of R than in the original populations. The R distributions are in fact quite similar if allowance is made for the difference in population standard deviations. Hence we can have quite a bit of confidence in using normal curve constants when making control charts for ranges for moderately skewed populations and small sample sizes.

THE STOCHASTIC CONVERGENCE OF A FUNCTION OF SAMPLE SUCCESSIVE DIFFERENCES[1]

By Lionel Weiss

University of Virginia

1. Summary and introduction. Let $f(x)$ be a bounded density function over the finite interval $[A, B]$ with at most a finite number of discontinuities. Let $X_1, X_2, \cdots, X_n$ be independent chance variables each with the density $f(x)$. Define $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ as the ordered values of $X_1, X_2, \cdots, X_n$, and T_i as $Y_{i+1} - Y_i$. Also define $R_n(t)$ as the proportion of the variates $T_1, \cdots, T_{n-1}$ not greater than $t/(n-1)$. We shall denote $[1 - \int_A^B f(x)e^{-tf(x)}\, dx]$ by $S(t)$, and $\sup_{t \geq 0} |R_n(t) - S(t)|$ by $V(n)$. Then it is shown that as n increases, $V(n)$ converges stochastically to zero. The relation of this result to other results is discussed.

2. Proof of the stochastic convergence of $V(n)$ to zero.

Lemma 1. *If for each given t, $R_n(t)$ converges stochastically to $S(t)$ as n increases, then $V(n)$ converges stochastically to zero.*

Proof. We must show that for any given positive numbers ϵ and δ, there is a positive integer $N(\epsilon, \delta)$ such that if $n > N(\epsilon, \delta)$, then $P[V(n) < \epsilon] > 1 - \delta$. We can find a finite set of values $t_0 < t_1 < \cdots < t_s$ such that

$$S(t_0) < \tfrac{1}{2}\epsilon, \qquad 1 - S(t_s) < \tfrac{1}{2}\epsilon, \qquad S(t_{i+1}) - S(t_i) < \tfrac{1}{2}\epsilon, \qquad i = 0, 1, \cdots, s-1.$$

Also, by the hypothesis of the lemma and other familiar considerations, we can find a positive integer, say $N(\epsilon, \delta)$, such that if $n > N(\epsilon, \delta)$,

$$P[|R_n(t_i) - S(t_i)| < \tfrac{1}{2}\epsilon \quad \text{for} \quad i = 0, \cdots, s] > 1 - \delta.$$

But then the lemma is proved, for it is easily verified that if $|R_n(t_i) - S(t_i)| < \frac{1}{2}\epsilon$ simultaneously for $i = 0, \cdots, s$, then $|R_n(t) - S(t)| < \epsilon$ simultaneously for all $t \geqq 0$.

Lemma 2. *Let $X_1, \cdots, X_n$ be independent chance variables each with a uniform distribution on* $[0, 1]$. *Let M denote the number of these variables falling in the closed*

Received August 6, 1954.

[1] Research under a grant from the Institute for Research in the Social Sciences, University of Virginia.

interval $[C, D]$, *where* $0 \leqq C < D \leqq 1$, *and let* $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_M$ *denote the ordered values of the variables in* $[C, D]$. *Define* $W_0 = Y_1 - A$, *and* $W_i = Y_{i+1} - Y_i$ *for* $i = 1, \cdots, M - 1$. *Finally, define* $L(n, t)$ *as the total number of values of* $W_1, \cdots, W_{M-1}$ *which are not greater than* $t / (n - 1)$ *for a given* $t \geqq 0$. *Then* $L(n, t) / (n - 1)$ *converges stochastically to* $(D - C)[1 - e^{-t}]$ *as n increases.*

PROOF. We denote $(D - C)$ by G, and by $K(n, t)$ the total number of $W_0, \cdots, W_{M-1}$ not greater than $t / (n - 1)$. Clearly, the lemma will be proved if we can show that $K(n, t) / (n - 1)$ converges stochastically to $G(1 - e^{-t})$ as n increases. The distribution of M is binomial, with parameters G and n, and the joint conditional density of $Y_1, \cdots, Y_M$ given M is $M! / G^M$ in the region $C \leqq Y_1 \leqq \cdots \leqq Y_M \leqq D$, and zero elsewhere. Thus the joint conditional density of $W_0, \cdots, W_{M-1}$ given M is $M! / G^M$ in the region $W_i \geqq 0$ and $\sum_{i=0}^{M-1} W_i \leqq G$.

Define Z_i to be 1 if $W_i \leqq t / (n - 1)$, and zero otherwise. By the symmetry of the joint conditional distribution of $W_0, \cdots, W_{M-1}$, $E\, K(n, t) = E\{M \cdot E[Z_0 \mid M]\}$. The conditional density of W_0 given M is $M(G - w)^{M-1} / G^M$ for $0 \leqq w \leqq G$. Thus $E[Z_0 \mid M] = 1 - (1 - t / (n - 1)G)^M$, assuming $G \geqq t / (n - 1)$, which involves no loss of generality, since G is fixed and we are interested in what happens as n increases. By routine manipulations of the moment generating function of M, we find that

$$\begin{aligned} EK(n, t) &= E\{M[1 - (1 - t / (n - 1)G)^M]\} \\ &= nG - [1 - t / (n - 1)]^{n-1}[nG - nt / (n - 1)], \end{aligned}$$

From this, we find that $E[K(n, t) / (n - 1)]$ approaches $G(1 - e^{-t})$ as n increases. Next we examine

$$\begin{aligned} E\left[\frac{K(n, t)}{n - 1}\right]^2 &= \left(\frac{1}{n - 1}\right)^2 E\left(\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} Z_i Z_j\right) \\ &= \left(\frac{1}{n - 1}\right)^2 E\left(\sum_{0}^{M-1} Z_i^2\right) + \left(\frac{1}{n - 1}\right)^2 E\left(\sum_{i \neq j}\sum Z_i Z_j\right). \end{aligned}$$

But $\sum Z_i^2 = \sum Z_i$, and from above we have that $E[\sum Z_i / (n - 1)]$ approaches $G(1 - e^{-t})$ an n increases. Therefore $E[K(n, t) / (n - 1)]^2$ has the same limit as

$$\left(\frac{1}{n - 1}\right)^2 E\left[\sum_{i \neq j}\sum Z_i Z_j\right] = \left(\frac{1}{n - 1}\right)^2 E[M(M - 1) \cdot E(Z_0 Z_1 \mid M)].$$

This last equality holds because of the symmetry of the distribution of $W_0, \cdots, W_{M-1}$. The joint conditional density of W_0, W_1 given M is $M(M - 1)(G - w_0 - w_1)^{M-2} / G^M$ for $w_0, w_1 \geqq 0$ and $w_0 + w_1 \leqq G$. Thus

$$E(Z_0 Z_1 \mid M) = 1 - 2\left(1 - \frac{t}{(n - 1)G}\right)^M + \left(1 - \frac{2t}{(n - 1)G}\right)^M,$$

provided $G \geqq 2t / (n - 1)$, which involves no loss of generality. Therefore $E[K(n, t) / (n - 1)]^2$ has the same limit as

$$\left(\frac{1}{n-1}\right)^2 E\left\{M(M-1)\left[1 - 2\left(1 - \frac{t}{(n-1)G}\right)^M + \left(1 - \frac{2t}{(n-1)G}\right)^M\right]\right\}$$
$$= \left[\frac{nG^2}{n-1}\right]\left[1 - 2\left(1 - \frac{t}{(n-1)G}\right)^2\left(1 - \frac{t}{(n-1)}\right)^{n-2} + \left(1 - \frac{2t}{(n-1)G}\right)^2\left(1 - \frac{2t}{(n-1)}\right)^{n-2}\right].$$

This last expression approaches $G^2[1 - 2e^{-t} + e^{-2t}] = [G(1 - e^{-t})]^2$ as n increases. But this proves Lemma 2, since the variance of $K(n, t) / (n - 1)$ approaches zero as n increases.

With a few simple changes in notation, Lemma 1 serves to show that $\sup_{t \geqq 0} |L(n, t) / (n - 1) - G(1 - e^{-t})|$ converges stochastically to zero as n increases. Also, when $G = 1$, Lemmas 1 and 2 prove that $V(n)$ converges stochastically to zero for the special case where $f(x) = 1$ on the interval [0, 1].

Now we turn to the proof that $V(n)$ converges to zero in the general case. We denote $\int_A^x f(x)\, dx$ by $F(x)$. By the assumptions about $f(x)$ listed in Section 1, given any positive number γ, we can break the interval $[A, B]$ into a finite number $k(\gamma)$ of subintervals $(a_0, a_1), \cdots, (a_{k(\gamma)-1}, a_{k(\gamma)})$, with $a_0 = A$ and $a_{k(\gamma)} = B$, such that in the interior of each subinterval (a_i, a_{i+1}), $f(x)$ is continuous, and for any x in the subinterval, $|f(x) - f(a_i)| < \gamma$ for $i = 0, \cdots, k(\gamma) - 1$. Choose any particular subinterval (a_i, a_{i+1}), and let $Q_1 \leqq Q_2 \leqq \cdots \leqq Q_M$ denote the ordered values of those variables $X_1, \cdots, X_n$ which fall in (a_i, a_{i+1}), while T'_j shall denote $Q_{j+1} - Q_j$ for $j = 1, \cdots, M - 1$. Denote $F(Q_{j+1}) - F(Q_j)$ by W_j. Then, defining $L_i(n, t)$ in terms of $W_1, \cdots, W_{M-1}$ as in Lemma 2, with the subscript i to show that we are dealing with the interval (a_i, a_{i+1}), we have that $\sup_{t \geqq 0} | L_i(n, t)/(n - 1) - \{F(a_{i+1}) - F(a_i)\}(1 - e^{-t}) |$ converges stochastically to zero as n increases. (This is so because $F(X_j)$ has the rectangular distribution over (0, 1) for any j.) By construction, we have

$$W_j = F(Q_{j+1}) - F(Q_j) = T'_j(f(a_i) + \theta_j), \qquad | \theta_j | < \gamma.$$

Therefore, if $T'_j \leqq t/(n - 1)$, then $W_j \leqq (f(a_i) + \gamma)t/(n - 1)$, and conversely. Then, letting $K_i(n, t)$ denote the number of the values T'_j which are not greater than $t/(n - 1)$, we have

$$L_i(n, t(f(a_i) - \gamma)) \leqq K_i(n, t) \leqq L_i(n, t(f(a_i) + \gamma)).$$

Using $R_n(t)$ as defined in Section 1, this becomes

$$\frac{1}{n-1}\sum_{i=0}^{k(\gamma)-1} L_i(n, t(f(a_i) - \gamma)) - k(\gamma) \leqq R_n(t)$$
$$\leqq \frac{1}{n-1}\sum_{i=0}^{k(\gamma)-1} L_i(n, t(f(a_i) + \gamma)) + k(\gamma).$$

Given any positive values ϵ, δ, we first choose γ so small that

$$\left| \sum_{i=0}^{k(\gamma)-1} (F(a_{i+1}) - F(a_i))e^{-tf(a_i)}e^{t\bar{\gamma}} - \int_A^B f(x)e^{-tf(x)}\,dx \right| < \tfrac{1}{2}\varepsilon$$

for $\bar{\gamma}$ equal to either γ or $-\gamma$. Then we choose $N(\epsilon, \delta)$ so large that if $n > N(\epsilon, \delta)$, then $k(\gamma)/(n - 1) < \frac{1}{4}\,\epsilon$, and also

$$P[\sup_{t\geqq 0} \mid L_i(n, t)/(n - 1) - \{F(a_{i+1}) - F(a_i)\}(1 - e^{-t}) \mid < \epsilon/4k(\gamma),$$
$$i = 0, \cdots, k(\gamma) - 1] > 1 - \delta.$$

But then if $n > N(\epsilon, \delta)$,

$$P\left[\begin{array}{l} \sum_{i=0}^{k(\gamma)-1} \{F(a_{i+1}) - F(a_i)\}(1 - e^{-tf(a_i)}e^{t\gamma}) - \frac{1}{2}\epsilon \leqq R_n(t) \\ \qquad \leqq \sum_{i=0}^{k(\gamma)-1} \{F(a_{i+1}) - F(a_i)\}(1 - e^{-tf(a_i)}e^{-t\gamma}) + \frac{1}{2}\varepsilon \end{array}\right] > 1 - \delta,$$

or $P\{ \mid R_n(t) - S(t) \mid < \epsilon \} > 1 - \delta$. This shows that for any given t, $R_n(t)$ converges stochastically to $S(t)$. Then by Lemma 1, $V(n)$ converges stochastically to zero.

The same results hold with only slight modifications in the argument when $A = -\infty$ and/or $B = \infty$, provided that there exist finite numbers A' and B', with $A' < B'$, such that $f(x)$ is nondecreasing in the interval $(-\infty, A')$ and is nonincreasing in the interval (B', ∞).

3. Relation to other results. The stochastic convergence of certain functions of $T_1, \cdots, T_{n-1}$ can be proved simply by the use of these results. For example, Sherman [1] studied the chance variable Ω_n defined as

$$\tfrac{1}{2}\sum_{i=1}^{n-1} \left| T_i - \frac{1}{n+1} \right| + \tfrac{1}{2}\left| Y_1 - A \right| + \tfrac{1}{2}\left| B - Y_n \right|.$$

Let us assume that A is the least upper bound of all numbers a such that $F(a) = 0$, and B is the greatest lower bound of all numbers b such that $F(b) = 1$. Then as n increases, $\mid Y_1 - A \mid$ and $\mid B - Y_n \mid$ converge stochastically to zero. Thus Ω_n converges stochastically to a constant if and only if $U_n = \frac{1}{2}\sum_{i=1}^{n-1} \mid T_i - 1/(n - 1) \mid$ converges stochastically to the same constant. We can write U_n as $S_n + V_n$ where

$$S_n = \tfrac{1}{2} \sum_{i: T_i \leqq 1/(n-1)} \left\{\frac{1}{n-1} - T_i\right\}, \qquad V_n = \tfrac{1}{2} \sum_{i: T_i > 1/(n-1)} \left\{T_i - \frac{1}{n-1}\right\}$$

But $\frac{1}{2}\sum_{i=1}^{n-1}\{T_i - 1/(n - 1)\} = V_n - S_n$ converges stochastically to $\frac{1}{2}(B - A - 1)$. Thus U_n converges stochastically to a constant if and only if S_n converges stochastically to a constant. We can write

$$S_n = \frac{n-1}{2}\int_0^1 \left(\frac{1}{n-1} - \frac{t}{n-1}\right) dR_n(t) = \tfrac{1}{2}\left[R_n(1) - \int_0^1 t\,dR_n(t)\right].$$

Integrating by parts, we find that $S_n = \frac{1}{2}\int_0^1 R_n(t)\,dt$. By the result proved in Section 2 this last expression converges stochastically to

$$\frac{1}{2}\int_0^1 \left[1 - \int_A^B f(x)\, e^{-tf(x)}\, dx\right] dt = \frac{1}{2}\left[1 + \int_A^B e^{-f(x)}\, dx - (B - A)\right].$$

Therefore Ω_n converges stochastically to $\frac{1}{2}(1 + A - B) + \int_A^B e^{-f(x)}\, dx$. For the special case $A = 0$ and $B = 1$, this is essentially the result contained in theorems 3 and 4 of [1].

REFERENCE

[1] B. Sherman, "A random variable related to the spacing of sample values," *Ann. Math. Stat.*, Vol. 21 (1950), pp. 339–361.

Note added in proof. Professor Julius Blum has pointed out that Lemma 2 holds with the words "converges stochastically" replaced by "converges with probability one." Then it is easily seen that all the results above hold when this replacement is made.

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Chapel Hill meeting of the Institute, April 22–23, 1955)

1. Estimation of Location and Scale Parameters by Order Statistics from Singly and Doubly Censored Samples. Part I. The Normal Distribution up to Samples of Size 10. A. E. Sarhan and B. G. Greenberg, University of North Carolina.

The variances and covariances of the order statistics for samples of sizes $\leqq 20$ from a normal distribution were calculated to 10 decimal places from Teichroew's tables of the expected value of the product of two order statistics. By the use of these values, and with the table of expected values of Rosser, the best linear estimates of the mean and standard deviation were calculated from singly and doubly censored samples up to samples of size 10. This was accomplished by applying the method of least squares to the linear combination of the ordered known observations to obtain unbiased estimates with minimum variance. The variances of the estimates were also calculated. An alternative linear estimate was derived for larger values of n which can be used to obtain estimates from doubly censored samples.

2. An Application of Chung's Lemma to the Kiefer-Wolfowitz Stochastic Approximation Procedure. Cyrus Derman, Syracuse University.

Let $M(x)$ be a strictly increasing regression function for $x < \theta$, and strictly decreasing regression function for $x > \theta$. Kiefer and Wolfowitz (Ann. Math. Stat., Vol. 23 (1952), pp. 462–466) suggested a recursive scheme for estimating θ. They proved, under certain regularity conditions, that their scheme converges stochastically to θ. Their conditions