1^* = 3(r-1), \\ b^* = 3(2r-1), & k^* = r, \quad \lambda_2^* = 0, \quad n_2^* = 2(r-1)^2, \\ P_1^* = \begin{pmatrix} r-2 & 2(r-1) \\ 2(r-1) & 2(r-1)(r-2) \end{pmatrix}, & P_2^* = \begin{pmatrix} 3 & 3(r-2) \\ 3(r-2) & 2r^2-7r+7 \end{pmatrix}. \end{cases}$$

It follows from Section 6 that corresponding designs of (7.1) and (7.2) are duals of each other and exist or fail to exist simultaneously. Note that series (7.2) is identical with (4.32). It follows from (4.31) that designs of (7.2) corresponding to values of r other than 2, 3, 5, and 11 do not exist, and the same therefore must hold for designs of (7.1). In what follows we shall show that designs of (7.1) with $r \geqq 6$ are impossible, which rules out the case $r = 11$. We shall also give a construction for the case $r = 5$ for which the corresponding design for (7.2) can be obtained by dualization. The designs corresponding to $r = 2$ and 3 are known, and will be considered briefly at the end of this section.

We write the r θ-blocks of (7.1) as

$$\begin{matrix} \theta & a_1 & a_2 \\ \theta & b_1 & b_2 \\ \cdots & \cdots & \cdots \\ \theta & l_1 & l_2 \end{matrix}$$

Each group consists of $2(r-1)$ blocks, the lettered treatments within a group being denoted by the same letter bearing the appropriate subscript. Each group contains two sets and each set contains $r-1$ blocks, the blocks within a set containing the same lettered treatment (Corollary 5.1). Since $n_2 = 4(r-1)$, each group must contain exactly one complete replication of the numbered treatments (Corollary 5.2). Without loss of generality, we may write the first group containing lettered treatments a_1 and a_2 as

$$\left.\begin{matrix} a_1 & 1 & 2 \\ a_1 & 3 & 4 \\ a_1 & 5 & 6 \\ \cdots & \cdots & \cdots \\ a_1 & 2r-3 & 2r-2 \end{matrix}\right\} \quad r-1 \text{ blocks,}$$

$$\left.\begin{matrix} a_2 & 2r-1 & 2r \\ a_2 & 2r+1 & 2r+2 \\ a_2 & 2r+3 & 2r+4 \\ \cdots & \cdots & \cdots \\ a_2 & 4r-5 & 4r-4 \end{matrix}\right\} \quad r-1 \text{ blocks.}$$

There is no loss in generality in writing the b_1-set as

b_1	1	$2r - 1$
b_1	3	$2r + 1$
b_1	5	$2r + 3$
$\cdots$	$\cdots$	$\cdots$
b_1	$2r - 3$	$4r - 5$

where the b_1-set contains only the odd-numbered second associates of θ (Corollaries 5.4 and 5.5). Then, the b_2-set must contain all the even-numbered second associates of θ (Corollary 5.2). The treatments 2, 4, 6, $\cdots$, $2r - 2$ must occur one in each block of the b_2-set, the same being true for the $r - 1$ treatments $2r$, $2r + 2$, $2r + 4$, $\cdots$, $4r - 4$.

We shall now show that the pairing off of the even-numbered treatments in the b_2-set is uniquely determined. Since no pair of numbered treatments appearing in the a_1- or a_2-set can occur together in any block of the b-, c-, $\cdots$, and l-groups ($\lambda_1 = 1$, $\lambda_2 = 0$, and Lemma 5.1), we form a lattice (Diagram 1) having horizontal coordinates 1, 2, 3, $\cdots$, $2r - 2$ and vertical coordinates $2r - 1$, $2r$, $2r + 1$, $\cdots$, $4r - 4$. The coordinates of the cells of the lattice diagram give all conceivably possible number-pairs which might occur in the b-, c-, $\cdots$, and

	1	2	3	4	5	6	$\cdots$	$2r-5$	$2r-4$	$2r-3$	$2r-2$
$2r - 1$	P	X	X		X		$\cdots$	X		X	
$2r$	X	P		X		X	$\cdots$		X		X
$2r + 1$	X		P	X	X		$\cdots$	X		X	
$2r + 2$		X	X	P		X	$\cdots$		X		X
$2r + 3$	X		X		P	X	$\cdots$	X		X	
$2r + 4$		X		X	X	P	$\cdots$		X		X
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$4r - 7$	X		X		X		$\cdots$	P	X	X	
$4r - 6$		X		X		X	$\cdots$	X	P		X
$4r - 5$	X		X		X		$\cdots$	X		P	X
$4r - 4$		X		X		X	$\cdots$		X	X	P

DIAGRAM 1

l-groups. The cells whose coordinates appear as a pair in some block of the design are marked with a P and the cells whose coordinates are ruled out by Lemma 5.1 or one of its corollaries are marked with a X. The cells corresponding to the number-pairs appearing in the b_1-set are those lying in the upper left corner of the subsquares of side 2 lying along the main diagonal of the lattice diagram.

The number-pairs whose coordinates are the numbers occurring with i in the a_1- and b_1-sets, where $i = 1, 3, 5, \cdots, 2r - 3$, are ruled out by Lemma 5.1. So are the number-pairs whose coordinates are the numbers occurring with j in the a_2- and b_1-sets, where $j = 2r - 1, 2r + 1, 2r + 3, \cdots, 4r - 5$. The cells corresponding to these number-pairs lie in the upper right and lower left corners of the subsquares of side 2 lying along the main diagonal of the lattice diagram. Furthermore, Lemma 5.1 rules out the occurrence of the number-pairs indicated in Diagram 2.

	1	3	5	$\cdots$	$2r - 5$	$2r - 3$
$2r - 1$	---	$(3, 2r - 1)$	$(5, 2r - 1)$	$\cdots$	$(2r - 5, 2r - 1)$	$(2r - 3, 2r - 1)$
$2r + 1$	$(1, 2r + 1)$	---	$(5, 2r + 1)$	$\cdots$	$(2r - 5, 2r + 1)$	$(2r - 3, 2r + 1)$
$2r + 3$	$(1, 2r + 3)$	$(3, 2r + 3)$	---	$\cdots$	$(2r - 5, 2r + 3)$	$(2r - 3, 2r + 3)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$4r - 7$	$(1, 4r - 7)$	$(3, 4r - 7)$	$(5, 4r - 7)$	$\cdots$	---	$(2r - 3, 4r - 7)$
$4r - 5$	$(1, 4r - 5)$	$(3, 4r - 5)$	$(5, 4r - 5)$	$\cdots$	$(2r - 5, 4r - 5)$	---

DIAGRAM 2

The corresponding cells in the lattice (Diagram 1) are those lying in the odd numbered columns and odd numbered rows, with the exception of those lying along the main diagonal of the lattice diagram. In other words, they are all cells lying in the upper left corners of the subsquares of side 2, with the exception of those lying along the main diagonal of the lattice diagram.

Treatments $2r - 1, 2r + 1, 2r + 3, \cdots, 4r - 5$ must each occur in each of the remaining $r - 2$ groups (Corollary 5.2). The available number-pairs involving these treatments are given in Diagram 3. Since there are only $r - 2$ number-

	2	4	6	$\cdots$	$2r - 4$	$2r - 2$
$2r - 1$	---	$(4, 2r - 1)$	$(6, 2r - 1)$	$\cdots$	$(2r - 4, 2r - 1)$	$(2r - 2, 2r - 1)$
$2r + 1$	$(2, 2r + 1)$	---	$(6, 2r + 1)$	$\cdots$	$(2r - 4, 2r + 1)$	$(2r - 2, 2r + 1)$
$2r + 3$	$(2, 2r + 3)$	$(4, 2r + 3)$	---	$\cdots$	$(2r - 4, 2r + 3)$	$(2r - 2, 2r + 3)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$4r - 7$	$(2, 4r - 7)$	$(4, 4r - 7)$	$(6, 4r - 7)$	$\cdots$	---	$(2r - 2, 4r - 7)$
$4r - 5$	$(2, 4r - 5)$	$(4, 4r - 5)$	$(6, 4r - 5)$	$\cdots$	$(2r - 4, 4r - 5)$	---

DIAGRAM 3

pairs involving each of the treatments $2r - 1, 2r + 1, 2r + 3, \cdots$, and $4r - 5$, each of the number-pairs in the Diagram 3 must occur in the design.

Comparing the available number-pairs of Diagram 3 with those in the a_2-set, and using Lemma 5.1, rules out the number-pairs indicated in Diagram 4. Thus

	2	4	6	$\cdots$	$2r-4$	$2r-2$
$2r$	---	$(4, 2r)$	$(6, 2r)$	$\cdots$	$(2r-4, 2r)$	$(2r-2, 2r)$
$2r+2$	$(2, 2r+2)$	---	$(6, 2r+2)$	$\cdots$	$(2r-4, 2r+2)$	$(2r-2, 2r+2)$
$2r+4$	$(2, 2r+4)$	$(4, 2r+4)$	---	$\cdots$	$(2r-4, 2r+4)$	$(2r-2, 2r+4)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$4r-6$	$(2, 4r-6)$	$(4, 4r-6)$	$(6, 4r-6)$	$\cdots$	---	$(2r-2, 4r-6)$
$4r-4$	$(2, 4r-4)$	$(4, 4r-4)$	$(6, 4r-4)$	$\cdots$	$(2r-4, 4r-4)$	---

DIAGRAM 4

in the lattice (Diagram 1) we cross out the cells lying in the lower right corners of all subsquares of side 2, except for those lying along the main diagonal of the lattice diagram.

Examination of the lattice diagram now shows that the only cells for which both coordinates are even numbers are those lying along the main diagonal, that is, cells whose coordinates are

$$(2, 2r), \quad (4, 2r+2), \quad \cdots, \quad (2r-2, 4r-4).$$

Thus the pairing off of the numbered treatments appearing in the b_2-set is uniquely determined. The $r-1$ blocks of the b_2-set are:

b_2	2	$2r$
b_2	4	$2r+2$
b_2	6	$2r+4$
$\cdots$	$\cdots$	$\cdots$
b_2	$2r-2$	$4r-4.$

The occurrence of the above number-pairs in the b_2-set rules out only number-pairs which have been previously excluded.

Each of the treatments $2r, 2r+2, 2r+4, \cdots, 4r-4$ must occur in each of the remaining $r-2$ groups (Corollary 5.2). The only available number-pairs involving these treatments are given in Diagram 5. Again, there are only $r-2$ pairs involving the treatments in question. Hence, each of the number-pairs in

	1	3	5	$\cdots$	$2r-5$	$2r-3$
$2r$	---	$(3, 2r)$	$(5, 2r)$	$\cdots$	$(2r-5, 2r)$	$(2r-3, 2r)$
$2r+2$	$(1, 2r+2)$	---	$(5, 2r+2)$	$\cdots$	$(2r-5, 2r+2)$	$(2r-3, 2r+2)$
$2r+4$	$(1, 2r+4)$	$(3, 2r+4)$	---	$\cdots$	$(2r-5, 2r+4)$	$(2r-3, 2r+4)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$4r-6$	$(1, 4r-6)$	$(3, 4r-6)$	$(5, 4r-6)$	$\cdots$	---	$(2r-3, 4r-6)$
$4r-4$	$(1, 4r-4)$	$(3, 4r-4)$	$(5, 4r-4)$	$\cdots$	$(2r-5, 4r-4)$	---

DIAGRAM 5

Diagram 5 must occur in the design. Furthermore, these number-pairs and those in Diagram 3 involving treatments $2r - 1, 2r + 1, 2r + 3, \cdots, 4r - 5$ are the only available number-pairs.

In each of the number-pair diagrams of available pairs, it is obvious that number-pairs lying in the same row or in the same column are not set compatible. Furthermore, the occurrence in any set of two unsymmetrically located number-pairs with respect to the main diagonal of the number-pair diagram would rule out the occurrence of the number-pair lying at the interesection of the rows and columns containing the two number-pairs in question (Corollary 5.5). Hence, only symmetrically located pairs with respect to the main diagonal of the number-pair diagram can possibly be set compatible.

Now each set contains $r - 1$ blocks and consequently requires $r - 1$ number-pairs. But we have available a maximum of 4 possibly set-compatible pairs, two from each number-pair diagram. Hence, all designs of the series (7.1) having $r \geqq 6$ are combinatorially impossible.

We shall now give a construction for the design of (7.1) corresponding to $r = 5$. Its parameters are

$$\text{(7.3)} \qquad \begin{cases} v = 27, & r = 5, \quad \lambda_1 = 1, \quad n_1 = 10, \\ b = 45, & k = 3, \quad \lambda_2 = 0, \quad n_2 = 16, \\ P_1 = \begin{pmatrix} 1 & 8 \\ 8 & 8 \end{pmatrix}, & P_2 = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}. \end{cases}$$

By the preceding argument it is seen that we may write, without loss of generality, the five θ-blocks, the eight mixed blocks of the a-group, the eight blocks of the b-group, and the distribution of the lettered treatments throughout the remaining mixed blocks as shown below. The placing of the underlined numbered treatments comes later.

θ	a_1	a_2
θ	b_1	b_2
θ	c_1	c_2
θ	d_1	d_2
θ	e_1	e_2

a-Group			b-Group			c-Group			d-Group			e-Group		
a_1	1	2	b_1	1	9	c_1	$\underline{4}$	$\underline{9}$	d_1	$\underline{6}$	$\underline{9}$	e_1	$\underline{8}$	$\underline{9}$
a_1	3	4	b_1	3	11	c_1	$\underline{2}$	$\underline{11}$	d_1	$\underline{2}$	$\underline{13}$	e_1	$\underline{2}$	$\underline{15}$
a_1	5	6	b_1	5	13	c_1	$\underline{7}$	$\underline{14}$	d_1	$\underline{7}$	$\underline{12}$	e_1	$\underline{5}$	$\underline{12}$
a_1	7	8	b_1	7	15	c_1	$\underline{5}$	$\underline{16}$	d_1	$\underline{3}$	$\underline{16}$	e_1	$\underline{3}$	$\underline{14}$
a_2	9	10	b_2	2	10	c_2	$\underline{8}$	$\underline{13}$	d_2	$\underline{8}$	$\underline{11}$	e_2	$\underline{6}$	$\underline{11}$
a_2	11	12	b_2	4	12	c_2	$\underline{6}$	$\underline{15}$	d_2	$\underline{4}$	$\underline{15}$	e_2	$\underline{4}$	$\underline{13}$
a_2	13	14	b_2	6	14	c_2	$\underline{3}$	$\underline{10}$	d_2	$\underline{5}$	$\underline{10}$	e_2	$\underline{7}$	$\underline{10}$
a_2	15	16	b_2	8	16	c_2	$\underline{1}$	$\underline{12}$	d_2	$\underline{1}$	$\underline{14}$	e_2	$\underline{1}$	$\underline{16}$

DIAGRAM 6.—Plan for design (7.3)

From Diagram 3 of available number-pairs it is seen that the only available number-pairs involving treatments 2, 4, 6, and 8 are those shown in Diagram 7.

	2	4	6	8
9	---	(4, 9)	(6, 9)	(8, 9)
11	(2, 11)	---	(6, 11)	(8, 11)
13	(2, 13)	(4, 13)	---	(8, 13)
15	(2, 15)	(4, 15)	(6, 15)	---

DIAGRAM 7

From Diagram 5 of available number-pairs we also see that the only available number-pairs involving treatments 1, 3, 5, and 7 are those shown in Diagram 8.

	1	3	5	7
10	---	(3, 10)	(5, 10)	(7, 10)
12	(1, 12)	---	(5, 12)	(7, 12)
14	(1, 14)	(3, 14)	---	(7, 14)
16	(1, 16)	(3, 16)	(5, 16)	---

DIAGRAM 8

Since these are the only available number-pairs and since each of the numbered treatments must occur in each of the c-, d-, and e-groups, by Corollary 5.2 the design must contain all the number-pairs in these two arrays (Diagrams 7 and 8).

Now, by Corollary 5.2 the number-pairs (4, 9), (6, 9), and (8, 9) must be distributed one in each of the last three groups. Suppose we arbitrarily form the three blocks

$$c_1 \quad 4 \quad 9$$
$$d_1 \quad 6 \quad 9$$
$$e_1 \quad 8 \quad 9.$$

Since each set must contain four number-pairs and since only symmetrically located pairs with respect to the main diagonal of the number-pair diagram are set compatible, we must also form the blocks:

$$c_1 \quad 2 \quad 11$$
$$d_1 \quad 2 \quad 13$$
$$e_1 \quad 2 \quad 15.$$

The occurrence of the pairs (4, 9) and (2, 11) in the c_1-set precludes, by Corollary 5.5, the occurrence in this set of pairs involving 1, 3, 10, and 12. Thus, the c_1-set must be completed with (5, 16) and (7, 14). Then by Corollary 5.2, we must complete the c_2-set by use of pairs (8, 13), (6, 15), (3, 10) and (1, 12).

The occurrence of pairs (6, 9) and (2, 13) in the d_1-set precludes, by Corollary 5.5, the occurrence in the same set of any pairs involving 1, 4, 5, 8, 10, 11, 14, and 15. Thus, it is seen that the only pairs which can occur with (6, 9) and (2, 13) in the d_1-set are (7, 12) and (3, 16). Then by Corollary 5.2 the d_2-set must contain pairs (8, 11), (4, 15), (5, 10), and (1, 14).

Since the e_1-set contains pairs (8, 9) and (2, 15), Corollary 5.5 rules out the occurrence in the e_1-set of any pair involving 1, 7, 10, 16, 4, 6, 11, or 13. Thus, it is seen that only (5, 12) and (3, 14) are available. Then by Corollary 5.2 the pairs (6, 11), (4, 13), (7, 10), and (1, 16) must occur in the e_2-set. This completes the construction of design (7.3). The conditions set forth in Lemma 5.1 and in Corollaries 5.1 through 5.5 are satisfied, and Theorems 3.1 and 3.2 are also satisfied.

When $r = 2$, the design (7.1) is a lattice design obtained by arranging 9 treatments in a 3×3 square and taking the rows and columns for blocks. The corresponding design of (7.2) is the dual of this and is a group divisible design [3] with 6 treatments divided into two groups, say 1, 3, 5 and 2, 4, 6. The 9 blocks are obtained by taking all possible pairs consisting of one treatment from each group.

When $r = 3$, both series (7.1) and (7.2) lead to the same symmetrical (self-dual) design with 15 treatments and 15 blocks. This is a triangular design with known solution [3]. Hence we may state the following theorem:

THEOREM 7.1. *The series of designs* (7.1) *contains only three combinatorially possible designs:*

(1) *the lattice design with* $v = 9$, $r = 2$, *and* $k = 3$;

(2) *the symmetrical triangular design with* $v = b = 15$, $r = k = 3$, $n_1 = 6$, $n_2 = 8$, $\lambda_1 = 1$, $\lambda_2 = 0$;

(3) *the design with parameters* (7.3) *and plan following* (7.3).

Combining the results of this section with those of Sections 4, 5, and 6, we may state the following theorem:

THEOREM 7.2. *Partially balanced incomplete block designs with two associate classes for which* $k > r = 3$, *with* $\lambda_1 = 1$ *and* $\lambda_2 = 0$, *must belong to one of the following classes:*

(a) *Designs obtained by dualizing balanced incomplete block designs with* $k = 3$ *and* $\lambda = 1$. *These designs belong to series* (4.34).

(b) *Lattice designs with three replications belonging to series* (4.33).

(c) *The design of series* (4.32) *for which* $k = 5$. *This is the dual of the design with parameters* (7.3).

REFERENCES

[1] R. C. BOSE, "Partially balanced incomplete block designs with two associate classes involving only two complete replications," *Bull. Calcutta Stat. Assn.*, Vol. 3 (1951), pp. 120–125.

[2] R. C. BOSE AND K. P. NAIR, "Partially balanced incomplete block designs," *Sankhyā*, Vol. 4 (1939), pp. 337–372.

[3] R. C. BOSE AND T. SHIMAMOTO, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.

[4] W. S. CONNOR AND W. H. CLATWORTHY, "Some theorems for partially balanced designs," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 100–112.

[5] K. R. NAIR AND C. R. RAO, "A note on partially balanced incomplete block designs," *Science and Culture*, Vol. 7 (1942), pp. 568–569.

[6] K. R. NAIR, "Certain inequality relationships among the combinatorial parameters of incomplete block designs," *Sankhyā*, Vol. 6 (1943), pp. 255–259.

[7] M. REISS, "Ueber eine steinerche combinatorische Aufgabe," *J. Reine Angew. Math.*, Vol. 56 (1895), pp. 326–344.

[8] S. S. SHRIKHANDE, "On the dual of some balanced incomplete block designs," *Biometrics*, Vol. 8 (1952), pp. 66–72.