STATISTICS AND SUBFIELDS[1]

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1. Introduction and summary. Let $(X, \mathbf{S}, \mu)$ be a probability measure space: X is set of points x, $\mathbf{S}$ is a field of subsets of X, and μ is a countably additive measure on $\mathbf{S}$ with $\mu(X) = 1$.[2] A *subfield* is a field $\mathbf{S}_0$ of subsets of X such that $\mathbf{S}_0 \subseteq \mathbf{S}$, that is, each $\mathbf{S}_0$-measurable set is also $\mathbf{S}$-measurable. A *statistic* is a function defined on X. There is no a priori restriction on the class of statistics; in particular, statistics are not necessarily real-valued, and a real-valued statistic is not necessarily an $\mathbf{S}$-measurable function. For any statistic f, let $\mathbf{S}_f$ denote the class of all sets which are $\mathbf{S}$-measurable and of the form $f^{-1}(B)$, where B is a subset of the range of f. The class $\mathbf{S}_f$ is clearly a subfield, and is called the subfield *induced by* f.

The induced subfield $\mathbf{S}_f$ plays a central role in the study of a statistic f, for the following reason. The probabilist or mathematical statistician is usually concerned not with the statistic f as such, but rather with the class of random variables (i.e., real-valued $\mathbf{S}$-measurable functions) which depend on x only through f, and, as is easily seen, this class of random variables is exactly the class of real-valued $\mathbf{S}_f$-measurable functions. In case the given statistic f is a random variable (and therefore itself an object of study), the argument just given continues to apply, because in this case f is necessarily an $\mathbf{S}_f$-measurable function.

This paper discusses certain measure-theoretic problems concerning the relations between subfields, subfields of the apparently special form $\mathbf{S}_f$, and statistics. The main problems, as also the main conclusions, are described in the following paragraphs. Most of the conclusions of the paper are valid only in the case when $(X, \mathbf{S})$ is (or may be taken to be) a euclidean sample space, that is, X is a Borel set of the m-dimensional euclidean space $(1 \leqq m \leqq \infty)$, and $\mathbf{S}$ is the field of Borel sets of X. It is assumed henceforth that this is the case under consideration.

There are two main problems. The first is whether every subfield is inducible by a statistic. This problem is discussed (in a more general setting) in [2], and the conclusions of the present paper complement those of [2].

It is shown here that every subfield is inducible by a statistic if and only if the sample space is discrete, that is to say, X is a countable set and $\mathbf{S}$ is the class of all subsets of X (Theorem 1). This result is, however, not quite relevant to situations where the natural equivalence relation between subfields is not identity but approximability to within sets of μ-measure zero. The equivalence relation

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[2] This paper uses some of the notation and terminology of the first part of [1]. In particular, all fields considered are understood to be countably additive.

referred to is defined as follows. A subfield $\mathbf{S}_1$ is a *contraction* of a subfield $\mathbf{S}_2$ if corresponding to each real-valued $\mathbf{S}_1$-measurable function f_1 there exists an $\mathbf{S}_2$-measurable function f_2 such that $f_2(x) = f_1(x)$ except on a set of μ-measure zero; we then write $\mathbf{S}_1 \subseteq \mathbf{S}_2$ $[\mathbf{S}, \mu]$. The subfields $\mathbf{S}_1$ and $\mathbf{S}_2$ are *equivalent* if each is a contraction of the other; we then write $\mathbf{S}_1 = \mathbf{S}_2$ $[\mathbf{S}, \mu]$. It is shown that, in fact, corresponding to any subfield $\mathbf{S}_0$ there exists an f such that $\mathbf{S}_f$ is equivalent to $\mathbf{S}_0$, and that this f may be taken to be a random variable (Theorem 2).

In the literature the notion of contraction (and the derived notion of equivalence) has been defined for statistics in two ways, which are here called contraction and functional contraction. A statistic f is a *contraction* of a statistic g if $\mathbf{S}_f$ is a contraction of $\mathbf{S}_g$ (that is, $\mathbf{S}_f \subseteq \mathbf{S}_g$ $[\mathbf{S}, \mu]$); f is a *functional contraction* of g (written $f \subseteq g$ $[\mathbf{S}, \mu]$) if there exists a function h on the range of g into that of f, and an $\mathbf{S}$-measurable set N with $\mu(N) = 0$, such that $f(x) = h(g(x))$ for x in $X - N$. (Cf. [3], [4].) It seems to the writer that for most (possibly all) technical purposes the relevant concept is contraction as just defined (cf. Lemmas 7.1 and 3.2 of [1]). However, functional contraction has simpler interpretations and greater intuitive appeal.

The second problem is the exact relation between contraction and functional contraction. It is shown that, in general, functional contraction does not imply contraction (Example 1), and also that contraction does not imply functional contraction (Example 2). If, however, both f and g are random variables, then $\mathbf{S}_f \subseteq \mathbf{S}_g$ $[\mathbf{S}, \mu]$ if and only if $f \subseteq g$ $[\mathbf{S}, \mu]$ (Theorem 3). It follows, in particular, that if the sample space is discrete, then contraction coincides with functional contraction.

The problems described above arose in connection with the theory of sufficiency, and the results have applications in that theory. It follows, for example (assuming that the sample space is euclidean and that the set of alternative distributions of the sample point is a dominated set), that if f is a necessary and sufficient statistic, then $\mathbf{S}_f$ is a necessary and sufficient subfield (Corollary 2).

The following are some general conclusions bearing on mathematical models for studies such as [1]. (a) The notion of a subfield, while certainly no less general than that of a statistic, is in fact no more general. (b) There is no loss of generality, or other disadvantage, in defining a statistic to be a random variable. On the contrary, admission of nonmeasurable functions to the discussion leads to inconsistencies between extension and functional extension—this seems undesirable. (c) If f is a random variable, it is immaterial whether f is regarded as a statistic or as a Borel-measurable transformation (cf. [1], p. 431). These satisfactory conclusions do not necessarily hold for an arbitrary space $(X, \mathbf{S}, \mu)$. An example given in [2] shows that at least (a) and (c) are not valid in the general case.

2. Theorems. Let R be the real line, and $\mathbf{R}$ be the class of Borel sets of R. In general, we shall denote the n-dimensional euclidean space by R^n and the class of Borel sets of R^n by $\mathbf{R}^n$ $(1 \leqq n \leqq \infty)$. The following well-known result (cf. [5], pp. 159–160) is stated here as a lemma for convenience of reference.

LEMMA 1. *There exists a one to one function* α_n *on* R^n *onto* R *such that* $A \varepsilon \mathbf{R}^n$ *implies* $\alpha_n(A) \varepsilon \mathbf{R}$ *and* $B \varepsilon \mathbf{R}$ *implies* $\alpha_n^{-1}(B) \varepsilon \mathbf{R}^n$ $(1 \leqq n \leqq \infty)$.

If f is a statistic on X into a space Y, and $\mathbf{C}$ is a class of subsets of Y, then $f^{-1}(\mathbf{C})$ denotes the class of all sets of X which are of the form $f^{-1}(B)$ with $B \varepsilon \mathbf{C}$. Clearly, $f^{-1}(\mathbf{C})$ is a field if and only if $\mathbf{C}$ is a field. A function f on X into R^n is said to be S-measurable if $f^{-1}(\mathbf{R}^n)$ is a subfield of S.

In this section and the following one, a number of results involving S-measurable functions (specifically, Lemmas 2, 4, 5, 6, 8, and 9; Theorems 2 and 3; Corollaries 1 and 3) are stated and proved in terms of real-valued functions. It can be seen from Lemma 1, or otherwise directly from the proofs, that these results are in fact valid for S-measurable functions in general.

LEMMA 2. *A function* f *on* X *into* R *is* S-*measurable if and only if* f *is an* $\mathbf{S}_f$-*measurable function.*

PROOF. Since $\mathbf{S}_f \subseteq \mathbf{S}$ in any case, $f^{-1}(\mathbf{R}) \subseteq \mathbf{S}_f$ implies $f^{-1}(\mathbf{R}) \subseteq \mathbf{S}$. Conversely, if f is S-measurable, then $A \varepsilon f^{-1}(\mathbf{R})$ implies $A \varepsilon \mathbf{S}_f$, by the definition of $\mathbf{S}_f$, so that $f^{-1}(\mathbf{R}) \subseteq \mathbf{S}_f$. This completes the proof.

THEOREM 1. *A necessary and sufficient condition that every subfield of* S *be inducible by a statistic is that* X *be a countable set.*

PROOF. Suppose first that X is countable, and let there be given a field $\mathbf{S}_0 \subseteq \mathbf{S}$. For each $x \varepsilon X$ let E_x be the intersection of all sets $A \subseteq X$ such that $x \varepsilon A$ and $A \varepsilon \mathbf{S}_0$. Let $D_1, D_2, \cdots$, be an enumeration of the sets E_x such that $D_i \cap D_j$ is empty for $i \neq j$ and $\bigcup_i D_i = X$. Define $f(x) = i$ for $x \varepsilon D_i$ $(i = 1, 2, \cdots)$. We shall show that $\mathbf{S}_0 = \mathbf{S}_f$.

Since X is separable in the discrete topology, the intersection of any collection of subsets of X equals the intersection of a countable subcollection. Hence $D_i \varepsilon \mathbf{S}_0$ for each i. Since $\mathbf{S}_f = \{f^{-1}(N) : N \subseteq I\}$ where I is the set of positive integers, and $f^{-1}(N) = \bigcup_{i \varepsilon N} D_i$, it follows that $A \varepsilon \mathbf{S}_f$ implies $A \varepsilon \mathbf{S}_0$. To prove the converse, choose and fix an $A \varepsilon \mathbf{S}_0$. Since $x \varepsilon E_x$ for all x, we have $A \subseteq \bigcup_{x \varepsilon A} E_x$; on the other hand, $E_x \subseteq A$ for each $x \varepsilon A$, so that $\bigcup_{x \varepsilon A} E_x \subseteq A$; hence $A = \bigcup_{x \varepsilon A} E_x = \bigcup_{i \varepsilon N} D_i = f^{-1}(N)$ for some $N \subseteq I$. Thus $A \varepsilon \mathbf{S}_0$ implies $A \varepsilon \mathbf{S}_f$. Hence $\mathbf{S}_f = \mathbf{S}_0$. Since $\mathbf{S}_0$ is arbitrary, the first part of the theorem is proved.

To prove the second part suppose that X is an uncountable set. Let $\mathbf{S}^*$ be the class of all sets A such that one of the sets A and $X - A$ is countable. Then $\mathbf{S}^*$ is a subfield of S such that for each $x \varepsilon X$ the set $\{x\}$ belongs to $\mathbf{S}^*$. Moreover, it can be shown that $\mathbf{S}^* \neq \mathbf{S}$, that is to say, there exists at least one $A \varepsilon \mathbf{S}$ such that neither A nor $X - A$ is countable. It follows from Lemma 1 of [2] that there exists no f such that $\mathbf{S}^* = \mathbf{S}_f$. This completes the proof.

A subfield $\mathbf{S}_0$ is *separable* if there exists a countable class $\mathbf{C}$ of subsets of X such that $\mathbf{S}_0$ is the field generated by $\mathbf{C}$. While S itself is separable, a given subfield $\mathbf{S}_0$ may or may not be separable.[3] However, we have:

LEMMA 3. *Corresponding to any subfield* $\mathbf{S}_0$ *there exists a separable subfield* $\mathbf{S}^*$ *such that* $\mathbf{S}^* = \mathbf{S}_0\ [\mathbf{S}, \mu]$.

PROOF. For the purposes of this proof only, for any two sets A and B in S

[3] The writer is indebted to Professor A. Dvoretzky for this remark.

write $A \backsim B$ if and only if $(A - B) \cup (B - A)$ is of μ-measure zero. Let $\{\mathcal{C}_\theta: \theta \,\varepsilon\, \Omega\}$ be the set of equivalence classes generated by the relation $\backsim$, where Ω is an index set of points θ. For any θ and δ in Ω define $\rho(\theta, \delta) = \mu(A - B) + \mu(B - A)$, where A and B are sets in $\mathcal{C}_\theta$ and $\mathcal{C}_\delta$ respectively. Since $\mathbf{S}$ is separable, Ω is a separable metric space under the metric ρ ([5], p. 168).

Let Ω_0 be the set of all θ such that $\mathcal{C}_\theta$ contains at least one $\mathbf{S}_0$-measurable set. Then Ω_0 is a nonempty subset of Ω and therefore separable. Let Ω^* be a countable subset of Ω_0 which is dense in Ω_0. For each θ in Ω^* let A_θ be an $\mathbf{S}_0$-measurable set in $\mathcal{C}_\theta$, and let $\mathbf{S}^*$ be the field generated by the class $\{A_\theta : \theta \,\varepsilon\, \Omega^*\}$. It is clear that $\mathbf{S}^*$ is a separable field, and that $\mathbf{S}^* \subseteq \mathbf{S}_0$. We proceed to show that $\mathbf{S}_0 \subseteq \mathbf{S}^*$ $[\mathbf{S}, \mu]$.

Choose and fix an $A \,\varepsilon\, \mathbf{S}_0$. By the definition of Ω_0, there exists a $\theta \,\varepsilon\, \Omega_0$ such that $A \,\varepsilon\, \mathcal{C}_\theta$. Since Ω^* is dense in Ω_0, there exists a sequence $\{\theta_n\}$ in Ω^* such that $\lim_{n\to\infty} \rho(\theta_n, \theta) = 0$. Letting f_0 denote the characteristic function of the set A, it follows from the definition of $\mathbf{S}^*$ that there exists a sequence $f_1, f_2, \cdots$ of $\mathbf{S}^*$-measurable characteristic functions such that $\lim_{n\to\infty} f_n = f_0$ in measure. Hence there exists a subsequence of $\{f_n\}$, say $\{g_n\}$, such that, except on an $\mathbf{S}$-μ-null set, $\lim_{n\to\infty} g_n(x) = f_0(x)$([5], p. 93). Let B be the set of all x such that $\lim_{n\to\infty} g_n(x) = 1$. Then B is $\mathbf{S}^*$-measurable, and $A \backsim B$. Since A is arbitrary, we conclude that $\mathbf{S}_0 \subseteq \mathbf{S}^*$ $[\mathbf{S}, \mu]$. This completes the proof.

LEMMA 4. *If* $\mathbf{S}^*$ *is a separable subfield, there exists an* $\mathbf{S}$*-measurable function* f *on* X *into* R *such that* $f^{-1}(\mathbf{R}) = \mathbf{S}^*$.

PROOF. Suppose that $\mathbf{S}^*$ is generated by $\mathbf{C} = \{A_1, A_2, \cdots\}$. Let ϕ_i be the characteristic function of A_i, and define $\psi(x) = (\phi_1(x), \phi_2(x), \cdots)$. Suppose that ψ takes values in the space R^n of points $r_{(n)} = (r_1, r_2, \cdots)$ for $1 \leqq n \leqq \infty$. Since $A_i = \{x : \phi_i(x) = 1\} = \psi^{-1}(B_i)$, where $B_i = \{r_{(n)} : r_i = 1\}$, we have $A_i \,\varepsilon\, \psi^{-1}(\mathbf{R}^n)$ for each i; hence $\mathbf{S}^* \subseteq \psi^{-1}(\mathbf{R}^n)$. On the other hand, since each ϕ_i is an $\mathbf{S}^*$-measurable function, ψ is $\mathbf{S}^*$-measurable also, so that $\psi^{-1}(\mathbf{R}^n) \subseteq \mathbf{S}^*$. Thus $\mathbf{S}^* = \psi^{-1}(\mathbf{R}^n)$; the lemma as stated now follows from Lemma 1 by taking $f = \alpha_n \psi$.

LEMMA 5. *If* f *is an* $\mathbf{S}$*-measurable function on* X *into* R, *then* $f^{-1}(\mathbf{R}) = \mathbf{S}_f$ $[\mathbf{S}, \mu]$.

PROOF. According to Lemma 2, $f^{-1}(\mathbf{R}) \subseteq \mathbf{S}_f$. We have therefore to show that $\mathbf{S}_f \subseteq f^{-1}(\mathbf{R})$ $[\mathbf{S}, \mu]$.

We recall that we have assumed $X \subseteq R^m$, $X \,\varepsilon\, \mathbf{R}^m$, and $\mathbf{S} = \{X \cap A : A \,\varepsilon\, \mathbf{R}^m\}$. Let α_m be the function described in Lemma 1, and write $\alpha_m(X) = Y$, $\alpha_m(\mathbf{S}) = \mathbf{T}$, $g(y) = f(\alpha_m^{-1}(y))$ for $y \,\varepsilon\, Y$. Then Y is a Borel set of the real line, $\mathbf{T}$ is the class of Borel sets of Y, g is a $\mathbf{T}$-measurable function on Y into R, and $f^{-1}(\mathbf{R}) = \alpha_m^{-1}(g^{-1}(\mathbf{R}))$, $\mathbf{S}_f = \alpha_m^{-1}(\mathbf{T}_g)$. Define $\nu(C) = \mu(\alpha_m^{-1}(C))$ for $C \,\varepsilon\, \mathbf{T}$. It is then easily seen that the desired conclusion is equivalent to $\mathbf{T}_g \subseteq g^{-1}(\mathbf{R})$ $[\mathbf{T}, \nu]$.

Choose and fix a set $A \,\varepsilon\, \mathbf{T}_g$. By definition of $\mathbf{T}_g$, there exists a set $B \subseteq R$ such that $g^{-1}(B) = A$. Now, since A is a Borel set, and g is a Borel measurable function, it follows from Lusin's theorem ([6], p. 72) that for each $k = 1, 2 \cdots$ there exists a set $A_k \,\varepsilon\, \mathbf{T}$ such that $A_k \subseteq A$, $\nu(A - A_k) < 1/k$, and $g(A_k) \,\varepsilon\, \mathbf{R}$.

Let $A_0 = \bigcup_k A_k$. Then $A_0 \varepsilon \mathbf{T}$, $A_0 \subseteq A$, $\nu(A - A_0) = 0$, and $g(A_0) = \bigcup_k g(A_k) = B_0$ (say) is a Borel set. Now, $g^{-1}(B_0) = g^{-1}(g(A_0)) \supseteq A_0$. Also, $B_0 = g(A_0) \subseteq g(A) = B$, so that $g^{-1}(B_0) \subseteq g^{-1}(B) = A$. Hence $C = g^{-1}(B_0)$ is a set such that $A_0 \subseteq C \subseteq A$, so that $\nu(A - C) = 0$; since C is a set in $g^{-1}(\mathbf{R})$, and since $A \ \varepsilon \mathbf{T}_g$ in this argument is arbitrary, it follows that $\mathbf{T}_g \subseteq g^{-1}(\mathbf{R})$ [$\mathbf{T}$, ν]. This completes the proof.

REMARK 1. The preceding argument shows that μ is a perfect measure, i.e., for each real-valued $\mathbf{S}$-measurable function f, corresponding to each set A in $\mathbf{S}_f$ there exists a B in $f^{-1}(\mathbf{R})$ such that $B \subseteq A$ and $\mu(A - B) = 0$. (Cf. [9], p. 18; also pp. 248–251.) Perfection is a little stronger than the property stated in Lemma 5. The fact that μ is perfect can be deduced, alternatively, from Theorem 1 of [9], p. 18, since $(X, \mathbf{S}, \mu)$ is a euclidean space.

REMARK 2. If X is an uncountable set, the "exact" form of Lemma 5 is false, that is to say, there do exist $\mathbf{S}$-measurable functions f for which $f^{-1}(\mathbf{R}) \neq \mathbf{S}_f$. This follows easily from the theory of analytic sets [7].

As an immediate consequence of Lemmas 3, 4, and 5 we have:

THEOREM 2. *Corresponding to any subfield* $\mathbf{S}_0$ *there exists an* $\mathbf{S}$*-measurable function f on X into R such that* $\mathbf{S}_f = \mathbf{S}_0$ [$\mathbf{S}$, μ].

The remainder of this section is devoted to showing that, for $\mathbf{S}$-measurable functions, contraction coincides with functional contraction (Theorem 3).

LEMMA 6. *If f is an* $\mathbf{S}$*-measurable function on X into R, and* $\mathbf{S}_f \subseteq \mathbf{S}_g$ [$\mathbf{S}$, μ], *then* $f \subseteq g$ [$\mathbf{S}$, μ].

PROOF. Since f is $\mathbf{S}_f$-measurable (cf. Lemma 2), the hypothesis $\mathbf{S}_f \subseteq \mathbf{S}_g$ [$\mathbf{S}$, μ] yields the existence of an $\mathbf{S}_g$-measurable function, h say, such that the set $\{x:f(x) \neq h(x)\}$ is $\mathbf{S}$-μ-null. Denote this last set by N, and let $g(X - N) = A$.

Since h is $\mathbf{S}_g$-measurable, it depends on x only through g (cf. Lemma 3.2 of [1]), say $h(x) = k(g(x))$ for all x. Define $k^* = k$ on A and $= \alpha$ on $g(X) - A$, where α is a point in $f(X)$. Then k^* is a function on the range of g into that of f such that $\{x:f(x) \neq k^*(g(x))\}$ is a subset of N; this completes the proof.

Let $\bar{\mathbf{S}}$ be the class of all sets of the form $A \cup C$ where A is $\mathbf{S}$-measurable and C is a subset of an $\mathbf{S}$-μ-null set, and define $\bar{\mu}(A \cup C) = \mu(A)$. Then $\bar{\mathbf{S}}$ is a field containing $\mathbf{S}$, $\bar{\mu}$ is a probability measure on $\bar{\mathbf{S}}$, and $\bar{\mu}(A) = \mu(A)$ for $A \ \varepsilon\ \mathbf{S}$. For any statistic f, $\bar{\mathbf{S}}_f$ is defined, as usual, as the class of all $\bar{\mathbf{S}}$-measurable sets of the form $f^{-1}(B)$. (Note. In general, $\bar{\mathbf{S}}_f$ is different from $\overline{(\mathbf{S}_f)}$.)

LEMMA 7. *If* $f \subseteq g$ [$\bar{\mathbf{S}}$, $\bar{\mu}$], *then* $\bar{\mathbf{S}}_f \subseteq \bar{\mathbf{S}}_g$ [$\bar{\mathbf{S}}$, $\bar{\mu}$].

PROOF. By hypothesis, there exists a function h on the range of g into that of f, and an $\bar{\mathbf{S}}$-$\bar{\mu}$-null set N such that $f(x) = h(g(x))$ on $X - N$. Choose and fix a set in $\bar{\mathbf{S}}_f$, say $A = f^{-1}(B)$. Define $A^* = g^{-1}(C)$, where $C = h^{-1}(B)$.

Write $N^* = \{x : f(x) \neq h(g(x))\}$. Then $N^* \subseteq N$, so that N^* is an $\bar{\mathbf{S}}$-$\bar{\mu}$-null set. We have $A^* \cap (X - N^*) = \{x:g \ \varepsilon\ h^{-1}(B), f = hg\} = \{x:hg \ \varepsilon\ B, f = hg\} = \{x: f \ \varepsilon\ B, f = hg\} = A \cap (X - N^*)$. Hence $A^* - A \subseteq N^*$ and $A - A^* \subseteq N^*$. Since A is $\bar{\mathbf{S}}$-measurable and $\bar{\mu}$ is complete on $\bar{\mathbf{S}}$, it follows that A^* is $\bar{\mathbf{S}}$-measurable (and therefore in $\bar{\mathbf{S}}_g$) and that A^* differs from A by a set of $\bar{\mu}$-measure zero. Since $A \ \varepsilon\ \bar{\mathbf{S}}_f$ is arbitrary, the lemma is proved.

LEMMA 8. *If g is an* $\mathbf{S}$*-measurable function on X into R, then* $\mathbf{S}_g = \bar{\mathbf{S}}_g$ [$\bar{\mathbf{S}}$, $\bar{\mu}$].

PROOF. Since $\mathrm{S} \subseteq \bar{\mathrm{S}}$, we have $\mathrm{S}_g \subseteq \bar{\mathrm{S}}_g$; and since g is S-measurable, $g^{-1}(\mathrm{R}) \subseteq \mathrm{S}_g$ by Lemma 2. Thus $g^{-1}(\mathrm{R}) \subseteq \mathrm{S}_g \subseteq \bar{\mathrm{S}}_g$. The desired conclusion can now be established by showing that $\bar{\mathrm{S}}_g \subseteq g^{-1}(\mathrm{R})$ $[\bar{\mathrm{S}}, \bar{\mu}]$. The demonstration of this last relation is essentially the same as the proof of the nontrivial part of Lemma 5, and so is omitted.

LEMMA 9. *If g is an* S*-measurable function on X into R, and $f \subseteq g$ $[\mathrm{S}, \mu]$, then* $\mathrm{S}_f \subseteq \mathrm{S}_g$ $[\mathrm{S}, \mu]$.

PROOF.
$$
\begin{aligned}
f \subseteq g\ [\mathrm{S}, \mu] &\leftrightarrow f \subseteq g\ [\bar{\mathrm{S}}, \bar{\mu}] \\
&\rightarrow \bar{\mathrm{S}}_f \subseteq \bar{\mathrm{S}}_g\ [\bar{\mathrm{S}}, \bar{\mu}] \qquad \text{(Lemma 7)} \\
&\leftrightarrow \bar{\mathrm{S}}_f \subseteq \mathrm{S}_g\ [\bar{\mathrm{S}}, \bar{\mu}] \qquad \text{(Lemma 8)} \\
&\rightarrow \mathrm{S}_f \subseteq \mathrm{S}_g\ [\bar{\mathrm{S}}, \bar{\mu}] \\
&\leftrightarrow \mathrm{S}_f \subseteq \mathrm{S}_g\ [\mathrm{S}, \mu].
\end{aligned}
$$

THEOREM 3. *Let f and g be* S*-measurable functions on X into R. Then* $\mathrm{S}_f \subseteq \mathrm{S}_g$ $[\mathrm{S}, \mu]$ *if an only if $f \subseteq g$* $[\mathrm{S}, \mu]$.

The proof is immediate from Lemmas 6 and 9.

It can be shown by the methods used in this section that Theorems 1, 2, and 3 are valid for any probability space (X, S, μ) which satisfies the following conditions: (i) for each x in X, $\{x\}$ is S-measurable, (ii) S is separable, and (iii) μ is perfect. However, such a space can differ but little from a euclidean sample space.

3. Applications to the theory of sufficiency. We suppose now that there is given a euclidean sample space (X, S), as before, and a dominated set P of probability measures on S. Definitions of the technical terms used here without explanation are given in the first part of [1]. The conclusions of this section are relevant to problem 3 of [1], p. 441.

Let μ be an arbitrary but fixed probability measure, not necessarily in P, such that for each S-measurable set A, $\mu(A) = 0$ if and only if $p(A) = 0$ for each p in P. The existence of such a μ is assured by Lemma 7 of [8].

COROLLARY 1. *There exists a function f on X into R such that:*

(a) *f is* S*-measurable,*

(b) *S_f is a necessary and sufficient subfield,*

(c) *f is a necessary and sufficient statistic.*

PROOF. Since P is dominated, it follows from Theorem 6.2 of [1] that there exists a subfield S_0 (say) which is necessary and sufficient. Let f be a function on X into R such that (a) holds, and such that $\mathrm{S}_f = \mathrm{S}_0$ $[\mathrm{S}, \mu]$; such an f exists, by Theorem 2. Property (b) is immediate (cf. Corollary 6.2 (iii) of [1]), and it remains to verify (c). Since S_f is sufficient ($\equiv f$ is sufficient) by (b), we have only to show that f is a necessary statistic. Let g be any sufficient statistic. Then $\mathrm{S}_f \subseteq \mathrm{S}_g$ $[\mathrm{S}, \mu]$, since S_g is sufficient by hypothesis and S_f is necessary by (b). Hence $f \subseteq g$ $[\mathrm{S}, \mu]$, by (a) and Lemma 6. This completes the proof.

REMARK. It is evident from Lemma 5 that Corollary 1 remains valid if S_f is replaced by $f^{-1}(\mathrm{R})$ in (b). It can be shown that this modified version of Corollary 1 is valid not only in the present case but in any framework (X, S), P

provided that P is a separable metric space under the metric $\delta(p, q) = \sup_{A \varepsilon \mathbf{S}} |p(A) - q(A)|$.

COROLLARY 2. *If g is a necessary and sufficient statistic, then $\mathbf{S}_g$ is a necessary and sufficient subfield.*

PROOF. We have only to show that if g is a necessary statistic, then $\mathbf{S}_g$ is a necessary subfield. Let f be a function on X into R such that conditions (a), (b), and (c) of Corollary 1 are satisfied. Since f is sufficient and g is necessary, we have $g \subseteq f$ $[\mathbf{S}, \mu]$. Hence $\mathbf{S}_g \subseteq \mathbf{S}_f$ $[\mathbf{S}, \mu]$ by Lemma 9. Since $\mathbf{S}_f$ is necessary, it follows that $\mathbf{S}_g$ is necessary, and the proof is complete.

It should be stated here that the converse of Corollary 2 is false (cf. Example 2 in Section 4), and also that the corollary itself is false in the general case (cf. [2]).

COROLLARY 3. *Let g be an* $\mathbf{S}$*-measurable function on X into R. Then g is a necessary and sufficient statistic if and only if $g^{-1}(\mathbf{R})$ is a necessary and sufficient subfield.*

PROOF. In view of Corollary 2 and Lemma 5, we have only to show that if $\mathbf{S}_g$ is a necessary subfield, then g is a necessary statistic; since g is $\mathbf{S}$-measurable, the desired result follows from Lemma 6 by the argument used in establishing part (c) of Corollary 1.

4. Two examples. In both examples, $X = U \times V$ is the set of all points $x = (u, v)$ with $-\infty < u < \infty$, $-\infty < v < \infty$; $\mathbf{S}$ is the field of Borel sets of X; $P = \{p_\theta: -\infty < \theta < \infty\}$, where p_θ is the measure on $\mathbf{S}$ corresponding to u and v being independent normally distributed random variables, with means θ and 0 respectively and variances 1; and $\mu = p_{\theta=0}$. Let U and V denote, respectively, the coordinate axes $v = 0$ and $u = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ denote, respectively, the Borel sets of U and V.

The first example shows that the following propositions are false:

(i) If $f \subseteq g$ $[\mathbf{S}, \mu]$, then $\mathbf{S}_f \subseteq \mathbf{S}_g$ $[\mathbf{S}, \mu]$. (Cf., however, Lemmas 7 and 9.)

(ii) If f is sufficient, and f is a functional contraction of g (that is, $f \subseteq g$ $[\mathbf{S}, \mu]$), then g is sufficient. (Cf. Theorem 6.4 of [1].)

EXAMPLE 1. Let $f(u, v) \equiv u$. To define g, let $N \subseteq V$ be a set such that N has linear measure zero but is not in $\mathbf{V}$. Let $g(u, v) = u$ for $v \varepsilon V - N$, and $g(u, v) = 0$ for $v \varepsilon N$. Then $f \subseteq g$ $[\mathbf{S}, \mu]$, and also $g \subseteq f$ $[\mathbf{S}, \mu]$ so that f and g are functionally equivalent. However, it is easily seen from Fubini's theorem ([6], p. 83) that $\mathbf{S}_f = f^{-1}(\mathbf{U})$ while $\mathbf{S}_g$ contains only X and the empty set.

The second example shows that the following propositions are false:

(iii) If $\mathbf{S}_f \subseteq \mathbf{S}_g$ $[\mathbf{S}, \mu]$, then $f \subseteq g$ $[\mathbf{S}, \mu]$. (Cf., however, Lemma 6.)

(iv) If S_f is a necessary and sufficient subfield, then f is a necessary and sufficient statistic. (Cf., however, Corollary 3 together with Lemma 5.)

EXAMPLE 2. Define $g(u, v) \equiv u$. To define f, let $M \subseteq V$ be a set which is not measurable with respect to linear measure on V, and let $f(u, v) = (u, 1)$ for $v \varepsilon M$ and $f(u, v) = (u, 2)$ for $v \varepsilon V - M$. We shall show that $\mathbf{S}_f = \mathbf{S}_g$, so that f and g are equivalent, but that f is not a functional contraction of g.

Let $U_1 = \{x: v = 1\}$, $U_2 = \{x: v = 2\}$. Since g is *exactly* a function of f,

$\mathbf{S}_g \subseteq \mathbf{S}_f$. To prove the converse, consider a fixed $C \varepsilon \mathbf{S}_f$. There exists a set $B_1 \subseteq U_1$ and a $B_2 \subseteq U_2$ such that $C = f^{-1}(B_1 \cup B_2)$. Let the perpendicular projections of B_1, B_2 on U be A_1, A_2, respectively. Then, by the definition of f, $C = E \cup F \cup G$, where $E = (A_1 \cap A_2) \times V$, $F = (A_1 - A_2) \times M$, and $G = (A_2 - A_1) \times (V - M)$. Since C is a Borel set while M and $V - M$ are not, it follows ([6], p. 83) that F and G must be empty. Hence $A_1 = A_2 = A$ say, and $C = A \times V = g^{-1}(A)$. It now follows ([6], p. 83) that A is in $\mathbf{U}$, so that $g^{-1}(A) = C$ is in $g^{-1}(\mathbf{U})$. Since C is arbitrary, we have $\mathbf{S}_f \subseteq g^{-1}(\mathbf{U})$; but $g^{-1}(\mathbf{U}) = \mathbf{S}_g$, so that $\mathbf{S}_f \subseteq \mathbf{S}_g$. Thus $\mathbf{S}_f = \mathbf{S}_g$.

To show that f is not a functional contraction of g, suppose to the contrary that $f \subseteq g$ [$\mathbf{S}$, μ]. Then $f \subseteq g$ [$\bar{\mathbf{S}}$, $\bar{\lambda}$], where $\bar{\mathbf{S}}$ denotes the Lebesgue measurable sets of X and $\bar{\lambda}$ is (planar) Lebesgue measure on $\bar{\mathbf{S}}$. In other words, there exists a function h on U into $U_1 \cup U_2$ and an $\bar{\mathbf{S}}$-$\bar{\lambda}$-null set N such that $f(x) = h(g(x))$ on $X - N$. Write $h^{-1}(U_1) = I$, $U \times M = J$, and $I \times V = K$. Then $J = f^{-1}(U_1)$ and $K = g^{-1}(h^{-1}(U_1))$, and it follows exactly as in the proof of Lemma 7 that the sets $J - K$ and $K - J$ are $\bar{\mathbf{S}}$-$\bar{\lambda}$-null. Hence $L = (J - K) \cup (K - J)$ is $\bar{\mathbf{S}}$-$\bar{\lambda}$-null.

For each $u \varepsilon U$, let E_u be the set of all $v \varepsilon V$ such that $(u, v) \varepsilon L$. Let $\bar{\lambda}_u$, $\bar{\lambda}_v$ denote linear measure on U, V, respectively. It follows from Fubini's theorem ([6], p. 81) that there exists a $C \subseteq U$ with $\bar{\lambda}_u(C) = 0$ such that, for each $u \varepsilon U - C$, the set E_u is $\bar{\lambda}_v$-measurable (and of $\bar{\lambda}_v$-measure zero). Since $u \varepsilon I$ implies $E_u = V - M$, and $u \varepsilon U - I$ implies $E_u = M$, and since at least one of the sets $I - C$, $U - I - C$ must be nonempty (because $\bar{\lambda}_u(C) = 0$), it follows that M is $\bar{\lambda}_v$-measurable, and this is a contradiction.

It can be shown by a slight elaboration of the preceding argument that in Example 2 we have $\bar{\mathbf{S}}_f \subseteq \bar{\mathbf{S}}_g$ [$\bar{\mathbf{S}}$, $\bar{\mu}$], but not $f \subseteq g$ [$\bar{\mathbf{S}}$, $\bar{\mu}$]. This, together with Lemma 7, shows that by completing a given probability space (X, $\mathbf{S}$, μ) to (X, $\bar{\mathbf{S}}$, $\bar{\mu}$) the inconsistency between contraction and functional contraction is reduced but not eliminated entirely.

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