URN MODELS OF CORRELATION AND A COMPARISON WITH THE MULTIVARIATE NORMAL INTEGRAL

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Summary. In a special case of Polya's urn scheme, the probability that the first n draws are all of the same color is interpreted as a function of the (single) correlation coefficient. A more general urn model is introduced in which the correlation between pairs of results may differ from pair to pair, and again the probability of consecutive colors is considered. This result is compared with the probability of coincidence in sign under the multivariate normal distribution. The comparison suggests a new approximation for the probability in the multivariate normal case. This approximation appears to be useful only in the Polya case, where the correlations are all equal.

1. Introduction. Consider n correlated random variables $x_1, x_2, \cdots, x_n$. If each variable x_i may assume only the values $+1$ and -1 and either result is equally probable (a priori), then, in terms of the correlation coefficients between pairs (x_i, x_j), what is the probability that all n variables are positive? An example of such a problem is provided by Polya's urn scheme and by a generalization given in Section 3.

A more difficult problem is the following: Consider n correlated *continuous* variables $\xi_1, \xi_2, \cdots, \xi_n$, with each having a mean value of zero and symmetry about the mean. If these variables obey a given distribution law (e.g., the multivariate normal distribution), what is the probability that all n variables are simultaneously positive? This second problem may be reduced, in principle, to the first by associating the signs of the ξ_i with the signs of the x_i; that is,

$$x_i = \begin{cases} 1, & \xi_i \geqq 0, \\ -1, & \xi_i < 0; \end{cases} \qquad i = 1, 2, \cdots, n. \tag{1}$$

The next two sections are concerned with examples of the first problem mentioned above.

2. Polya's urn scheme. Consider the symmetric case of Polya's urn scheme ([1], [2], [3]), in which an urn contains initially a black balls and a red balls. Successive drawings are performed, with replacement, and with the further provision that Δ extra balls are added after each drawing, all of the same color as the ball most recently drawn. Δ may be negative, but it must obey the inequality,

$$a + (n - 1)\Delta \geqq 0, \tag{2}$$

where n is the total number of draws, in order that neither color may be overdrawn.

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The probability of drawing a black ball in the first trial is, of course, $a/2a = \frac{1}{2}$. The probability of drawing two black balls in the first two trials is $a(a + \Delta) / 2a(2a + \Delta)$. The probability of drawing n black balls in the first n trials is

$$P_n = (a/\Delta)_n / (2a/\Delta)_n, \tag{3}$$

where $(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1)$, and $(\alpha)_0 = 1$.

Let $x_i = +1$ if the ith draw is black, and let $x_i = -1$ if the ith draw is red. Then Polya has shown ([3], p. 140) that the correlation coefficient between x_i and x_j $(i \neq j)$ is

$$r = \Delta / (2a + \Delta). \tag{4}$$

The result is the same for all possible pairs (i, j). As Δ varies from $-a / (n - 1)$ to ∞, r varies from $-1 / (2n - 3)$ to 1.

Equation (4) may be verified easily for the case $i = 1, j = 2$. Since the mean values $E(x_i)$ are all zero and the variances $E(x_i^2)$ are all unity, the correlation coefficient between x_i and x_j is simply the expectation $E(x_i x_j)$. For the first two draws, there are four possibilities: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, and $(-1, -1)$. Then

$$r = E(x_1 x_2) = 2\frac{a(a + \Delta)}{2a(2a + \Delta)} - 2\frac{a^2}{2a(2a + \Delta)} = \frac{\Delta}{2a + \Delta},$$

which agrees with (4). The same procedure may be carried out for other pairs (i, j).

a/Δ may now be eliminated from (3) by equation (4); then, in terms of the correlation alone, the probability that all the x_i are equal is given by

$$P_n = ([1 - r] / 2r)_n / ([1 - r] / r)_n. \tag{5}$$

At this point the integers a and Δ may be forgotten. Let r assume *any* value in the range from $-1 / (2n - 3)$ to 1—not only the fractional values given by equation (4).

P_n may be expressed in terms of the beta function, as follows:

$$P_n = \frac{B(n + [1 - r] / 2r, [1 - r] / 2r)}{B([1 - r] / 2r, [1 - r] / 2r)}, \tag{6}$$

or, equivalently, as a terminating hypergeometric series:

$$\begin{aligned} P_n &= 2^{-n} F(-n/2, [1 - n] / 2; 1/2r; 1) \\ &= 2^{-n}\left\{1 + \frac{n(n-1)}{2(1)} r + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(1)(1 + 2r)} r^2 + \cdots\right\} \end{aligned} \tag{7}$$

[The identity between (6) and (7) follows from the theorem on $F(a, b; c; 1)$—see [4], p. 282—and from the multiplication theorem for gamma functions, [4], p. 240. See also [4], p. 262, problem 37.]

Note that if $r \to 0$, the probability (7) approaches 2^{-n}, which is the usual result for a sequence of Bernoulli trials when the individual probabilities are $\frac{1}{2}$.

If $r \to 1$, the expression in braces in (7) becomes the binomial series for $\frac{1}{2}[(1+1)^n + (1-1)^n]$ and therefore $P_n \to \frac{1}{2}$. This is the case of perfect coherence, which leaves only two possibilities: all black or all red.

3. A generalized urn scheme. It was noted in Section 2 that the Polya scheme yields a correlation matrix in which all elements except those on the main diagonal have the value (4). The following model exhibits a general correlation matrix.

As before, the urn contains initially a black balls and a red balls. In contrast with the single addition parameter Δ, this scheme makes use of a *matrix* of elements Δ_{ij}. One ball is drawn and replaced, and Δ_{12} balls are added of the color drawn. Again one ball is drawn and replaced; then Δ_{13} are added of the first color drawn and Δ_{23} of the second color drawn. After the $(k-1)$th draw (and replacement), Δ_{1k} are added of the first color drawn, Δ_{2k} of the second, etc., and $\Delta_{k-1,k}$ of the $(k-1)$th color drawn.

To simplify the algebra, let

$$D_{mk} = \sum_{i=m+1}^{k} \Delta_{mi}, \qquad m = 1, 2, \cdots, k-1; \tag{8}$$

thus, immediately preceding the kth draw, D_{mk} is the total number of balls which have been added up to that time *because of* the mth draw. Some of the Δ's may be negative, but to prevent overdrawing they must obey the inequality,

$$a + \sum_{m=1}^{k-1} D_{mk} \geqq 0, \tag{9}$$

for all integral k between 2 and n, inclusive. (n is again the total number of draws.)

The probabilities of the sequences black-black and black-red in the first two draws are, respectively,

$$P_{++} = \frac{a(a + D_{12})}{2a(2a + D_{12})}, \qquad P_{+-} = \frac{a(a)}{2a(2a + D_{12})}. \tag{10}$$

By symmetry, $P_{--} = P_{++}$ and $P_{-+} = P_{+-}$, as in the Polya scheme. For three draws the probabilities are

$$\begin{aligned} P_{+++} &= \frac{a(a + D_{12})(a + D_{13} + D_{23})}{2a(2a + D_{12})(2a + D_{13} + D_{23})}, \\ P_{++-} &= \frac{a(a + D_{12})(a)}{2a(2a + D_{12})(2a + D_{13} + D_{23})}, \\ P_{+-+} &= \frac{a(a)(a + D_{13})}{2a(2a + D_{12})(2a + D_{12} + D_{23})}, \\ P_{+--} &= \frac{a(a)(a + D_{23})}{2a(2a + D_{12})(2a + D_{13} + D_{23})}, \end{aligned} \tag{11}$$

and the other four may be obtained by symmetry.

What are the correlation coefficients in this scheme? Again let $x_i = 1$ or -1 if the ith draw is black or red, respectively, so that $r_{ij} = E(x_i x_j)$. For the first two draws,

$$r_{12} = E(x_1 x_2) = 2(P_{++} - P_{+-}) = D_{12} / (2a + D_{12}). \tag{12}$$

The last equality follows from the substitution of (10). Equation (12) conforms with (4), since for two draws the two urn schemes are identical.

For the first three draws, by (10) and (11),

$$\begin{aligned} r_{13} &= 2(P_{+++} - P_{++-} - P_{+--} + P_{+-+}) \\ &= 2(P_{+++} - P_{++-}) + 2(P_{+-+} - P_{+--}) \\ &= 2P_{++} \frac{D_{13} + D_{23}}{2a + D_{13} + D_{23}} + 2P_{+-} \frac{D_{13} - D_{23}}{2a + D_{13} + D_{23}} \\ &= 2\,\frac{D_{13}(P_{++} + P_{+-}) + D_{23}(P_{++} - P_{+-})}{2a + D_{13} + D_{23}} \\ &= \frac{D_{13} + r_{12} D_{23}}{2a + D_{13} + D_{23}}, \end{aligned} \tag{13}$$

and by a similar calculation,

$$r_{23} = \frac{r_{12} D_{13} + D_{23}}{2a + D_{13} + D_{23}}. \tag{14}$$

The above method is easily generalized for the first n draws. The result is

$$r_{in} = \frac{\sum_{j=1}^{n-1} r_{ij} D_{jn}}{2a + \sum_{j=1}^{n-1} D_{jn}}, \qquad i = 1, 2, \cdots, n - 1, \tag{15}$$

where $r_{ii} = 1$ for all i. Notice that if all the correlation coefficients are known for the first $(n - 1)$ draws, then equation (15) gives the remaining coefficients necessary to correlate the nth draw.

The next quantity to be calculated is the ratio $2P_n / P_{n-1}$, where P_n is again the probability of drawing n black balls in the first n trials. ($P_1 = P_+ = \frac{1}{2}$, $P_2 = P_{++}$, $P_3 = P_{+++}$, etc.) By the first of equations (10),

$$2\,\frac{P_{++}}{P_+} = \frac{2(a + D_{12})}{2a + D_{12}} = 1 + \frac{D_{12}}{2a + D_{12}}. \tag{16}$$

By equations (10) and (11),

$$2\,\frac{P_{+++}}{P_{++}} = 1 + \frac{D_{13} + D_{23}}{2a + D_{13} + D_{23}}. \tag{17}$$

Similarly, for n draws,

$$2\frac{P_n}{P_{n-1}} = 1 + \frac{\sum_{j=1}^{n-1} D_{jn}}{2a + \sum_{j=1}^{n-1} D_{jn}}, \qquad n = 2, 3, \cdots. \tag{18}$$

The next objective is to express the ratios (18) as functions *of the correlation coefficients alone*. A new variable is introduced:

$$G_{kn} = \frac{D_{kn}}{2a + \sum_{j=1}^{n-1} D_{jn}}, \qquad k = 1, 2, \cdots, n-1. \tag{19}$$

Then equations (15) may be written

$$r_{in} = \sum_{j=1}^{n-1} r_{ij} G_{jn}, \qquad i = 1, 2, \cdots, n-1 \tag{20}$$

and equation (18) may be written in the form

$$-1 = \sum_{j=1}^{n-1} G_{jn} - 2P_n / P_{n-1}. \tag{21}$$

Now equations (20) and (21) constitute n equations in the n unknowns, $G_{1n}, G_{2n}, \cdots, G_{n-1,n}$, and $2P_n / P_{n-1}$. The equations may be solved directly for the last quantity; then a simple manipulation of the determinants yields the result,

$$2\frac{P_n}{P_{n-1}} = \frac{\begin{vmatrix} 1 + r_{1n} & r_{12} + r_{1n} & \cdots & r_{1,n-1} + r_{1n} \\ r_{12} + r_{2n} & 1 + r_{2n} & \cdots & r_{2,n-1} + r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1,n-1} + r_{n-1,n} & r_{2,n-1} + r_{n-1,n} & \cdots & 1 + r_{n-1,n} \end{vmatrix}}{\begin{vmatrix} 1 & r_{12} & \cdots & r_{1,n-1} \\ r_{12} & 1 & \cdots & r_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1,n-1} & r_{2,n-1} & \cdots & 1 \end{vmatrix}}, \qquad n = 2, 3, \cdots. \tag{22}$$

The denominator is simply the determinant of the $(n-1)$-variate correlation matrix.

As in the case of the Polya scheme, the correlation coefficients may now be regarded as continuous rather than discrete. These coefficients may assume any values between -1 and 1 which do not violate (9), i.e., which do not lead to negative probabilities. By comparison with equation (18), it follows that the inequality (9) may be rewritten

$$P_k / P_{k-1} \geqq 0, \tag{23}$$

for $k = 2, 3, \cdots, n$, where P_k / P_{k-1} is given by (22).

Finally, by induction, the probability P_n is given by

$$P_n = 2^{-n}\frac{(1+r_{12})\begin{vmatrix}1+r_{13} & r_{12}+r_{13}\\ r_{12}+r_{23} & 1+r_{23}\end{vmatrix}}{1\begin{vmatrix}1 & r_{12}\\ r_{12} & 1\end{vmatrix}}$$

(24)

$$\cdots\frac{\begin{vmatrix}1+r_{1n} & r_{12}+r_{1n} & \cdots & r_{1,n-1}+r_{1n}\\ r_{12}+r_{2n} & 1+r_{2n} & \cdots & r_{2,n-1}+r_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ r_{1,n-1}+r_{n-1,n} & r_{2,n-1}+r_{n-1,n} & \cdots & 1+r_{n-1,n}\end{vmatrix}}{\begin{vmatrix}1 & r_{12} & \cdots & r_{1,n-1}\\ r_{12} & 1 & \cdots & r_{2,n-1}\\ \cdots & \cdots & \cdots & \cdots\\ r_{1,n-1} & r_{2,n-1} & \cdots & 1\end{vmatrix}}, \quad n = 2, 3, \cdots.$$

When all the coefficients r_{ij} are equal for $i, j = 1, 2, \cdots, n$ $(i \neq j)$, equation (24) reduces to the Polya result (5).

When $n = 2$ and 3 the result (24) becomes simply

(25) $$P_2 = \tfrac{1}{4}(1 + r_{12})$$

(26) $$P_3 = \tfrac{1}{8}(1 + r_{12} + r_{13} + r_{23}).$$

For higher values of n the complete expansion of (24) is quite complicated. However P_n may be expanded in a power series in the r's. To second order, for $n = 4$,

(27) $$P_4 = \tfrac{1}{16}(1 + r_{12} + r_{13} + r_{23} + r_{14} + r_{24} + r_{34} + r_{12}r_{34} + r_{13}r_{24} + r_{14}r_{23}) + O(r^3).$$

By induction, it can be shown that, for general n,

(28) $$P_n = 2^{-n}\left[1 + \sum_{j>i\geq 1}^{n} r_{ij} + \sum_{l>k>j>i\geq 1}^{n} (r_{ij}r_{kl} + r_{ik}r_{jl} + r_{il}r_{jk}) + O(r^3)\right].$$

When all the coefficients r_{ij} are equal $(i \neq j)$, the number of first-order terms in (28) is the binomial coefficient $\binom{n}{2}$. The number of second-order terms is $3\binom{n}{4}$; therefore this special case of (28) checks (to second order) with the result (7) of the Polya model.

If P_n is expanded to a higher order, then the series is no longer symmetric with respect to interchange of the variables $x_1, x_2, \cdots, x_n$.

4. The multivariate normal distribution. The following example belongs to the second type of problem given in the introduction.

Suppose that $\xi_1, \xi_2, \cdots, \xi_n$ obey the multivariate normal distribution law with correlation matrix

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix} \tag{29}$$

If all the mean values $E(\xi_i)$ are zero, what is the probability P_n that all n variables are simultaneously positive?

As stated in the introduction, this question may be reduced to the corresponding problem in the discrete variables $x_1, x_2, \cdots, x_n$ by means of equation (1); however, it must be remembered that the correlation r_{ij} between x_i and x_j is not the same as the correlation ρ_{ij} between ξ_i and ξ_j.

Various writers have investigated the probability P_n for the multivariate normal distribution. For $n = 2$ there is the Stieltjes-Sheppard result [5],

$$P_2 = \tfrac{1}{4}\left(1 + \frac{2}{\pi}\sin^{-1}\rho_{12}\right) \tag{30}$$

For $n = 3$ the result is (see Kendall [7] and David [8]):

$$P_3 = \tfrac{1}{8}\left[1 + \frac{2}{\pi}(\sin^{-1}\rho_{12} + \sin^{-1}\rho_{13} + \sin^{-1}\rho_{23})\right] \tag{31}$$

For $n > 3$ no solution has been given in closed form, but there exists the infinite series of Aitken, Kendall ([6], [7]), and Moran [9]. For $n = 4$, their series may be written, to second order in the ρ_{ij},

$$P_4 = \tfrac{1}{16}\left[1 + \frac{2}{\pi}\sum_{j>i\geq 1}^{4}\sin^{-1}\rho_{ij} + \frac{4}{\pi^2}\{\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23} + O(\rho^3)\}\right]. \tag{32}$$

For general n, the probability is

$$P_n = 2^{-n}\left[1 + \frac{2}{\pi}\sum_{j>i\geq 1}^{n}\sin^{-1}\rho_{ij} + \frac{4}{\pi^2}\sum_{l>k>j>i\geq 1}^{n}(\rho_{ij}\rho_{kl} + \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk}) + O(\rho^3)\right] \tag{33}$$

(This result will be derived in Section 6.)

Equation (33) may be compared with the corresponding probability (28) for the generalized urn scheme. Note that under the transformation,

$$r_{ij} = \frac{2}{\pi}\sin^{-1}\rho_{ij}, \tag{34}$$

the two expressions (28) and (33) agree to second order. [Note that r_{ij}, given by (34), is actually the correlation between the discrete variables x_i and x_j, given by equation (1), when the ξ's are normal. See equation (41).] This agreement suggests that the substitution of (34) into the *closed form* (24) of the result for the generalized urn scheme might provide an approximation for P_n in the multivariate normal case, to be used in place of the poorly converging series of Aitken, Kendall, and Moran. (See David's remarks [8] on convergence.)

For $n > 3$ and arbitrary ρ_{ij}, the agreement between the two power series does not extend beyond the second-order terms.

5. Numerical results. The approximation indicated above has been tested by a comparison with several known results for the multivariate normal integral.

When all the ρ_{ij} have the same value, defined by ρ, then r is obtained from equation (34); then this value of r is used in the closed expression (5) of the Polya scheme.

When $n = 2$ or 3, the result obtained from (5) is exact for all values of ρ; that is, equation (30) and the special case of (31) follow immediately.

When $\rho = 0$, (5) gives $P_n = 2^{-n}$; when $\rho = 1$, $P_n = \frac{1}{2}$. (See the explanation at the end of Section 2.) It appears, therefore, that when $\rho = 0$ or 1 the results are exact for all values of n.

When $\rho = \frac{1}{2}$, then $r = \frac{1}{3}$ and equation (5) gives $P_n = 1 / (n + 1)$. This result is also exact for all n. {See Ruben [10], p. 214, equation (70). In fact, Ruben's (70) holds for a more general class of distributions, as shown by Foster and Stuart [11], p. 22.}

When $1/\rho = 2, 3, \cdots, 12$, the results obtained from equation (5) may be compared with those of Ruben ([10], pp. 222–223). For the case $n = 4$, the comparison is shown in Table I. (Ruben's values have been rounded off to seven decimal places.)

The best agreement in Table I occurs for small ρ, as one might have predicted after a comparison of the corresponding power series.

For a given value of ρ, the approximation grows steadily worse as n increases. A comparison for $\rho = \frac{1}{4}$ is shown in Table II.

TABLE I

($n = 4$)

$1/\rho$	P_4 [from (5)]	Ruben's $\bar{u}_4(1/\rho)$
2	0.20000 00	0.20000 00
3	0.14975 57	0.14973 77
4	0.12649 38	0.12647 92
5	0.11302 30	0.11301 25
6	0.10423 15	0.10422 40
7	0.09804 22	0.09803 67
8	0.09344 92	0.09344 51
9	0.08990 58	0.08990 27
10	0.08708 94	0.08708 71
11	0.08479 73	0.08479 5
12	0.08289 56	0.08289 4

TABLE II

($\rho = \frac{1}{4}$)

n	P_n [from (5)]	Ruben's $\bar{a}_n(4)$
2	0.29021 53	0.29021 53
3	0.18532 30	0.18532 30
4	0.12649 38	0.12647 92
5	0.09069 62	0.09065 98
6	0.06754 16	0.06748 27
7	0.05183 56	0.05175 69
8	0.04076 86	0.04067 37
9	0.03272 29	0.03261 57
10	0.02671 93	0.02660 32

TABLE III

ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	P_4 [from (24)]	Plackett's Φ_4^0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0.13393	0.13333
						0.13194	
						0.13194	
						0.13393	
$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0.16369	0.16667
						0.16369	
						0.16369	
						0.16369	
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0.15000	0.15000
						0.15278	
						0.14881	
						0.14881	

It appears from Tables I and II that the Polya urn approximation might be useful in many problems, at least where ρ is not greater than $\frac{1}{2}$ and where n is not much greater than 4. Formulas (5) and (34) are certainly more easily applicable than Ruben's integral recursion formulas or an interpolation in Ruben's table.

In the general case of unequal ρ_{ij}, the results are much less satisfactory. This fact can be illustrated by a comparison with several exact values given by Plackett ([12], p. 360) for the quadrivariate case. The comparison is shown in Table III.

To obtain P_4, one substitutes the values of r_{ij} from (34) into the closed expression (24). When $n > 3$, (24) is not symmetric with respect to interchange of the indices; therefore different results are possible. The four values of P_4 given in Table III (for each correlation matrix) are those obtained when x_4, x_3, x_2, and x_1, respectively, are considered as the fourth draw from the urn. Without doubt the lack of symmetry in (24) is partly responsible for the poor agreement.

It appears from Table III (and from other comparisons with Plackett's figures) that the generalized urn approximation is not a satisfactory method for computing the general multivariate normal integral. The only possible exceptions would be those situations in which the ρ_{ij} are very small or are all nearly equal to each other, i.e., approaching the Polya case.

Recently Plackett [12] has given a numerical method for evaluating the quadrivariate normal integral. It involves more labor than the urn scheme but yields considerably greater accuracy.

6. General remarks. This paper will be concluded by a discussion of the general problem described in the introduction, in which no specific model or distribution law is assumed.

Consider again the n mutually interacting (discrete) random variables x_1, $x_2, \cdots, x_n$. Let the *observed values* of these variables (in a given experiment) be $y_1, y_2, \cdots, y_n$, where $y_i = \pm 1$. Then there are 2^n possible combinations for the n results y_i, and each result has the probability

$$(35) \qquad P(x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n).$$

For any given distribution law there are 2^n product moments, i.e., the expected values $E(1)$, $E(x_j)$, $E(x_jx_k)$, $E(x_jx_kx_l)$, $\cdots$, $E(x_1x_2 \cdots x_n)$. All other moments degenerate to one of these, since $x_i^2 = 1$[e.g., $E(x_1^2) = E(1)$; $E(x_1^2x_2) = E(x_2)$].

The 2^n moments may be expressed in terms of the 2^n probabilities, as follows:

$$(36) \quad \begin{aligned} E(1) &= \sum_{y_i} 1 \cdot P(x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n), \\ E(x_j) &= \sum_{y_i} y_j P(x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n), \quad j = 1, 2, \cdots, n, \\ E(x_jx_k) &= \sum_{y_i} y_j y_k P(x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n), \\ &\qquad\qquad j, k = 1, 2, \cdots, n(j < k), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ E(x_1x_2 \cdots x_n) &= \sum_{y_i} y_1y_2 \cdots y_n P(x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n), \end{aligned}$$

where the sums are taken over all combinations $y_1 = \pm 1$, $y_2 = \pm 1$, $\cdots$, $y_n = \pm 1$.

If all the equations (36) are added together, then all the probabilities cancel except $P(x_1 = x_2 = \cdots = x_n = 1)$, which was previously called P_n. Then

$$(37) \quad P_n = 2^{-n}\left[1 + \sum_{i=1}^{n} E(x_i) + \sum_{j>i\geq 1}^{n} E(x_ix_j) + \cdots + E(x_1x_2 \cdots x_n)\right].$$

Now suppose that the symmetry of the distribution law is such that all product moments *of odd order* are zero, i.e.,

$$(38) \quad \begin{aligned} E(x_1) = E(x_2) = \cdots = E(x_n) = 0, \\ E(x_1x_2x_3) = E(x_1x_2x_4) = \cdots = E(x_{n-2}x_{n-1}x_n) = 0, \end{aligned}$$

etc. Then (37) becomes

$$P_n = 2^{-n}\left[1 + \sum_{j>i\geqq 1}^{n} E(x_i x_j) + \sum_{l>k>j>i\geqq 1}^{n} E(x_i x_j x_k x_l) + \cdots\right], \tag{39}$$

ending with a product moment of order n if n is even, or with moments of order $(n - 1)$ if n is odd.

It is now evident that P_2 and P_3 in equations (25) and (26) could be obtained directly from the general formula (39) by the substitution of $n = 2$ or 3 and of the definition $E(x_i x_j) = r_{ij}$. In other words, it is only for $n > 3$ that a specific urn model must be assumed, and this specialization is reflected in the values of the higher-order moments $E(x_i x_j x_k x_l)$, etc. in (39).

On the other hand, suppose ξ_i are *continuous* random variables obeying a given distribution law with symmetry as in (38). Then for the calculation of P_2 and P_3, $E(x_i x_j)$ must be obtained as a function of the parameters of the original distribution. For P_4 and P_5, $E(x_i x_j x_k x_l)$ must be obtained, etc., and all other P's will follow two at a time from the higher moments.

It is now possible to derive equations (31) and (33). Assume that P_2 is given by (30). By matching (30) with (39) when $n = 2$, it follows that

$$E(x_1 x_2) = \frac{2}{\pi}\sin^{-1}\rho_{12}. \tag{40}$$

Then by the symmetry of the normal distribution,

$$E(x_i x_j) = \frac{2}{\pi}\sin^{-1}\rho_{ij}, \tag{41}$$

for all i and j, $i \neq j$, and equation (31) follows by the substitution of the moments (41) into the general expression (39) with $n = 3$.

Now assume that P_4 is given correctly by equation (32). Then (32) may be matched with (39) when $n = 4$, with the aid of (41), and the result is given by

$$E(x_1 x_2 x_3 x_4) = \frac{4}{\pi^2}(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}) + O(\rho^3). \tag{42}$$

Then, by symmetry, the general fourth-order product moment is

$$E(x_i x_j x_k x_l) = \frac{4}{\pi^2}(\rho_{ij}\rho_{kl} + \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk}) + O(\rho^3), \qquad i < j < k < l, \tag{43}$$

and, since all higher-order moments are of higher order in the ρ_{ij}, equation (33) follows. The last operation is the substitution of the moments (41) and (43) into the general expression (39).

This process is equivalent to a method used by David [8] to obtain P_{n+1} from P_n when n is even.

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