BALANCED INCOMPLETE BLOCK DESIGNS AND TACTICAL CONFIGURATIONS

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1. Summary and Introduction. A balanced incomplete block design (BIB design) is an arrangement of v varieties or treatments in b blocks of k distinct varieties each, so that each variety is contained in r blocks and every pair of varieties is contained in λ blocks. Various methods of constructing such designs are discussed in [2], and certain designs are listed in [3], [4], [5], [7], [14]. If $v = b$, the design is said to be symmetric; the impossibility of certain symmetric designs was proved in [10].

Although in [8] certain tactical configurations are discussed, it seems that the relationship between BIB designs and tactical configurations, and in particular, the Steiner system, has been overlooked. It is the purpose of this note to point out this relationship and to discuss the properties of designs arising from such configurations.

2. Tactical configurations. A complete α-β-k-v configuration is an arrangement of v elements in blocks of k so that each set of β elements occurs in exactly α blocks. A Steiner system is a complete 1-β-k-v configuration, that is, v elements arranged in blocks of k so that each set of β elements occurs exactly once. Various systems of this kind are discussed in [6], [9], [12], [13]. We shall use the notation $S(\beta, k, v)$ to denote a Steiner system; thus $S(2, 3, v)$ is a triple system, $S(2, p^n + 1, 1 + p^n + p^{2n})$ is a finite two-dimensional projective geometry, and $S(2, p^n, p^{2n})$ is a finite two-dimensional Euclidean geometry.

A list of some of the properties of Steiner systems is:

(1) The existence of $S(\beta, k, v)$ implies the existence of $S(\beta - r, k - r, v - r)$ for $r < \beta$, [12], [13]. (Similarly, the existence of the α-β-k-v configuration implies the existence of the α-$(\beta - r)$-$(k - r)$-$(v - r)$ configuration.)

(2) The existence of $S(2, p^n, v_1)$ and $S(2, p^n, v_2)$ implies the existence of $S(2, p^n, v_1v_2)$ [12], [13].

(3) The existence of $S(2, p^n + 1, v)$ implies the existence of $S(2, p^n + 1, p^nv + 1)$ [12], [13].

(4) The existence of $S(3, 4, v)$ implies the existence of $S(3, 4, 2v)$ [12], [13].

(5) $S(3, p^n + 1, p^{nf} + 1)$ exists if p is prime [12], [13].

(6) $S(2, 2^{f-1}, \frac{1}{2}(4^f - 2^f))$ exists [12], [13].

(7) The triple system $S(2, 3, v)$ exists for $v = 6m + 1$ or $6m + 3$, [2], [6], [9].

(8) The 2-3-4-$(p + 1)$ configuration exists if p is a prime of the form $6m + 1$. If m is odd, say $m = 2n + 1$, the system subdivides into two 1-3-4-$(6n + 4)$ configurations [6].

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3. The α-β-k-v configuration and BIB designs.

THEOREM 3.1. *The α-β-k-v configuration is a* BIB *design with parameters*

$$v, \quad b = \alpha\binom{v}{\beta}/\binom{k}{\beta}, \quad r = \alpha\binom{v-1}{\beta-1}/\binom{k-1}{\beta-1}, \quad k,$$

$$\lambda = \alpha\binom{v-2}{\beta-2}/\binom{k-2}{\beta-2} \text{ if } \beta \geqq 3, \quad \lambda = \alpha \text{ if } \beta = 2.$$

PROOF. Each set of β elements determines α and only α blocks. There are $\alpha\binom{v}{\beta}$ such sets, each block giving $\binom{k}{\beta}$ of them. Hence the number of blocks is

$$b = \alpha\binom{v}{\beta}/\binom{k}{\beta}.$$

The number of replications is the block number for the α-$(\beta - 1)$-$(k - 1)$-$(v - 1)$ configuration, and so

$$r = \alpha\binom{v-1}{\beta-1}/\binom{k-1}{\beta-1}.$$

Similarly, λ is the block number for the α-$(\beta - 2)$-$(k - 2)$-$(v - 2)$ configuration, and so

$$\lambda = \alpha\binom{v-2}{\beta-2}/\binom{k-2}{\beta-2}.$$

If $\beta = 2$, every pair of elements occurs α times by definition. Finally, every block contains k different elements.

COROLLARY. *Every set of u elements occurs in λ_u blocks where*

$$\lambda_u = \alpha\binom{v-u}{\beta-u}/\binom{k-u}{\beta-u}.$$

PROOF. This is just the block number for the α-$(\beta - u)$-$(k - u)$-$(v - u)$ configuration.

4. Series of BIB designs formed from tactical configurations.

THEOREM 4.1. *If a* BIB *design with parameters $v, b, r, k, \lambda = \lambda_2$ has the additional property that every set of u elements occurs exactly λ_u times, then the design splits into two smaller designs with parameters*

$$(1) \quad v' = v - 1, \quad b' = b - r, \quad r' = r - \lambda_2, \quad k' = k, \quad \lambda_i' = \lambda_i - \lambda_{i+1};$$

$$(2) \quad v'' = v - 1, \quad b'' = r, \quad r'' = \lambda_2, \quad k'' = k - 1, \quad \lambda_i'' = \lambda_{i+1}.$$

PROOF. Consider the blocks remaining when, from the original design, all blocks containing a given element are deleted. In the original design, every set of i elements occurred λ_{i+1} times with the given element, and λ_i times in all. Since only blocks containing the given element were deleted, we get

$$\lambda_i' = \lambda_i - \lambda_{i+1}, \quad r' = r - \lambda_2, \quad b' = b - r.$$

This is the first design of the theorem. If the given element is deleted from all blocks which contain it, design (2) is obtained with $v'' = v - 1$, $b'' = r$, $k'' = k - 1$. λ_i'' is the number of times that every set of i elements occurs, and this is equal to the number of times that every set of i elements occurred with the given element in the original design, that is to λ_{i+1}. Obviously, $r'' = \lambda_2$.

COROLLARY. *Any α-β-k-v configuration with $\beta > 2$ splits in the manner described in Theorem* 4.1.

PROOF. Theorem 3.1 and the definition of the complete α-β-k-v configuration ensure that the required conditions are satisfied.

Example 4.1. The 2-3-4-$(p + 1)$ configuration exists if p is a prime of the form $6m + 1$. Hence we obtain from it the design

$$v = p + 1, \quad b = p(p^2 - 1)/12, \quad r = p(p - 1)/3, \quad k = 4, \quad \lambda_2 = p - 1, \quad \lambda_3 = 2.$$

If $m = 1$ we have

$$v = 14, \quad b = 182, \quad r = 52, \quad k = 4, \quad \lambda_2 = 12, \quad \lambda_3 = 2,$$

which splits into the two designs (13, 130, 40, 4, 10) and (13, 52, 12, 3, 2).

Example 4.2. A great many Steiner systems with $\beta = 3$ are known; however, only four are known with $\beta > 3$, namely, $S(5, 6, 12)$, $S(5, 8, 24)$, together with their derived systems $S(4, 5, 11)$ and $S(4, 7, 23)$. The method of constructing these systems is outlined in [6]. Thus the design $S(5, 6, 12)$, with parameters

$$v = 12, \quad b = 132, \quad r = 66, \quad k = 6, \quad \lambda_2 = 30, \quad \lambda_3 = 12, \quad \lambda_4 = 4, \quad \lambda_5 = 1,$$

splits to form the following series of designs:

$S(4, 5, 11)$	$v = 11$,	$b = 66$,	$r = 30$,	$k = 5$,	$\lambda_2 = 12$,	$\lambda_3 = 4$,	$\lambda_4 = 1$
$S(3, 4, 10)$	$v = 10$,	$b = 30$,	$r = 12$,	$k = 4$,	$\lambda_2 = 4$,	$\lambda_3 = 1$,	
$S(2, 3, 9)$	$v = 9$,	$b = 12$,	$r = 4$,	$k = 3$,	$\lambda_2 = 1$,		
	$v = 11$,	$b = 66$,	$r = 36$,	$k = 6$,	$\lambda_2 = 18$,	$\lambda_3 = 4$,	$\lambda_4 = 3$
	$v = 10$,	$b = 36$,	$r = 18$,	$k = 5$,	$\lambda_2 = 8$,	$\lambda_3 = 3$,	
	$v = 10$,	$b = 30$,	$r = 18$,	$k = 6$,	$\lambda_2 = 10$,	$\lambda_3 = 5$,	
	$v = 9$,	$b = 8$,	$r = 8$,	$k = 4$,	$\lambda_2 = 3$,		
	$v = 9$,	$b = 18$,	$r = 10$,	$k = 5$,	$\lambda_2 = 5$,		
	$v = 9$,	$b = 12$,	$r = 8$,	$k = 6$,	$\lambda_2 = 5$.		

A similar series of nine designs arises from $S(5, 8, 24)$.

5. The construction of configurations from BIB designs.

Using the BIB designs.

(1) $(2k, 2r, r, k, \lambda)$ or

(2) $(2k - 1, b, 2\lambda, k, \lambda)$,

it is possible to construct a complete configuration consisting of blocks B_i of either (1) or (2) together with blocks B_i'; if (1) is used, blocks B_i' are the complements of blocks B_i, i.e., they contain the elements not contained in blocks B_i; if (2) is used, blocks B_i' are the complements of blocks B_i, and in addition contain the element ∞. It is obvious that in either case the resulting configuration has $v = 2k$ elements and all blocks contain k distinct elements.

THEOREM 5.1. *If in the previous construction series* (1) *is used, a complete* $(3\lambda - r)$-3-k-$2k$ *configuration is obtained; if series* (2) *is used, a complete* $(b - 3\lambda)$-3-k-$2k$ *configuration is obtained.*

PROOF. Let B_i be the blocks of a general design (v, b, r, k, λ); then B_i' are the blocks of a design $(v, b, b - r, v - k, b - 2r + \lambda)$. Suppose that a specified triplet x, y, z, occurs in exactly c blocks B_i ; then exactly two of x, y, z, occur together in $\lambda - c$ blocks B_i, and exactly one of x, y, z, occurs in $r - 2(\lambda - c) - c = r - 2\lambda + c$ blocks B_i. Hence the total number of blocks B_i containing x, y, or z is $3(r - 2\lambda + c) + 3(\lambda - c) + c = 3r - 3\lambda + c$, and the number of blocks B_i not containing x, y, or z is $b - 3r + 3\lambda - c$, which is equal to the number of blocks B_i' containing x, y, and z. Therefore any triplet x, y, z, occurs in $b - 3r + 3\lambda$ blocks B_i and B_i'. In series (1), $b - 3r + 3\lambda = 3\lambda - r$. In series (2), $b - 3r + 3\lambda = b - 3\lambda$. Also, ∞ occurs in all blocks B_i', and hence occurs $b - 2r + \lambda = b - 3\lambda$ times with every pair of elements. Therefore all triplets occur $b - 3\lambda$ times.

Example 5.1. In series (2) choose $k = 2\lambda$, that is, (2) is the series $(4\lambda - 1, 4\lambda - 1, 2\lambda, 2\lambda, \lambda)$, which exists if $4\lambda - 1$ is a prime power (Theorem 4.1, [11]). The theorem allows us to form the complete $(\lambda - 1)$-3-2λ-4λ configuration. This is a series of BIB designs with parameters

$$v = 4\lambda, \quad b = 2(4\lambda - 1), \quad r = 4\lambda - 1, \quad k = 2\lambda, \quad \lambda_2 = 2\lambda - 1, \quad \lambda_3 = \lambda - 1.$$

This series of designs is affine resolvable (see section 7).

Example 5.2. In series (2) choose $k = \lambda$, that is, (2) is the series $(2\lambda - 1, 4\lambda - 2, 2\lambda, \lambda, \lambda)$ which exists if $(2\lambda - 1)$ is a prime power. The theorem gives us the complete $(\lambda - 2)$-3-λ-2λ configuration, which is the resolvable series of BIB designs

$$v = 2\lambda, \quad b = 4(2\lambda - 1), \quad r = 2(2\lambda - 1), \quad k = \lambda, \quad \lambda_2 = 2\lambda - 2, \quad \lambda_3 = \lambda - 1.$$

Some special cases of the preceding series are constructed in [15], where possible applications of such designs are also considered.

6. Symmetrical BIB designs derived from Steiner systems. In the next two sections the symmetry and resolvability of designs arising from Steiner systems will be discussed.

THEOREM 6.1. *If $S(2, k, v)$ exists, then a necessary and sufficient condition for it to be a symmetrical design is*

$$v = k^2 - k + 1.$$

The proof of this theorem is obvious.

THEOREM 6.2. *Except for the trivial case $S(3, 3, 4)$, $S(3, k, v)$ is not a symmetrical design.*

PROOF. The existence of $S(3, k, v)$ implies the existence of $S(2, k - 1, v - 1)$; applying the necessary condition $b \geqq v$ to the latter design, we get

$$v - 2 \geqq (k - 1)(k - 2).$$

Also, for $S(3, k, v)$ to be symmetric we must have

$$b = v = v(v - 1)(v - 2)/k(k - 1)(k - 2).$$

This can be written as $(v - 2)^2 + (v - 2) - k(k - 1)(k - 2) = 0$. Hence $v - 2 = \frac{1}{2}\{-1 + \sqrt{1 + 4k(k - 1)(k - 2)}\} \geqq (k - 1)(k - 2)$, or

$$(k - 1)^2(k - 2)(k - 3) \leqq 0,$$

admitting $k = 1, 2$, and 3. Since $\beta = 3$, the only possible value is $k = 3$, and then $v = 4$.

THEOREM 6.3. *The design* (1) *obtained from* $S(3, k, v)$ *in Theorem* 4.1 *is not symmetric unless* $k = 4$.

PROOF. For this design to be symmetric, we must have

$$v - 1 = b - r = \frac{v(v - 1)(v - 2)}{k(k - 1)(k - 2)} - \frac{(v - 1)(v - 2)}{(k - 1)(k - 2)}.$$

Therefore, $(v - 2)^2 - (v - 2)(k - 2) - k(k - 1)(k - 2) = 0$, and $v - 2 = \frac{1}{2}\{(k - 2) + \sqrt{(k - 2)^2 + 4k(k - 1)(k - 2)}\} \geqq (k - 1)(k - 2)$ as in Theorem 6.2. Thus $(k - 1)^2(k - 2)(k - 4) \leqq 0$, admitting $k = 1, 2, 3$, and 4. Since $\beta = 3$, the only possible values are 3 and 4, and $k = 3$ is trivial. For $k = 4$, we have $v = 8$, so that $S(3, 4, 8)$ is the only system $S(3, k, v)$ that splits to give a symmetric design.

It happens that in the case of $S(3, 4, 8)$, the derived series (2) in Theorem 4.1 is also symmetric; the two sub-designs are (7, 7, 4, 4, 2) and (7, 7, 3, 3, 1). $S(3, 4, 8)$ is the configuration of series in example 5.1 with $\lambda = 2$, and is a BIB design (8, 14, 7, 4, 3).

7. Resolvability of designs derived from Steiner systems. A BIB design is resolvable if the blocks can be grouped so that each group contains one complete replication. These designs are discussed in [1], where it is proved that the necessary conditions for resolvability are (1) v is divisible by k; (2) $b \geqq v + r - 1$. If a design is resolvable and if $b = v + r - 1$, then it is affine resolvable and k^2 is divisible by v; in this case, pairs of blocks chosen from two different replications have a constant number k^2/v of elements in common. We have at once that for $n > 0$, $S(2, k, k^2 + nk)$ fulfils the necessary conditions for resolvability; $S(2, k, k^2 - nk)$ is not resolvable; if $S(2, k, k^2)$ is resolvable, then it is affine resolvable.

THEOREM 7.1. *All designs* $S(3, k, nk)$ *satisfy the necessary conditions for resolvability; the only possible affine resolvable design* $S(3, k, nk)$ *is* $S(3, 4, 8)$.

PROOF. Because of resolvability, $b \geqq v + r - 1$, which implies

$$\frac{v(v - 1)(v - 2)}{k(k - 1)(k - 2)} \geqq v + \frac{(v - 1)(v - 2)}{(k - 1)(k - 2)} - 1,$$

that is, $(v - 2)^2 - (k - 2)(v - 2) - k(k - 1)(k - 2) \geqq 0$. Therefore

$$(v - 2) \geqq \tfrac{1}{2}\{(k - 2) + \sqrt{(k - 2)^2 + 4k(k - 1)(k - 2)}\}.$$

For affine resolvability equality holds, and then $(k-1)(k-2) \leqq v-2 = \frac{1}{2}\{(k-2)+\sqrt{(k-2)^2+4k(k-1)(k-2)}\}$. Thus $(k-1)^2(k-2)(k-4) \leqq 0$, admitting $k = 1, 2, 3$, and 4. Since $\beta = 3$, the only possible values are 3 and 4, and $k = 3$ is trivial. Hence the only possible non trivial affine resolvable design $S(3, k, v)$ is $S(3, 4, 8)$; since $S(3, 4, 8)$ is a member of the series of example 5.1, it actually is affine resolvable.

THEOREM 7.2. *Design* (1) *of Theorem* 4.1 *derived from* $S(3, k, v)$ *is not affine resolvable, but satisfies the condition for resolvability.*

PROOF. Here $b \geqq v + r - 1$ implies

$$\frac{(v-1)(v-2)(v-k)}{k(k-1)(k-2)} \geqq \frac{(v-2)(v-k)}{(k-1)(k-2)} + v - 2.$$

Proceeding as in Theorem 7.1, we get

$$(k-1)(k-2) \leqq (v-2) \geqq \tfrac{1}{2}\{2k-3+\sqrt{(2k-3)^2+4(k-1)^2(k-2)}\}.$$

For affine resolvability this simplifies to give $(k-1)(k-2)(k^2-6k+6) \leqq 0$, admitting $k = 1, 2, 3$, and 4. Since $\beta = 3$, the only possible values are 3 and 4. The design corresponding to $k = 3$ is trivial, and the design derived from $S(3, 4, 8)$ is not resolvable. For $k > 4$, then $b > v + r - 1$.

8. S(3, 4, 8). S(3, 4, 8) is a BIB design with parameters (8, 14, 7, 4, 3). Numbering the elements from 1 to 8, the design may be written:

(1, 6, 7, 8)	(2, 3, 4, 5)	(1, 4, 5, 8)	(2, 3, 6, 7)
(1, 3, 5, 7)	(2, 4, 6, 8)	(1, 3, 4, 6)	(2, 5, 7, 8)
(1, 2, 5, 6)	(3, 4, 7, 8)	(1, 2, 4, 7)	(3, 5, 6, 8)

(1, 2, 3, 8) (4, 5, 6, 7).

Each pair of blocks is a replication, and any block from one replication has exactly $k^2/v = 2$ elements in common with all blocks in the other replications.

This is the only Steiner system that splits to give symmetrical designs. The design (7, 7, 3, 3, 1), that is, $S(2, 3, 7)$ is obtained by selecting blocks containing a given element, say 8: (1, 6, 7) (1, 2, 3) (2, 4, 6) (3, 4, 7) (1, 4, 5) (2, 5, 7) (3, 5, 6). The design (1) of Theorem 4.1 is formed by the remaining blocks: (1, 3, 5, 7) (1, 2, 5, 6) (2, 3, 6, 7) (4, 5, 6, 7) (1, 3, 4, 6) (1, 2, 4, 7) (2, 3, 4, 5); the parameters are (7, 7, 4, 4, 2).

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