ON AN APPLICATION OF KRONECKER PRODUCT OF MATRICES TO STATISTICAL DESIGNS[1]

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1. Summary. By a statistical design (or simply, a design) we mean an arrangement of a certain number of "treatments" in a certain number of "blocks" in such a way that some prescribed combinatorial conditions are fulfilled. With every design is associated a unique matrix called the incidence matrix of the design (definitions, etc., in subsequent sections). In many instances, e.g., [7], [8], [10], [12], [16], information regarding certain kinds of designs such as BIB, PBIB designs is obtained from properties of the matrix NN' or of its determinant $|NN'|$ where N is the incidence matrix of the design under consideration. On the other hand in a few cases, such as [4], [5], [11], [14], [15], the incidence matrix N itself has been used to investigate properties of designs. This paper gives a method of using incidence matrices of known designs to obtain new designs.

In Section 2 we have defined the Kronecker product of matrices. This definition and some properties of the Kronecker product of matrices are given in [1]. Section 3 is devoted to a general discussion of an application of the concept of the Kronecker product of matrices to define the Kronecker product of designs. This section also contains two theorems which illustrate the use of the method of obtaining Kronecker products of designs. Definitions of some well-known designs are given in Section 4, which also contains a number of results giving explicit forms of certain Kronecker products. Finally some illustrations of a few results of Section 4 are given in Section 5.

2. The Kronecker product of matrices. Let

$$A = (a_{ij}), \qquad B = (b_{kl}), \qquad I_u, \qquad O_{m\times n} \tag{2.1}$$

be respectively an $m \times n$ matrix, a $p \times q$ matrix, the identity matrix of order u, the null or zero matrix of order $m \times n$, all defined over the set of non-negative integers. The Kronecker product of matrices A and B is defined as follows.

Definition 2.1. The Kronecker product $A \times B$ of matrices A and B of (2.1) is defined by

$$A \times B = \begin{bmatrix} a_{11}\,B & a_{12}\,B & \cdots & a_{1n}\,B \\ a_{21}\,B & a_{22}\,B & \cdots & a_{2n}\,B \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}\,B & a_{m2}\,B & \cdots & a_{mn}\,B \end{bmatrix}, \tag{2.2}$$

where $a_{ij}B$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ is itself a $p \times q$ matrix.

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We shall always use an "$\times$" in the product of matrices to denote the Kronecker product. The ordinary product of matrices A and B (when it exists) will be denoted by $A \cdot B$ or AB.

It is clear from Definition 2.1 that the Kronecker product always exists and that $A \times B$ is an $mp \times nq$ matrix defined over the set of non-negative integers. Also it is obvious that the Kronecker product of two matrices reduces to the ordinary product if and only if one of the matrices is a scalar.

The result contained in the following theorem will be used later in Section 3.

THEOREM 2.1. *For any matrices A and B as in* (2.1) *we must have*

$$A \times B = P \cdot (B \times A) \cdot Q \tag{2.3}$$

where the matrices P and Q are obtained from the identity matrices I_{mp} and I_{nq} respectively by permuting rows and columns.

It should be noted that P and Q are nonsingular matrices whose elements consist only of 0's and 1's, and that the matrices P and Q are the same for any A and B as defined in (2.1).

A proof of Theorem 2.1 can be constructed from that of a similar result proved by Murnaghan [1], who gives various other properties of the Kronecker product.

3. The Kronecker product of designs. Let D_p, $p = 1, 2$, be a design in which v_p treatments are arranged in b_p blocks. Let N_p, the incidence matrix of the design D_p, be defined by

$$N_p = (n_{i_p j_p}^{(p)}), \qquad i_p = 1, 2, \cdots, v_p, \qquad j_p = 1, 2, \cdots, b_p, \tag{3.1}$$

where $n_{i_p j_p}^{(p)}$ is the number of times the i_pth treatment of D_p occurs in the j_pth block of D_p. Clearly $n_{i_p j_p}^{(p)}$ is a non-negative integer so that N_p is defined over the set of non-negative integers. Since a design uniquely determines its incidence matrix and vice versa, we may denote both a design and its incidence matrix by the same symbol. Also the treatments and blocks of a design correspond respectively to the rows and columns of the incidence matrix of the design.

Let N_1 and N_2 be the designs defined by (3.1). Then

$$N_{12} = N_1 \times N_2 \tag{3.2}$$

uniquely determines a design and so does

$$N_{21} = N_2 \times N_1. \tag{3.3}$$

Theorem 2.1 at once leads to the following theorem.

THEOREM 3.1. *If N_1 and N_2 are designs defined by* (3.1), *then the designs N_{12} and N_{21} defined respectively by* (3.2) *and* (3.3) *are structurally the same, i.e., one of them can be obtained from the other by simply renaming the treatments and renumbering the blocks.*

This theorem enables us to designate the designs N_{12} and N_{21} by a common symbol N, the incidence matrix of N being taken to be $N_1 \times N_2$ or $N_2 \times N_1$, whichever is convenient.

Since the incidence matrix of the design N obtained above is the Kronecker product of the incidence matrices of the designs N_1 and N_2, we may say that the design N is the Kronecker product of the designs N_1 and N_2.

We now examine a few matrices and the corresponding designs.

3(a). Let N_1 be a row n-vector

$$N_1 = (1\ 1\ \cdots\ 1) \tag{3.4}$$

there being n 1's on the right-hand side. The design N_1 clearly consists of n blocks each of size one, each block being treated by the same single treatment.

3(b). Let N_2 be a column m-vector

$$N_2 = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \tag{3.5}$$

there being m 1's on the right-hand side. The design N_2 is a single replication of m treatments in one block of size m.

If N_0 be any design, then with N_1 as in (3.4), we have

$$N^{(1)} = N_1 \times N_0 = (N_0\, N_0\ \cdots\ N_0),$$

there being n N_0's on the right-hand side. This means that the design $N^{(1)}$ is nothing but n replications of the design N_0 as a whole. Again if N_0 be any design, then with N_2 as in (3.5), we have

$$N^{(2)} = N_0 \times N_2,$$

where clearly $N^{(2)}$ defines a design which is obtained from N_0 by replacing each treatment of N_0 by a group of m treatments. Also the rows of $N^{(2)}$ consist only of m repetitions of each row of N_0.

These two results can be combined into the following theorem.

THEOREM 3.2. *If N_0 be any design and if N_1 and N_2 be as defined in* (3.4) *and* (3.5) *respectively, then the designs $N^{(1)} = N_1 \times N_0$ and $N^{(2)} = N_0 \times N_2$ are respectively*

(i) *n replications of the design N_0 as a whole, and*

(ii) *the design obtained from N_0 by replacing each of its treatments by a group of m treatments so that the rows of $N^{(2)}$ consist only of m repetitions of each row of N_0.*

The following corollaries to Theorem 3.2 are trivial.

COROLLARY 3.2.1. *A randomized block design, N_3, with m treatments and n blocks, each block being a complete replication, is the Kronecker product*

$$N_3 = N_1 \times N_2 \tag{3.6}$$

of the designs N_1 and N_2 defined in (3.4) *and* (3.5) *respectively.*

COROLLARY 3.2.2. *If N_0 be any design and N_3 be the randomized block design defined in* (3.6), *then $N_3 \times N_0$ defines a design which contains n replications of the design derived from N_0 by replacing each of its treatments by a group of m treatments.*

3(c). The design corresponding to I_u, the identity matrix of order u, contains u treatments and u blocks each of size one, and the ith block contains a single plot to which the ith treatment is applied, $i = 1, 2, \cdots, u$.

The following corollary to Theorem 3.2 is also trivial.

COROLLARY 3.2.3. *With* N_2 *as defined in* (3.5), *we have*

$$N = I_u \times N_2,$$

which defines a design N *useful for confounding with blocks the effects of certain treatment combinations of a factorial design when* u *and* m *have suitable values.*

It may be noted that if N_0 be any design, then the Kronecker product $I_u \times N_0$ is always a disconnected design, and therefore no further illustrations involving I_u will be given.

4. Special cases of Kronecker products of designs. We first define a few designs.

4(a). Design N_2 is already defined in (3.5).

4(b). A balanced incomplete block (BIB) design N_{BIB} with parameters v^*, b^*, r^*, k^*, λ^* is defined to be the one in which the v^* treatments are arranged in the b^* blocks of size k^* each, such that

(i) the treatments in any block are all distinct,

(ii) each treatment is replicated r^* times, and

(iii) every pair of treatments occurs together in λ^* blocks (cf. [12]).

4(c). A partially balanced incomplete block (PBIB) design $N_{\text{PBIB}}^{(s)}$ with s associate classes and with parameters

$$v, \quad b, \quad r, \quad k, \quad n_i, \quad \lambda_i, \quad p_{jk}^i, \qquad i, j, k = 1, 2, \cdots, s,$$

is defined as follows.

(i) There are v treatments arranged in b blocks each of size k such that each treatment is replicated r times and the treatments in any block are all distinct.

(ii) There can be established a relation of association between any two treatments, satisfying the following conditions.

(α) Two treatments are either 1st, 2nd, $\cdots$, or sth associates.

(β) Each treatment has n_i ith associates, $i = 1, 2, \cdots, s$.

(γ) Given any two treatments which are ith associates, the number of treatments which are common to the jth associates of the first and the kth associates of the second is p_{jk}^i and is independent of the pair of treatments with which we start.

(iii) Two treatments which are ith associates occur together in exactly λ_i blocks, $i = 1, 2, \cdots, s$ (cf. [8]).

PBIB designs with two associate classes have been extensively investigated by Bose and Shimamoto [9].

When the parameters $\lambda_1, \lambda_2, \cdots, \lambda_s$ of a PBIB design are not all different, the s associate classes of the PBIB design may not be all distinct. The following lemma, which is a modification of a remark by Rao [3], gives a criterion to determine whether the PBIB design has s or fewer distinct associate classes when its parameters $\lambda_1, \lambda_2, \cdots, \lambda_s$ are not all different.

LEMMA 4.1. *Let a* PBIB *design* $N^{(s)}_{\text{PBIB}}$ *with s associate classes and with parameters*

$$v, \quad b, \quad r, \quad k, \quad n_i, \quad \lambda_i, \quad p^i_{jk}, \qquad i, j, k = 1, 2, \cdots, s,$$

be such that $\lambda_1, \lambda_2, \cdots, \lambda_s$ *are not all different so that at least two of them are equal. Without loss of generality we can assume that* $\lambda_1 = \lambda_2$. *In this case the number of associate classes of the design* N^s_{PBIB} *can be reduced from s to s − 1 by combining its first two associate classes if and only if*

$$\begin{bmatrix} \sum_{u,w=1}^{2} p^1_{uw} & \sum_{u=1}^{2} p^1_{u3} & \cdots & \sum_{u=1}^{2} p^1_{us} \\ \sum_{w=1}^{2} p^1_{3w} & p^1_{33} & \cdots & p^1_{3s} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{w=1}^{2} p^1_{sw} & p^1_{s3} & \cdots & p^1_{ss} \end{bmatrix} = \begin{bmatrix} \sum_{u,w=1}^{2} p^2_{uw} & \sum_{u=1}^{2} p^2_{u3} & \cdots & \sum_{u=1}^{2} p^2_{us} \\ \sum_{w=1}^{2} p^2_{3w} & p^2_{33} & \cdots & p^2_{3s} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{w=1}^{2} p^2_{sw} & p^2_{s3} & \cdots & p^2_{ss} \end{bmatrix} \tag{4.1}$$

Further if (4.1) *holds, then the parameters of the reduced* PBIB *design with s − 1 associate classes are*

$$v' = v, \qquad b' = b, \qquad r' = r, \qquad k' = k,$$
$$n'_1 = n_1 + n_2, \qquad n'_2 = n_3, \cdots, n'_{s-1} = n_s,$$
$$\lambda'_1 = \lambda_1 = \lambda_2, \qquad \lambda'_2 = \lambda_3, \cdots, \lambda'_{s-1} = \lambda_s,$$

$$(p'^1_{yz}) = \begin{bmatrix} \sum_{u,w=1}^{2} p^t_{uw} & \sum_{u=1}^{2} p^t_{u3} & \cdots & \sum_{u=1}^{2} p^t_{us} \\ \sum_{w=1}^{2} p^t_{3w} & p^t_{33} & \cdots & p^t_{3s} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{w=1}^{2} p^t_{sw} & p^t_{s3} & \cdots & p^t_{ss} \end{bmatrix},$$

$$(p'^x_{yz}) = \begin{bmatrix} \sum_{u,w=1}^{2} p^{x+1}_{uw} & \sum_{u=1}^{2} p^{x+1}_{u3} & \cdots & \sum_{u=1}^{2} p^{x+1}_{us} \\ \sum_{w=1}^{2} p^{x+1}_{3w} & p^{x+1}_{33} & \cdots & p^{x+1}_{3s} \\ \cdot & \cdot & \cdots & \cdot \\ \sum_{w=1}^{2} p^{x+1}_{sw} & p^{x+1}_{s3} & \cdots & p^{x+1}_{ss} \end{bmatrix} \tag{4.2}$$

where $t = 1$ *or* 2; $x = 2, 3, \cdots, s - 1$; $y, z = 1, 2, \cdots, s - 1$.

It follows that repeated applications of Lemma 4.1 to any PBIB design will ultimately give a PBIB design whose associated classes are all distinct.

The following results give the Kronecker products of various pairs of designs chosen from 4(a), 4(b), and 4(c).

THEOREM 4.1. (a) *The Kronecker product* $N = N_2 \times N^{(s)}_{\text{PBIB}}$ *of the design* N_2 *of* (3.5) *and a* PBIB *design* $N^{(s)}_{\text{PBIB}}$ *with s associate classes and with parameters*

$$v, \quad b, \quad r, \quad k, \quad n_i, \quad \lambda_i, \quad p^i_{jk}, \quad i, j, k = 1, 2, \cdots, s,$$

is a PBIB *design with at most* $s + 1$ *associate classes.*

(b) *The design N defined above has* $s + 1$ *distinct associate classes if the design* $N^{(s)}_{\text{PBIB}}$ *has s distinct associate classes and* $\lambda_i < r$ *for all* $i = 1, 2, \cdots, s$.

(c) *In any case the parameters of the design N can be expressed in terms of those of the designs* N_2 *and* $N^{(s)}_{\text{PBIB}}$ *by the equations*:

$$v' = mv, \qquad b' = b, \qquad r' = r, \qquad k' = mk,$$
$$n'_i = mn_i, \qquad n'_{s+1} = m - 1, \qquad \lambda'_i = \lambda_i, \qquad \lambda'_{s+1} = r,$$

$$(4.3) \qquad (p'^{i}_{yz}) = \left[\begin{array}{c|c} m(p^i_{jk}) & (m-1)(\delta_{ji}) \\ \hline (m-1)(\delta_{ij}) & 0 \end{array}\right], \qquad (p'^{s+1}_{yz}) = \left[\begin{array}{c|c} m(n_j\delta_{jk}) & O_{s\times 1} \\ \hline O_{1\times s} & m-2 \end{array}\right]$$

where $i, j, k = 1, 2, \cdots, s$; $y, z = 1, 2, \cdots, s + 1$; $\delta_{\alpha\beta} = 0$ *if* $\alpha \neq \beta$ *and* $\delta_{\alpha\beta} = 1$ *if* $\alpha = \beta$ *for all* $\alpha, \beta = 1, 2, \cdots$.

(d) *Lemma* 4.1 *can be applied to the cases in which the conditions in* (b) *above are not fulfilled.*

The essence of Theorem 4.1 appears in a paper by Zelen [17].

COROLLARY 4.1.1. *The Kronecker product* $N = N_2 \times N_{\text{BIB}}$ *of the design* N_2 *of* (3.5) *and a* BIB *design* N_{BIB} *with parameters* $v^*, b^*, r^*, k^*, \lambda^*$ *is a singular* GD *(group divisible) design with parameters*

$$(4.4) \qquad \begin{array}{llll} v' = mv^*, & b' = b^*, & r' = r^*, & k' = mk^*, \\ m' = v^*, & n' = m, & \lambda'_1 = r^*, & \lambda'_2 = \lambda^*. \end{array}$$

It should be noted that singular GD designs can be obtained only by using Corollary 4.1.1 as was shown by Bose and Connor [8].

THEOREM 4.2. (a) *The Kronecker product* $N = N^{(s)}_{\text{PBIB}} \times N^{(t)}_{\text{PBIB}}$ *of two* PBIB *designs* $N^{(s)}_{\text{PBIB}}$ *and* $N^{(t)}_{\text{PBIB}}$ *with s and t associate classes respectively and with respective sets of parameters*

$$(4.5) \qquad v_1, b_1, r_1, k_1, n_{1i_1}, \lambda_{1i_1}, p^{i_1}_{1j_1k_1}; \; i_1, j_1, k_1 = 1, 2, \cdots, s,$$

$$(4.6) \qquad v_2, b_2, r_2, k_2, n_{2i_2}, \lambda_{2i_2}, p^{i_2}_{2j_2k_2}; \; i_2, j_2, k_2 = 1, 2, \cdots, t,$$

is a PBIB *design with at most* $t + s + ts$ *associate classes.*

(b) *The design N defined above has $t + s + ts$ distinct associate classes if*

(i) *the s associate classes of $N^{(s)}_{\text{PBIB}}$ and the t associate classes of $N^{(t)}_{\text{PBIB}}$ are all distinct,*

(ii) $\lambda_{1i_1} < r_1$, $\lambda_{2i_2} < r_2$, *and*

(iii) $r_1\lambda_{2i_2} \neq r_2\lambda_{1i_1}$ *for all* $i_1 = 1, 2, \cdots, s$ *and* $i_2 = 1, 2, \cdots, t$.

(c) *In any case the parameters of the design N can be expressed in terms of those of $N^{(s)}_{\text{PBIB}}$ and $N^{(t)}_{\text{PBIB}}$ by the equations:*

$$v' = v_1 \cdot v_2, \qquad b' = b_1 \cdot b_2, \qquad r' = r_1 \cdot r_2, \qquad k' = k_1 \cdot k_2,$$

$$n'_{i_2} = n_{2i_2}, \qquad n'_{t+i_1} = n_{1i_1}, \qquad n'_{t+i_1+i_2 s} = n_{2i_2} \cdot n_{1i_1},$$

$$\lambda'_{i_2} = r_1 \cdot \lambda_{2i_2}, \qquad \lambda'_{t+i_1} = r_2 \cdot \lambda_{1i_1}, \qquad \lambda'_{t+i_1+i_2 s} = \lambda_{2i_2} \cdot \lambda_{1i_1},$$

$$(p'^{i_2}_{yz}) = \left[\begin{array}{c|c|c} (p^{i_2}_{2j_2k_2}) & O_{t\times s} & O_{t\times st} \\ \hline O_{s\times t} & O_{s\times s} & (\delta_{i_2j_2}) \times (n_{1j_1}\delta_{j_1k_1}) \\ \hline O_{st\times t} & (\delta_{j_2i_2}) \times (n_{1j_1}\delta_{j_1k_1}) & (p^{i_2}_{2j_2k_2}) \times (n_{1j_1}\delta_{j_1k_1}) \end{array}\right]$$

$$(4.7) \quad (p'^{t+i_1}_{yz}) = \left[\begin{array}{c|c|c} O_{t\times t} & O_{t\times s} & (n_{2j_2}\delta_{j_2k_2}) \times (\delta_{i_1j_1}) \\ \hline O_{s\times t} & (p^{i_1}_{1j_1k_1}) & O_{s\times st} \\ \hline (n_{2j_2}\delta_{j_2k_2}) \times (\delta_{j_1i_1}) & O_{st\times s} & (n_{2j_2}\delta_{j_2k_2}) \times (p^{i_1}_{1j_1k_1}) \end{array}\right]$$

$$(p'^{t+i_1+i_2 s}_{yz}) = \left[\begin{array}{c|c|c} O_{t\times t} & (\delta_{j_2i_2}) \times (\delta_{i_1j_1}) & (p^{i_2}_{2j_2k_2}) \times (\delta_{i_1j_1}) \\ \hline (\delta_{i_2j_2}) \times (\delta_{j_1i_1}) & O_{s\times s} & (\delta_{i_2j_2}) \times (p^{i_1}_{1j_1k_1}) \\ \hline (p^{i_2}_{2j_2k_2}) \times (\delta_{j_1i_1}) & (\delta_{j_2i_2}) \times (p^{i_1}_{1j_1k_1}) & (p^{i_2}_{2j_2k_2}) \times (p^{i_1}_{1j_1k_1}) \end{array}\right]$$

where $i_1, j_1, k_1 = 1, 2, \cdots, s$; $i_2, j_2, k_2 = 1, 2, \cdots, t$;

$$y, z = 1, 2, \cdots, t + s + ts;$$

$\delta_{\alpha\beta} = 0$ *if* $\alpha \neq \beta$ *and* $\delta_{\alpha\beta} = 1$ *if* $\alpha = \beta$ *for all* $\alpha, \beta = 1, 2, \cdots$.

(d) *Lemma* 4.1 *can be applied to the cases in which the conditions in* (b) *above are not fulfilled.*

PROOF. Let us consider the Kronecker product of the design $N^{(s)}_{\text{PBIB}}$ and $N^{(t)}_{\text{PBIB}}$ in the form $N = N^{(s)}_{\text{PBIB}} \times N^{(t)}_{\text{PBIB}}$. Since $N^{(s)}_{\text{PBIB}}$ and $N^{(t)}_{\text{PBIB}}$ are PBIB designs with parameters (4.5) and (4.6) respectively, it follows that their incidence matrices $N^{(s)}_{\text{PBIB}}$ and $N^{(t)}_{\text{PBIB}}$ are of orders $v_1 \times b_1$ and $v_2 \times b_2$, respectively. Also the elements of these matrices consist only of 0's and 1's, there being r_1 1's in every row and

k_1 1's in every column of $N_{\text{PBIB}}^{(s)}$ and r_2 1's in every row and k_2 1's in every column of $N_{\text{PBIB}}^{(t)}$. In obtaining N from $N_{\text{PBIB}}^{(s)}$ and $N_{\text{PBIB}}^{(t)}$ we replace every 1 in $N_{\text{PBIB}}^{(s)}$ by the matrix $N_{\text{PBIB}}^{(t)}$ and every 0 in $N_{\text{PBIB}}^{(s)}$ by the null matrix $O_{v_2 \times b_2}$. From this it follows that the incidence matrix of N is a $v_1 \cdot v_2 \times b_1 \cdot b_2$ matrix whose elements consist only of 0's and 1's, there being $r_1 \cdot r_2$ 1's in every row and $k_1 \cdot k_2$ 1's in every column of N. This means that the parameters v', b', r', k' of N given by

$$(4.8) \qquad v' = v_1 \cdot v_2, \qquad b' = b_1 \cdot b_2, \qquad r' = r_1 \cdot r_2, \qquad k' = k_1 \cdot k_2$$

have their usual significance for the design N.

We shall now identify the various associate classes of a treatment of N. Let the first row of $N_{\text{PBIB}}^{(s)}$ correspond to the treatment Θ of $N_{\text{PBIB}}^{(s)}$, that of $N_{\text{PBIB}}^{(t)}$ to the treatment Θ' of $N_{\text{PBIB}}^{(t)}$, and that of N to the treatment θ of N. We shall identify the various associate classes of θ in N.

The row corresponding to Θ in $N_{\text{PBIB}}^{(s)}$ contains r_1 1's, all other elements in the row being 0. In obtaining N each of these r_1 1's is replaced by the matrix $N_{\text{PBIB}}^{(t)}$ and each of the 0's is replaced by the null matrix $O_{v_2 \times b_2}$. Hence the first v_2 rows of N contain exactly r_1 replications of the design $N_{\text{PBIB}}^{(t)}$ as a whole and nothing else. Consider one of these r_1 replications. Its first row, which corresponds to a section of θ in N, also corresponds to Θ' in $N_{\text{PBIB}}^{(t)}$. Consider the i_2th associates, $i_2 = 1, 2, \cdots, t$, of Θ' in $N_{\text{PBIB}}^{(t)}$ which occur in the replication of $N_{\text{PBIB}}^{(t)}$ under consideration. These are n_{2i_2} in number and each of them occurs together with Θ' in λ_{2i_2} blocks of the replication of $N_{\text{PBIB}}^{(t)}$. When we take into account all the r_1 replications of $N_{\text{PBIB}}^{(t)}$ in the first v_2 rows of N, we find that each of the n_{2i_2} i_2th associates of Θ' in $N_{\text{PBIB}}^{(t)}$, considered as treatments of N, will occur together with θ in $r_1 \cdot \lambda_{2i_2}$ blocks of N. We take these n_{2i_2} treatments of N to be the i_2th associates of θ in N. This identifies the first t associate classes of θ in N and we see that the parameters n'_{i_2}, λ'_{i_2} of N given by

$$(4.9) \qquad n'_{i_2} = n_{2i_2}, \qquad \lambda_{i_2} = r_1 \lambda_{2i_2}, \qquad i_2 = 1, 2, \cdots, t,$$

have their usual significance for the design N.

Now consider the i_1th associates, $i_1 = 1, 2, \cdots, s$, of Θ in $N_{\text{PBIB}}^{(s)}$. These are n_{1i_1} in number, and each of them occurs together with Θ in λ_{1i_1} blocks of $N_{\text{PBIB}}^{(s)}$. Consider one of the n_{1i_1} i_1th associates of Θ. The row in $N_{\text{PBIB}}^{(s)}$ corresponding to this i_1th associate also contains r_1 1's and all other elements in the row are 0's. In obtaining N each of these r_1 1's is replaced by the matrix $N_{\text{PBIB}}^{(t)}$ and each of the 0's is replaced by the null matrix $O_{v_2 \times b_2}$. Hence the row in $N_{\text{PBIB}}^{(s)}$ corresponding to the i_1th associate of Θ under consideration gives rise to only r_1 replications in N of the design $N_{\text{PBIB}}^{(t)}$ as a whole and nothing else. Out of these r_1 replications of $N_{\text{PBIB}}^{(t)}$ only λ_{1i_1} can be paired off with similar replications of $N_{\text{PBIB}}^{(t)}$ in N arising out of the row corresponding to Θ, because Θ occurs together with any of its i_1th associates in λ_{1i_1} blocks of $N_{\text{PBIB}}^{(s)}$. Consider one of these λ_{1i_1} pairs of replications of $N_{\text{PBIB}}^{(t)}$. The first row in each component replication of $N_{\text{PBIB}}^{(t)}$ in this pair is identical with that corresponding to Θ'. One of these first rows is a section of that corresponding to θ in N; we define the treatment in N

corresponding to the other first row to be a $(t + i_1)$th associate of θ in N. Since Θ' is replicated r_2 times in $N_{\text{PBIB}}^{(t)}$, it follows that in the pair of replications of $N_{\text{PBIB}}^{(t)}$ considered above θ and the other treatment, which is defined to be a $(t + i_1)$th associate of θ in N, occur together in r_2 blocks of N. Taking into account the λ_{1i_1} pairs of replications of $N_{\text{PBIB}}^{(t)}$ brought to notice earlier, it follows that θ and its $(t + i_1)$th associate in N occur together in $r_2 \cdot \lambda_{1i_1}$ blocks of N. Also remembering that the number of the i_1th associates of Θ in $N_{\text{PBIB}}^{(s)}$ is n_{1i_1} we see that the number of the $(t + i_1)$th associates of θ in N is also n_{1i_1}. This identifies s more associate classes of θ in N and we see that the parameters n'_{t+i_1}, λ'_{t+i_1} of N given by

$$n'_{t+i_1} = n_{1i_1}, \qquad \lambda'_{t+i_1} = r_2 \cdot \lambda_{1i_1}, \qquad i_1 = 1, 2, \cdots, s, \tag{4.10}$$

have their usual significance for the design N.

Consider again for a moment the above pair of replications of $N_{\text{PBIB}}^{(t)}$. Consider in particular that component $N_{\text{PBIB}}^{(t)}$ in this pair which contains the $(t + i_1)$th associate of θ in N. The first row of this $N_{\text{PBIB}}^{(t)}$ corresponds to Θ'. Consider the i_2th associates of Θ' in this $N_{\text{PBIB}}^{(t)}$. These, considered as treatments of N, are defined to be $(t + i_1 + i_2 s)$th associates of θ in N. In the replication of $N_{\text{PBIB}}^{(t)}$ under consideration they are n_{2i_2} in number and each of them occurs together with θ in λ_{2i_2} blocks of N. Remembering that there are λ_{1i_1} such replications of $N_{\text{PBIB}}^{(t)}$ corresponding to each of the n_{1i_1} i_1th associates of Θ in $N_{\text{PBIB}}^{(s)}$ it follows that there are $n_{1i_1} \cdot n_{2i_2}$ $(t + i_1 + i_2 s)$th associates of θ in N and each of them occurs together with θ in $\lambda_{1i_1} \cdot \lambda_{2i_2}$ blocks of N. This identifies ts further associate classes of θ in N, and the parameters $n'_{t+i_1+i_2 s}$, $\lambda'_{t+i_1+i_2 s}$ of N given by

$$n'_{t+i_1+i_2 s} = n_{1i_1} \cdot n_{2i_2}, \qquad \lambda'_{t+i_1+i_2 s} = \lambda_{1i_1} \cdot \lambda_{2i_2} \tag{4.11}$$

are seen to have their usual significance for the design N.

Now, since $\sum_{i_1=1}^{s} n_{1i_1} = v_1 - 1$ and $\sum_{i_2=1}^{t} n_{2i_2} = v_2 - 1$ (cf. [2]), the number of treatments of N accounted for in the above identification of the various associate classes of θ in N is

$$\sum_{i_2=1}^{t} n_{2i_2} + \sum_{i_1=1}^{s} n_{1i_1} + \sum_{i_1=1}^{s} \sum_{i_2=1}^{t} n_{1i_1} \cdot n_{2i_2} = \left\{1 + \sum_{i_1=1}^{s} n_{1i_1}\right\}\left\{1 + \sum_{i_2=1}^{t} n_{2i_2}\right\} - 1$$
$$= v_1 v_2 - 1,$$

which together with θ exhausts all the $v_1 \cdot v_2$ treatments of N.

It may also be observed that if $1 \leqq i_1 \leqq s$ and $1 \leqq i_2 \leqq t$, then any integer m such that $t + s < m \leqq t + s + ts$ can be uniquely written in the form $m = t + i_1 + i_2 \cdot s$, that is, if $m = t + i_1 + i_2 s$ and if $m = t + i'_1 + i'_2 \cdot s$, then we must have $i_1 = i'_1$, and $i_2 = i'_2$. This fact ensures the uniqueness of the enumeration of the associate classes of θ in N, as described above.

This proves that the design N has at most $t + s + ts$ associate classes.

We now calculate the parameters p'^{x}_{yz}, $x, y, z = 1, 2, \cdots, t + s + ts$, of the design N.

Consider the treatment θ of N which corresponds to the first row of N. Let ϕ be an i_2th associate of θ in N, $i_2 = 1, 2, \cdots, t$. Then clearly the row in N corresponding to ϕ is contained in the first v_2 rows of N and these v_2 rows of N contain among themselves exactly r_1 replications of $N_{\text{PBIB}}^{(t)}$ as a whole and nothing else. The first row in any one of these replications of $N_{\text{PBIB}}^{(t)}$, which is a section of that corresponding to θ in N, corresponds to the treatment Θ' of $N_{\text{PBIB}}^{(t)}$. This replication of $N_{\text{PBIB}}^{(t)}$ also contains a row which is a section of the row corresponding to ϕ in N, and this row of $N_{\text{PBIB}}^{(t)}$ corresponds to the treatment Φ' of $N_{\text{PBIB}}^{(t)}$. Clearly Θ' and Φ' are i_2th associates of each other in $N_{\text{PBIB}}^{(t)}$. Now there are $p_{2j_2k_2}^{i_2}$ treatments of $N_{\text{PBIB}}^{(t)}$ which are in common with the j_2th associates of Θ' and the k_2th associates of Φ' in $N_{\text{PBIB}}^{(t)}$, for $j_2, k_2 = 1, 2, \cdots, t$. It is clear that exactly these $p_{2j_2k_2}^{i_2}$ treatments in the replication of $N_{\text{PBIB}}^{(t)}$ under consideration, considered as treatments of N, are those which are in common with the j_2th associates of θ and the k_2th associates of ϕ in N. Hence

$$p'^{i_2}_{j_2k_2} = p^{i_2}_{2j_2k_2}, \qquad i_2, j_2, k_2 = 1, 2, \cdots, t. \tag{4.12}$$

Also observe that the first v_2 rows of N contain all the first t associate classes of θ, and also of ϕ, and only these. Hence we must have

$$p'^{i_2}_{j_2u} = 0; \qquad i_2, j_2 = 1, 2, \cdots, t; u = t+1, t+2, \cdots, t+s+ts. \tag{4.13}$$

Next consider the $(t + j_1)$th associates of θ in N. They are the treatments of N corresponding to those rows in N which correspond to Θ' in the replications of $N_{\text{PBIB}}^{(t)}$ arising from each of the j_1th associates of Θ in $N_{\text{PBIB}}^{(s)}$. Similarly the $(t + k_1)$th associates of ϕ in N are the treatments of N which correspond to those rows in N which correspond to Φ' in the replications of $N_{\text{PBIB}}^{(t)}$ arising from each of the k_1th associates of Θ in $N_{\text{PBIB}}^{(s)}$. Since the treatments Θ' and Φ' are distinct we must have

$$p'^{i_2}_{t+j_1, t+k_1} = 0, \qquad j_1, k_1 = 1, 2, \cdots, s. \tag{4.14}$$

Again the $(t + k_1 + k_2 \cdot s)$th associates of ϕ in N are the k_2th associates of Φ' in the replications of $N_{\text{PBIB}}^{(t)}$ arising out of the k_1th associates of Θ in $N_{\text{PBIB}}^{(s)}$. To calculate the value of $p'^{i_2}_{t+j_1, t+k_1+k_2s}$ we have to count the number of treatments of N which are in common with the n_{1j_1} $(t + j_1)$th associates of θ and the $n_{1k_1} \cdot n_{2k_2}$ $(t + k_1 + k_2s)$th associates of ϕ in N. It is clear, from the way in which these associate classes are defined, that

$$p'^{i_2}_{t+j_1, t+k_1+k_2s} = 0 \quad \text{if} \quad j_1 \neq k_1,$$
$$p'^{i_2}_{t+j_1, t+j_1+k_2s} = 0 \quad \text{if} \quad i_2 \neq k_2,$$
$$p'^{i_2}_{t+j_1, t+j_1+i_2s} = n_{1j_1}.$$

It may be easily seen that the above relations can be written in the form

$$p'^{i_2}_{t+j_1, t+k_1+k_2s} = \delta_{i_2k_2} \cdot n_{1j_1} \cdot \delta_{j_1k_1}, \qquad j_1, k_1 = 1, 2, \cdots, s; k_2 = 1, 2, \cdots, t,$$

and since the indices j_1, k_1 have to run over their entire ranges before the index

k_2 can change its value, we can write the above equation in the matrix form

$$(4.15) \qquad (p'^{i_2}_{t+j_1, t+k_1+k_2s}) = (\delta_{i_2k_2}) \times (n_{1j_1}\delta_{j_1k_1}).$$

Finally let us consider the $(t + j_1 + j_2 \cdot s)$th associates of θ in N and the

$$(t + k_1 + k_2 \cdot s)\text{th}$$

associates of ϕ in N. The number of treatments of N in common with these two associate classes is $p'^{i_2}_{t+j_1+j_2 \cdot s, t+k_1+k_2 \cdot s}$. From the definitions of these two associate classes we find that

$$p'^{i_2}_{t+j_1+j_2s, t+k_1+k_2s} = 0 \quad \text{if} \quad j_1 \neq k_1,$$

$$p'^{i_2}_{t+j_1+j_2s, t+j_1+k_2s} = n_{1j_1}p^{i_2}_{2j_2k_2}.$$

These relations are easily seen to be equivalent to writing

$$p'^{i_2}_{t+j_1+j_2s, t+k_1+k_2s} = p^{i_2}_{2j_2k_2} \cdot n_{1j_1}\delta_{j_1k_1}, \quad j_1, k_1 = 1, 2, \cdots, s; j_2, k_2 = 1, 2, \cdots, t,$$

and since here also the indices j_1, k_1 have to run over their entire ranges before the indices j_2, k_2 can change their values, it follows that we can write the above equations in the matrix form

$$(4.16) \qquad (p'^{i_2}_{t+j_1+j_2s, t+k_1+k_2s}) = (p^{i_2}_{2j_2k_2}) \times (n_{1j_1}\delta_{j_1k_1}).$$

Combining the calculation in (4.12) to (4.16) and remembering that $p'^{i_2}_{yz} = p'^{i_2}_{zy}$, $i_2 = 1, 2, \cdots, t$; $y, z = 1, 2, \cdots, t + s + ts$, we get the first of the three matrices in (4.7).

Similar calculations will give the other two matrices in (4.7).

Thus the argument so far together with the results in (4.8) to (4.16) prove the statements (a) and (c) of Theorem 4.2.

Also from the way in which we have defined the various associate classes in N, we find that if the s associate classes of $N^{(s)}_{\text{PBIB}}$ and the t associate classes of $N^{(t)}_{\text{PBIB}}$ are all distinct, then the first t associate classes of N are all distinct, the next s associate classes of N are all distinct, and the last ts associate classes of N are all distinct. Further suppose that $\lambda_{1i_1} = r_1$ for some i_1, $1 \leqq i_1 \leqq s$. Then from (4.9) and (4.11) we find that $\lambda'_{i_2} = \lambda'_{t+i_1+i_2s}$, i_1 fixed; $i_2 = 1, 2, \cdots, t$; hence it may be possible to combine some of the corresponding associate classes. Similarly if $\lambda_{2i_2} = r_2$ for some i_2, $1 \leqq i_2 \leqq t$, then from (4.10) and (4.11) we find that $\lambda'_{t+i_1} = \lambda'_{t+i_1+i_2s}$, i_2 fixed; $i_1 = 1, 2, \cdots, s$; hence it may be possible to combine some of the corresponding associate classes. But if $\lambda_{1i_1} < r_1$ and $\lambda_{2i_2} < r_2$ for all $i_1 = 1, 2, \cdots, s$ and $i_2 = 1, 2, \cdots, t$, no such situation can arise and then the first t associate classes and the next s associate classes are distinct from the last ts associate classes of N. Finally if $r_1 \cdot \lambda_{2i_2} \neq r_2\lambda_{1i_1}$ for all $i_1 = 1, 2, \cdots, s$ and $i_2 = 1, 2, \cdots, t$, then the first t associate classes are distinct from the next s associate classes of N because of (4.9) and (4.10). This means that if the conditions in the statement (b) of Theorem 4.2 are satisfied, then the $t + s + ts$ associate classes of N are all distinct. This proves the statement (b) of Theorem 4.2.

Lastly the statement (d) of Theorem 4.2 is simply a provision for the cases to which the statement (b) of Theorem 4.2 does not apply.

This completes the proof of Theorem 4.2.

Although the following corollary is an obvious special case of Theorem 4.2 we state it separately because we shall require it for further investigation.

COROLLARY 4.2.1. (a) *The Kronecker product* $N = N_{(1)\mathrm{BIB}} \times N_{(2)\mathrm{BIB}}$ *of the two* BIB *designs* $N_{(1)\mathrm{BIB}}$ *and* $N_{(2)\mathrm{BIB}}$ *defined by the respective sets of parameters*

$$v_1^*,\ b_1^*,\ r_1^*,\ k_1^*,\ \lambda_1^*, \tag{4.17}$$

$$v_2^*,\ b_2^*,\ r_2^*,\ k_2^*,\ \lambda_2^* \tag{4.18}$$

is a PBIB *design* N *with at most three associate classes.*

(b) *The three associate classes of the design* N *defined above are all distinct if* $r_1^* \cdot \lambda_2^* \neq r_2^* \cdot \lambda_1^*$.

(c) *In any case the parameters of the design* N *can be expressed in terms of those of* $N_{(1)\mathrm{BIB}}$ *and* $N_{(2)\mathrm{BIB}}$ *by the equations*

$$\begin{aligned} &v' = v_1^* \cdot v_2^*, \quad b' = b_1^* \cdot b_2^*, \quad r' = r_1^* \cdot r_2^*, \quad k' = k_1^* \cdot k_2^*, \\ &n_1' = v_2^* - 1, \quad n_2' = v_1^* - 1, \quad n_3' = (v_1^* - 1)(v_2^* - 1), \\ &\lambda_1' = r_1^* \cdot \lambda_2^*, \quad \lambda_2' = r_2^* \cdot \lambda_1^*, \quad \lambda_3' = \lambda_1^* \cdot \lambda_2^*, \end{aligned} \tag{4.19}$$

$$(p_{yz}'^{1}) = \begin{bmatrix} v_2^* - 2 & 0 & 0 \\ 0 & 0 & v_1^* - 1 \\ 0 & v_1^* - 1 & (v_1^* - 1)(v_2^* - 2) \end{bmatrix}$$

$$(p_{yz}'^{2}) = \begin{bmatrix} 0 & 0 & v_2^* - 1 \\ 0 & v_1^* - 2 & 0 \\ v_2^* - 1 & 0 & (v_1^* - 2)(v_2^* - 1) \end{bmatrix}$$

$$(p_{yz}'^{3}) = \begin{bmatrix} 0 & 1 & v_2^* - 2 \\ 1 & 0 & v_1^* - 2 \\ v_2^* - 2 & v_1^* - 2 & (v_1^* - 2)(v_2^* - 2) \end{bmatrix}$$

where $y, z = 1, 2, 3$.

(d) *Lemma* 4.1 *can be applied to the cases in which the condition in* (b) *above is not fulfilled.*

We shall now obtain the conditions under which the Kronecker product N of two BIB designs $N_{(1)\mathrm{BIB}}$ and $N_{(2)\mathrm{BIB}}$, defined in Corollary 4.2.1.(a), is a PBIB design with only two distinct associate classes.

Since $\lambda_1^* < r_1^*$ and $\lambda_2^* < r_2^*$, it is clear that the first necessary condition is that $r_1^*\lambda_2^* = r_2^*\lambda_1^*$. In this case applying Lemma 4.1 to the first two matrices in (4.19) we find that the second necessary condition is that $v_1^* = v_2^*$. It is clear from Lemma 4.1 that these conditions are also sufficient for the design (4.19) to have only two distinct associate classes.

If the conditions $v_1^* = v_2^*$ and $r_1^* \cdot \lambda_2^* = r_2^* \cdot \lambda_1^*$ are satisfied, then from the relations among the parameters of BIB designs (cf. [11]), we must have $k_1^* = k_2^*$. Conversely, if we assume that $v_1^* = v_2^*$ and $k_1^* = k_2^*$, then we can deduce that $r_1^* \cdot \lambda_2^* = r_2^* \cdot \lambda_1^*$. This means that the conditions

$$v_1^* = v_2^*, \qquad k_1^* = k_2^* \tag{4.20}$$

are equivalent to the conditions

$$v_1^* = v_2^*, \qquad r_1^* \cdot \lambda_2^* = r_2^* \cdot \lambda_1^*, \tag{4.21}$$

and hence either (4.20) or (4.21) are necessary and sufficient conditions for the design (4.19) to have only two distinct associate classes. Under (4.20) or (4.21) we can further deduce that

$$\frac{b_2^*}{b_1^*} = \frac{r_2^*}{r_1^*} = \frac{\lambda_2^*}{\lambda_1^*} \tag{4.22}$$

These results are stated in the following corollary.

COROLLARY 4.2.2. *The necessary and sufficient conditions for the Kronecker product* $N = N_{(1)\mathrm{BIB}} \times N_{(2)\mathrm{BIB}}$ *of the* BIB *designs* $N_{(1)\mathrm{BIB}}$ *and* $N_{(2)\mathrm{BIB}}$ *with respective sets of parameters*

$$v_1^*, b_1^*, r_1^*, k_1^*, \lambda_1^*,$$
$$v_2^*, b_2^*, r_2^*, k_2^*, \lambda_2^*,$$

to have only two distinct associate classes are

$$v_1^* = v_2^* = v, \quad \text{say}, \qquad k_1^* = k_2^* = k, \quad \text{say}.$$

If these conditions are satisfied then we have $b_2^*/b_1^* = r_2^*/r_1^* = \lambda_2^*/\lambda_1^* = \mu$, *say, where* μ *is a positive fraction, and in this case the parameters of* N *are expressed in terms of those of* $N_{(1)\mathrm{BIB}}$ *and* $N_{(2)\mathrm{BIB}}$ *by the equations*

$$\begin{aligned} &v' = v^2, \quad b' = \mu \cdot (b_1^*)^2, \quad r' = \mu \cdot (r_1^*)^2, \quad k' = k^2, \\ &n_1' = 2(v-1), \quad n_2' = (v-1)^2, \quad \lambda_1' = \mu \cdot r_1^* \cdot \lambda_1^*, \quad \lambda_1' = \mu \cdot (\lambda_1^*)^2, \end{aligned} \tag{4.23}$$

$$(p_{yz}'^1) = \begin{pmatrix} v-2 & v-1 \\ v-1 & (v-1)(v-2) \end{pmatrix}, \qquad (p_{yz}'^2) = \begin{pmatrix} 2 & 2(v-2) \\ 2(v-2) & (v-2)^2 \end{pmatrix}.$$

The following definition of a cyclic design is given by Bose and Shimamoto [9].

Consider a PBIB design $N_{\mathrm{PBIB}}^{(2)}$ with two associate classes and with parameters $v, b, r, k, n_i, \lambda_i, p_{jk}^i, \quad i, j, k = 1, 2$. Let its treatments be designated by the integers $1, 2, \cdots, v$. The design $N_{\mathrm{PBIB}}^{(2)}$ is said to be a C (cyclic) design if the first associates of the treatment i of $N_{\mathrm{PBIB}}^{(2)}$ regarded as a PBIB design are the treatments

$$i + d_1, \quad i + d_2, \cdots, i + d_{n_1} \mod v$$

where the d's satisfy the conditions:

(i) the d's are all different and $0 < d_j < v$ for $j = 1, 2, \cdots, n_1$;

(ii) among the $n_1(n_1 - 1)$ differences $d_j - d_{j'}$, $j, j' = 1, 2, \cdots, n_1; j \neq j'$; reduced mod v each of the numbers $d_1, d_2, \cdots, d_{n_1}$ occurs α times whereas each of the numbers $e_1, e_2, \cdots, e_{n_2}$ occurs β times where $d_1 d_2, \cdots, d_{n_1}, e_1, e_2, \cdots, e_{n_2}$ are all the different $v - 1$ numbers $1, 2, \cdots, v - 1$.

Clearly it is necessary that

$$n_1\alpha + n_2\beta = n_1(n_1 - 1). \tag{4.24}$$

The parameters p^i_{jk}, $i, j, k = 1, 2$, of $N^{(2)}_{\mathrm{PBIB}}$ are in this case given by

$$(p^1_{jk}) = \begin{pmatrix} \alpha & n_1 - \alpha - 1 \\ n_1 - \alpha - 1 & n_2 - n_1 + \alpha + 1 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} \beta & n_1 - \beta \\ n_1 - \beta & n_2 - n_1 + \beta + 1 \end{pmatrix}. \tag{4.25}$$

If we take $\alpha = v - 2$ and $\beta = 2$, then we find that the necessary conditions (4.24) and (4.25) are satisfied by the corresponding parameters in (4.23). Let the treatments of the design N of (4.23) be designated by integers $1, 2, \cdots, v'$. Then, according to the method of identification of the various associate classes in N described in the proof of Theorem 4.2, the first associates of the treatment 1 in N are treatments $2, 3, \cdots, v, v + 1, 2v + 1, \cdots, v^2 - v + 1$. The corresponding d's are clearly $1, 2, \cdots, v - 1, v, 2v, \cdots, v^2 - v$. If we form the

$$2(v - 1)(2v - 3)$$

differences $d_j - d'_j$, $j, j' = 1, 2, \cdots, 2(v - 1); j \neq j'$; of these $2(v - 1)$ d's, it is obvious that $d_1 = 1$ will occur in these differences exactly $v - 1$ times whereas if the design N of (4.23) were a cyclic design, $d_1 = 1$ must occur only $\alpha = v - 2$ times. Hence we find that the design N of (4.23) cannot be a cyclic design even though its parameters satisfy the necessary conditions (4.24) and (4.25).

5. Construction of certain PBIB designs. From the results of Section 4 we find that two Kronecker products which give PBIB designs with two associate classes are

(i) $N_2 \times N_{\mathrm{BIB}}$ (Corollary 4.1.1),

(ii) $N_{(1)\mathrm{BIB}} \times N_{(2)\mathrm{BIB}}$ where $N_{(1)\mathrm{BIB}}$ and $N_{(2)\mathrm{BIB}}$ are two BIB designs for which $v_1^* = v_2^*$ and $k_1^* = k_2^*$ (Corollary 4.2.2).

Bose, Shrikhande, and Bhattacharya [13] have obtained certain singular GD designs by applying Corollary 4.1.1, which is the only way of getting them. The following example illustrates Corollary 4.2.2.

EXAMPLE 5.1. Let us take $N_{(1)\mathrm{BIB}}$ to be the BIB design defined by the parameters

$$v^* = b^* = 4, \qquad r^* = k^* = 3, \qquad \lambda^* = 2 \tag{5.1}$$

(cf. Cochran and Cox [6]). Let $N_{(2)\mathrm{BIB}}$ be the same as $N_{(1)\mathrm{BIB}}$ so that the value of μ in Corollary 4.2.2 is 1. Then clearly the parameters v', b', r', k' of

$$N = N_{(1)\mathrm{BIB}} \times N_{(2)\mathrm{BIB}}$$

are

$$v' = b' = 16, \qquad r' = k' = 9. \tag{5.2}$$

Let the treatments of N be designated by the integers $1, 2, \cdots, 16$. The proof of Theorem 4.2 contains a description of the method of identifying the various associate classes of N. According to this method we get the following identifications.

Treatment	1	2	6
First associates	2, 3, 4, 5, 9, 13.	1, 3, 4, 6, 10, 14.	2, 5, 7, 8, 10, 14.
Second associates	6, 7, 8, 10, 11, 12, 14, 15, 16.	5, 7, 8, 9, 11, 12, 13, 15, 16.	1, 3, 4, 9, 11, 12, 13, 15, 16.

It is clear that the parameters n_1', n_2', λ_1', λ_2' of N are given by

$$n_1' = 6, \qquad n_2' = 9, \qquad \lambda_1' = 6, \qquad \lambda_2' = 4. \tag{5.3}$$

Further the comparisons of the associate classes of treatment 1 with those of treatments 2 and 6 respectively lead to

$$(p_{jk}^{\prime 1}) = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}, \qquad (p_{jk}^{\prime 2}) = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix} \tag{5.4}$$

where $p_{jk}^{\prime i}$, $i, j, k = 1, 2$, are parameters of N. It may be noted that the design N is not cyclic even though the necessary conditions (4.24) and (4.25) are satisfied by its parameters.

The equations (5.2) to (5.4) give all the parameters of the design N. The blocks of the design N are shown below.

(1, 2, 3, 5, 6, 7, 9, 10, 11),
(1, 2, 4, 5, 6, 8, 9, 10, 12),
(1, 3, 4, 5, 7, 8, 9, 11, 12),
(2, 3, 4, 6, 7, 8, 10, 11, 12),
(1, 2, 3, 9, 10, 11, 13, 14, 15),
(1, 2, 4, 9, 10, 12, 13, 14, 16),
(1, 3, 4, 9, 11, 12, 13, 15, 16),
(2, 3, 4, 10, 11, 12, 14, 15, 16),
(1, 2, 3, 5, 6, 7, 13, 14, 15),
(1, 2, 4, 5, 6, 8, 13, 14, 16),
(1, 3, 4, 5, 7, 8, 13, 15, 16),
(2, 3, 4, 6, 7, 8, 14, 15, 16),
(5, 6, 7, 9, 10, 11, 13, 14, 15),
(5, 6, 8, 9, 10, 12, 13, 14, 16),
(5, 7, 8, 9, 11, 12, 13, 15, 16),
(6, 7, 8, 10, 11, 12, 14, 15, 16).

From the remaining results of Section 4 we find that the following Kronecker products give PBIB designs with more than two associate classes.

(i) $N_2 \times N_{\mathrm{PBIB}}^{(s)}$ (Theorem 4.1),

(ii) $N_{\mathrm{PBIB}}^{(s)} \times N_{\mathrm{PBIB}}^{(t)}$ (Theorem 4.2),

(iii) $N_{(1)\text{BIB}} \times N_{(2)\text{BIB}}$ (Corollary 4.2.1).

The following example illustrates Corollary 4.2.1.

EXAMPLE 2. Let $N_{(1)\text{BIB}}$ and $N_{(2)\text{BIB}}$ be the BIB designs defined by the sets of parameters

$$(5.5) \qquad v_1^* = b_1^* = 3, \quad r_1^* = k_1^* = 2, \quad \lambda_1^* = 1,$$

$$(5.6) \qquad v_2^* = 5, \quad b_2^* = 10, \quad r_2^* = 4, \quad k_2^* = 2, \quad \lambda_2^* = 1$$

respectively (cf. Cochran and Cox [6]). Then clearly the parameters v', b', r', k' of $N = N_{(1)\text{BIB}} \times N_{(2)\text{BIB}}$ are given by

$$(5.7) \qquad v' = 15, \quad b' = 30, \quad r' = 8, \quad k' = 4.$$

Let the treatments of N be designated by integers 1, 2, $\cdots$, 15. According to the method of identifying the various associate classes of N described in the proof of Theorem 4.2, we get the following identifications.

Treatment	1	2	6	7
First associates	2, 3, 4, 5.	1, 3, 4, 5.	7, 8, 9, 10.	6, 8, 9, 10.
Second associates	6, 11.	7, 12.	1, 11.	2, 12.
Third associates	7, 8, 9, 10, 12, 13, 14, 15.	6, 8, 9, 10, 11, 13, 14, 15.	2, 3, 4, 5, 12, 13, 14, 15.	1, 3, 4, 5, 11, 13, 14, 15.

Also it is clear that the parameters n_1', n_2', n_3', λ_1', λ_2', λ_3' of N are given by

$$(5.8) \qquad n_1' = 4, \quad n_2' = 2, \quad n_3' = 8, \quad \lambda_1' = 4, \quad \lambda_2' = 2, \quad \lambda_3' = 1.$$

Further, the comparisons of the associate classes of treatment 1 with those of treatments 2, 6, and 7 respectively lead to

$$(5.9) \qquad (p_{jk}'^1) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 6 \end{pmatrix}, \quad (p_{jk}'^2) = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix}, \quad (p_{jk}'^3) = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

where $p_{jk}'^i$, $i, j, k = 1, 2, 3$, are parameters of N.

The equations (5.7) to (5.9) give all the parameters of the design N. The blocks of the design are as shown below.

(1, 2, 6, 7),	(1, 3, 6, 8),	(1, 4, 6, 9),	(1, 5, 6, 10),	(2, 3, 7, 8),
(2, 4, 7, 9),	(2, 5, 7, 10),	(3, 4, 8, 9),	(3, 5, 8, 10),	(4, 5, 9, 10),
(1, 2, 11, 12),	(1, 3, 11, 13),	(1, 4, 11, 14),	(1, 5, 11, 15),	(2, 3, 12, 13),
(2, 4, 12, 14),	(2, 5, 12, 15),	(3, 4, 13, 14),	(3, 5, 13, 15),	(4, 5, 14, 15),
(6, 7, 11, 12),	(6, 8, 11, 13),	(6, 9, 11, 14),	(6, 10, 11, 15),	(7, 8, 12, 13),
(7, 9, 12, 14),	(7, 10, 12, 15),	(8, 9, 13, 14),	(8, 10, 13, 15),	(9, 10, 14, 15).

EXAMPLE 5.3. Consider the PBIB design N with three associate classes and with parameters:

$$v = b = pq, \quad r = k = p + q - 1,$$

$$n_1 = p - 1, \quad n_2 = q - 1, \quad n_3 = (p - 1)(q - 1), \tag{5.10}$$

$$\lambda_1 = p, \quad \lambda_2 = q, \quad \lambda_3 = 2,$$

$$(p^1_{jk}) = \begin{bmatrix} p-2 & 0 & 0 \\ 0 & 0 & q-1 \\ 0 & q-1 & (p-2)(q-1) \end{bmatrix},$$

$$(p^2_{jk}) = \begin{bmatrix} 0 & 0 & p-1 \\ 0 & q-2 & 0 \\ p-1 & 0 & (p-1)(q-2) \end{bmatrix},$$

$$(p^3_{jk}) = \begin{bmatrix} 0 & 1 & p-2 \\ 1 & 0 & q-2 \\ p-2 & q-2 & (p-2)(q-2) \end{bmatrix}$$

where p, q are positive integers $\geqq 2$ and $j, k = 1, 2, 3$.

This design taken from Bose and Nair [2] very much resembles the Kronecker product of two BIB designs. Let us suppose that the above design is the Kronecker product of the two BIB designs defined by the sets of parameters

$$v_1^* = q, \quad b_1^*, \quad r_1^*, \quad k_1^*, \quad \lambda_1^*$$

and

$$v_2^* = p, \quad b_2^*, \quad r_2^*, \quad k_2^*, \quad \lambda_2^*.$$

From Corollary 4.2.1 it follows that we must have

$$\lambda_1 = r_1^*\lambda_2^* = p, \quad \lambda_2 = r_2^*\lambda_1^* = q, \quad \lambda_3 = \lambda_1^* \cdot \lambda_2^* = 2.$$

Hence

$$pq = \lambda_1 \cdot \lambda_2 = \lambda_1^*\lambda_2^* r_1^* r_2^* = 2r_1^* r_2^* = 2k_1^* k_2^* = 2(p + q - 1),$$

which leads to

$$(p - 2)(q - 2) = 2. \tag{5.11}$$

This means that a necessary condition for the design N of (5.10) to be the Kronecker product of two BIB designs is (5.11).

Also since p and q are positive integers, we must have from (5.11) either $p = 3$ and $q = 4$ or $p = 4$ and $q = 3$. It is enough to consider one case, say, $p = 4$, $q = 3$. With these values it is clear that (5.11) is satisfied. The corresponding

BIB designs are defined by the sets of parameters

$$v_1^* = b_1^* = 3, \qquad r_2^* = k_2^* = 2, \qquad \lambda_1^* = 1,$$

and

$$v_2^* = b_2^* = 4, \qquad r_2^* = k_2^* = 3, \qquad \lambda_2^* = 2.$$

Now applying Corollary 4.2.1 to these two BIB designs it is easily verified that their Kronecker product has the parameters of design N of (5.10) for which $p = 4$ and $q = 3$. Thus we find that the condition (5.11) is also sufficient for the design N of (5.10) to be constructible as the Kronecker product of two BIB designs.

It has been remarked by Bose and Nair [2] that the design N of (5.10) with three associate classes reduces to a PBIB design with two associate classes if $p = 2$ or $q = 2$. Because of Lemma 4.1 we can further add that the design N with three associate classes reduces to a PBIB design with two associate classes if $p = q$.

The method of taking Kronecker product of designs has been used to prove the impossibility of a certain class of PBIB designs and to analyse some other class of designs. It is hoped to publish at a later date some results in this direction.

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