MULTI-LEVEL CONTINUOUS SAMPLING PLANS[1]

By Gerald J. Lieberman and Herbert Solomon

Stanford University and Teachers College, Columbia University

1. Summary and introduction. In 1943 Dodge [1] published a sampling inspection plan for a continuous production line. He assumed the production process to be in statistical control and also assumed the items were classified, after measurement, as "defective" or "non-defective". Dodge derived the Average Outgoing Quality (AOQ) function for his plan, obtained the Average Outgoing Quality Limit (AOQL), and provided a graphical procedure for choosing the parameters of the plan which guarantee a specified AOQL. Wald and Wolfowitz [2], in 1945, discussed a sampling inspection plan for continuous production which insures a prescribed limit on the outgoing quality even when production is not in statistical control. However, they demonstrate an awareness of the penalty involved in accomplishing this end and discuss other desirable features an optimal plan should enjoy, namely, a minimum amount of inspection to reduce inspection costs, and protection to insure what they term "local stability", i.e., the ability to detect quickly "too many long sequences" of poor quality. Dodge in his paper also discusses minimum inspection and an idea similar to "local stability" which he calls "protection against spotty quality".

An inconvenient feature of both plans is the abrupt change between partial inspection and 100% inspection. This can lead to hardships in personnel assignments in the administration of an inspection program. For example, in an item such as aircraft engines, a smoother transition to 100% inspection is needed. Both plans also tend to produce a form of tightened inspection when the process average may not warrant it. In a later paper [3] Dodge considers two modifications of his plan which delay the beginning of 100% inspection and also add some insurance for local stability. He derived the AOQ function for each of the two plans.

The primary purpose of this paper is to consider an extension of Dodge's first plan which (*a*) allows for smoother transition between sampling inspection and 100% inspection, (*b*) requires 100% inspection only when the quality submitted is quite inferior, and (*c*) allows for a minimum amount of inspection when quality is definitely good. This aim is accomplished by the introduction of a multi-level sampling plan which specifically allows for any number of sampling levels subject to the provision that transitions can only occur between adjacent levels. This inspection plan will be recognized as a random walk model with reflecting barriers. The first Dodge plan is easily recognized as a special case containing only one sampling level.

Received September 21, 1954, revised May 16, 1955.

[1] Done under Office of Naval Research sponsorship, Contract N6onr-25126; reproduction in whole or in part permitted for any U. S. Government purpose.

The AOQ function for the plan is derived and contours of constant AOQL are developed for a two-level and an infinite-level plan. These are added to the contours of constant AOQL for Dodge's single-level plan to present a picture in Figs. 1, 2 and 3 reflecting the relationship between a fixed AOQL contour and the number of sampling levels used in the plan. In addition an approximation procedure is presented for determining contours of constant AOQL when the number of sampling levels lies between three and infinity.

For a desired AOQL and a given process average, criteria for selecting a specific multi-level plan are discussed.

2. The Multi-Level Inspection Plan (MLP). The plan proposed in this paper is as follows:

0) At the outset inspect 100 percent of the units consecutively as produced and continue such inspection until i units in succession are found clear of defects.

1) When i units in succession are found clear of defects, discontinue 100 per cent inspection and inspect only a fraction f of the units (i.e., one out of every $1/f$ where $1/f$ is an integer). If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to 100 per cent inspection.

2) When at rate f, i inspected units are found clear of defects, discontinue sampling at rate f and proceed to sampling at rate f^2. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f.

3) When at rate, f^2, i inspected units are found clear of defects, discontinue sampling at rate f^2 and proceed to sampling at rate f^3. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f^2.

. . . .

. . . .

. . . .

$k - 1$) When at rate f^{k-2}, i inspected units are found clear of defects, discontinue sampling at rate f^{k-2} and proceed to sampling at rate f^{k-1}. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f^{k-2}.

k) When at rate of f^{k-1}, i inspected units are found clear of defects, discontinue sampling at rate f^{k-1} and proceed to sampling at rate f^k. If a defective occurs, revert immediately to sampling at rate f^{k-1}, otherwise, continue sampling at rate f^k.

Whenever sampling is in operation, one item should be selected at random from each segment of $1/f^j (j = 0, 1, 2, \cdots, k)$ production items. During both sampling inspection and 100 per cent inspection all defective items found should either be corrected or replaced with good items.

This plan will be called the Multi-Level Continuous Inspection Plan (MLP). For $k = 1$, it reduces to the first Dodge Plan. The MLP plan is one of a general class of multi-level plans which have the property that provision is made for smaller sampling rates when quality is good. It was specifically chosen because it made mathematical and computational analysis tractable and still maintained all the fundamental ideas of multi-level sampling plans. In fact, results for more general type multi-level plans are given in Sections 3 and 4.

3. The AOQ function for MLP. Suppose the inspection plan is generalized so that at the j^{th} sampling level ($j = 0, 1, 2, \cdots, k$) there is a sampling rate f_j and i_j non-defectives must occur to proceed to rate f_{j+1} ; $f_0 = 1$ (100 per cent inspection), i_k is infinite. While on 100 per cent inspection i_0 successive units must be non-defective before proceeding to the first sampling level. In MLP, $f_j = f^j$; and $i_j = i_0 = i, j \neq k$. The AOQ function for this more general inspection plan can be derived without any more complexity than the AOQ for MLP and this will now be done. It will be assumed, of course, that the production process is in control, i.e., qualities of the items are mutually independent binomial random variables with constant parameter p.

Let a "unit" be a group of $f_j^{-1}(j = 0, 1, \cdots, k)$ successive production items from which one is to be chosen at random for inspection. After the inspection of any item the size of the unit from which the next item is to be chosen for inspection is determined by past history according to the given rule. Suppose we represent the result of the m^{th} inspection by the random variable x_m where x_m is zero if the inspected item is non-defective and is one if it is defective. Then a sequence $(x_1, x_2, \cdots, x_m, \cdots)$ represents results on successive inspection trials and can be considered a point in sample space. A particular sampling plan attaches an integer from the set $f_0^{-1}, f_1^{-1}, f_2^{-1}, \cdots, f_k^{-1}$ to each coordinate (x_m) of the sample point. Which integer gets attached to a particular x_m depends on $x_1, x_2, \cdots, x_{m-1}$. The integer attached to x_m is the number of production items in the unit from which a member is inspected with result x_m.

If $f_{j_m}^{-1}$ is the integer attached to x_m in the sequence $(x_1, x_2, \cdots, x_m, \cdots)$ then the reciprocal of the average fraction inspected *for that sequence* is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f_{j_m}^{-1} \tag{1}$$

provided the limit exists. Equation (1) can be written as

$$\lim_{n\to\infty} \sum_{j=0}^{k} \frac{f_j^{-1} g_j^{(n)}}{n} \tag{2}$$

where $g_j^{(n)}$ is the number of times that sampling from a sequence of individual production items of length f_j^{-1} occurs in the first n trials. Now, define the reciprocal of the average fraction inspected, F, as

$$F^{-1} = \sum_{j=0}^{k} f_j^{-1} P_j \tag{3}$$

where P_j is the probability that, for a "randomly chosen" m, x_m is the result of inspection of an item selected from a unit of size f_j^{-1}.

If it can be shown that $\lim_{n\to\infty} g_j^{(n)}/n = P_j$ almost everywhere equations (1) and (3) become identical. Thus, the average fraction inspected can be represented in terms of the steady state probabilities. In order to prove these results, it is useful to relate this process to a Markov chain.

Consider a sequence of trials. At anytime m, (after the m^{th} observation) the system is in state

$$E_v(v = 0, 1, 2, \cdots, i_0, i_0 + 1, \cdots, i_0 + i_1, i_0 + i_1 + 1, \cdots, i_0 + i_1 + i_2, i_0 + i_1 + i_2 + 1, \cdots, i_0 + i_1 + \cdots + i_{k-1} + 1)$$

where

E_0 = state where the m^{th} trial resulted in beginning 100 per cent inspection. It signifies a defective item observed while sampling at rate f_1 or during 100 per cent inspection.

E_1 = state where the $(m - 1)^{\text{th}}$ trial resulted in beginning 100 per cent inspection and where the m^{th} trial resulted in a non-defective.

E_2 = state where the $(m - 2)^{\text{nd}}$ trial resulted in beginning 100 per cent inspection and where the $(m - 1)^{\text{th}}$ and m^{th} trial resulted in non-defectives.

.
.
. ...

E_{i_0} = state where the $(m - i_0)^{\text{th}}$ trial resulted in beginning 100 per cent inspection and where the next i_0 trials resulted in non-defectives. This means that sampling at rate f_1 is to begin.

E_{i_0+1} = state where the $(m - 1)^{\text{th}}$ trial resulted in allowing sampling at rate f_1 to begin and where the m^{th} trial resulted in a non-defective.

E_{i_0+2} = state where the $(m - 2)^{\text{nd}}$ trial resulted in allowing sampling at rate f_1 to begin and where the next two trials resulted in non-defectives.

.
.
. ...

$E_{i_0+i_1+\cdots+i_{k-1}+1}$ = state where the $(m - 1)^{\text{th}}$ trial resulted in allowing sampling at rate f_k to begin or sampling at rate f_k is in operation, and where the m^{th} trial resulted in a non-defective.

The matrix of transition probabilities is given by

$$
\begin{aligned}
p_{r,0} &= p && \text{for } r = 0, 1, \cdots, i_0 + i_1 - 1\\
p_{r,i_0} &= p && \text{for } r = i_0 + i_1, i_0 + i_1 + 1, \cdots, i_0 + i_1 + i_2 - 1\\
p_{r,i_0+i_1} &= p && \text{for } r = i_0 + i_1 + i_2, \cdots, i_0 + i_1 + i_2 + i_3 - 1\\
&\cdot && \\
&\cdot && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdot && \\
p_{r,i_0+i_1+\cdots+i_{k-1}} &= p && \text{for } r = i_0 + i_1 + \cdots + i_{k-1} + 1\\
p_{r,s} &= q = 1 - p && \text{for } r = s - 1, \quad s = 1, 2, \cdots, i_0 + i_1 + \cdots + i_{k-1} + 1\\
p_{i_0+i_1+\cdots+i_{k-1}+1, i_0+i_1+\cdots+i_{k-1}+1} &= q
\end{aligned}
$$

all other $p_{r,s} = 0$.

A result[2] of a theorem of Chung [5] indicates that this type of Markov chain has the property that

$$\lim_{n\to\infty} g_v^{*(n)}/n \text{ exists and is equal to a unique } P_v^*$$

almost everywhere, where $g_v^{*(n)}$ is the number of items in state E_v in the first n trials and P_v^* is the "steady state" probability that for a "randomly chosen" m, x_m is in state E_v. Since $g_j^{(n)}$ is a finite sum of $g_v^{*(n)}$ and P_j is a finite sum of P_v^*, we get $\lim_{n\to\infty} \sum_{j=0}^{k} f_j^{-1} g_j^{(n)}/n$ exists almost everywhere and $\lim_{n\to\infty} g_j^{(n)}/n = P_j$ almost everywhere. Furthermore, since the process is an irreducible, aperiodic, finite Markov chain, the limiting stationary probabilities are independent of the initial probabilities. This implies that the AOQ does not depend on which sampling rate is used initially, e.g., the process can start at sampling rate f_j, $j \neq 0$.

In order to calculate the values of P_j, $j = 0, 1, \cdots, k$, it is necessary to introduce some definitions. P_j has already been defined as the probability that, for a "randomly chosen" m, x_m is the result of a choice from a unit of size f_j^{-1}. Let a prime attached to the P_j's denote the probability that for a "randomly chosen" m, x_m is the result of a choice from a unit of size f_j^{-1}, and $x_{m-1} = 0$ while sampling from a unit of size f_{j-1}^{-1} or $x_{m-1} = 1$ while sampling from a unit of size f_{j+1}^{-1}, $j = 1, 2, \cdots, k - 1$. P_0' is the probability that for a "randomly chosen" m, x_m is the result of a choice from a unit of size $f_0^{-1} = 1$ and $x_{m-1} = 1$ while sampling at rate f_0 or f_1. P_k' is the probability that for a "randomly chosen" m, x_m is the result of a choice from a unit of size f_k^{-1} and $x_{m-1} = 0$ while sampling from a unit of size f_{k-1}^{-1}. In other words, P_j' denotes the probability of beginning sampling at level j.

---

[2] The authors are indebted to Professor Samuel Karlin for pointing out the applicability of Chung's paper.

Then we may write

$$P_0' = P_0'[p + qp + q^2p + \cdots + q^{i_0-1}p] + P_1'[p + pq + q^2p + \cdots + q^{i_1-1}p] \tag{4}$$

$$P_0' = P_0'[1 - q^{i_0}] + P_1'[1 - q^{i_1}]$$

$$P_0 = P_0'[1 + q + q^2 + \cdots + q^{i_0-1}]$$

$$P_0 = P_0'\left[\frac{1 - q^{i_0}}{p}\right] \tag{5}$$

$$P_1' = P_0'q^{i_0} + P_2'[p + qp + \cdots + q^{i_2-1}p]$$

$$P_1' = P_0'q^{i_0} + P_2'[1 - q^{i_2}] \tag{6}$$

$$P_1 = P_1'[1 + q + q^2 + \cdots + q^{i_1-1}]$$

$$P_1 = P_1'\left[\frac{1 - q^{i_1}}{p}\right] \tag{7}$$

In a similar manner we get

$$P_2' = P_1'q^{i_1} + P_3'[1 - q^{i_3}] \tag{8}$$

$$P_2 = P_2'\left[\frac{1 - q^{i_2}}{p}\right] \tag{9}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{k-2}' = P_{k-3}'q^{i_{k-3}} + P_{k-1}'[1 - q^{i_{k-1}}] \tag{10}$$

$$P_{k-2} = P_{k-2}'\left[\frac{1 - q^{i_{k-2}}}{p}\right] \tag{11}$$

$$P_{k-1}' = P_{k-2}'q^{i_{k-2}} + P_k' \tag{12}$$

$$P_{k-1} = P_{k-1}'\left[\frac{1 - q^{i_{k-1}}}{p}\right] \tag{13}$$

$$P_k' = P_{k-1}'q^{i_{k-1}} \tag{14}$$

$$P_k = P_k'[1 + q + q^2 + \cdots + \cdots] = \frac{P_k'}{p} \tag{15}$$

Also

$$\sum_{j=0}^{k} P_j = 1.$$

Solving equations (4)–(15) for P_0' we then find $P_0, P_1, \cdots, P_k$ since each is a multiple of P_0', namely

$$P_0 = \frac{1 - q^{i_0}}{q^{i_0}} \cdot \frac{1}{D} \qquad P_1 = \frac{1}{D}$$

$$P_2 = \frac{q^{i_1}}{1 - q^{i_1}} \cdot \frac{1}{D} \qquad P_3 = \frac{q^{i_1} q^{i_2}}{(1 - q^{i_1})(1 - q^{i_2})} \cdot \frac{1}{D}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{k-1} = \frac{q^{i_1} q^{i_2} \cdots q^{i_{k-2}}}{(1 - q^{i_1})(1 - q^{i_2}) \cdots (1 - q^{i_{k-2}})} \cdot \frac{1}{D}$$

$$P_k = \frac{q^{i_1} q^{i_2} \cdots q^{i_{k-1}}}{(1 - q^{i_1})(1 - q^{i_2}) \cdots (1 - q^{i_{k-1}})} \cdot \frac{1}{D}$$

and

$$\frac{1}{D} = P_0' \frac{q^{i_0}}{p}.$$

The AOQ function can be written as

$$\text{AOQ} = p(1 - F) = p \frac{\sum_{j=1}^{k} \left(\frac{1}{f_j} - 1\right) P_j}{\sum_{j=0}^{k} \left(\frac{P_j}{f_j}\right)}. \tag{16}$$

Substituting all the values we get

$$\text{AOQ} = p \left\{ \frac{\left(\frac{1}{f_1} - 1\right)\left(\frac{q^{i_0}}{1 - q^{i_0}}\right) + \left(\frac{1}{f_2} - 1\right) \frac{q^{i_0} q^{i_1}}{(1 - q^{i_0})(1 - q^{i_1})} + \cdots + \left(\frac{1}{f_k} - 1\right) \frac{q^{i_0} q^{i_1} \cdots q^{i_{k-1}}}{(1 - q^{i_0})(1 - q^{i_1}) \cdots (1 - q^{i_{k-1}})}}{1 + \frac{1}{f} \frac{q^{i_0}}{1 - q^{i_0}} + \frac{1}{f_2} \frac{q^{i_0} q^{i_1}}{(1 - q^{i_0})(1 - q^{i_1})} + \cdots + \frac{1}{f_k} \frac{q^{i_0} q^{i_1} \cdots q^{i_{k-1}}}{(1 - q^{i_0})(1 - q^{i_1}) \cdots (1 - q^{i_{k-1}})}} \right\}. \tag{16a}$$

Now consider MLP, i.e., $f_j = f^j$; $i_j = i (j \neq k)$, $i_k = \infty$, $j = 0, 1, 2, \cdots, k$, then

$$\text{AOQ} = p \left\{ \frac{\left(\frac{1}{f} - 1\right) \frac{q^i}{1 - q^i} + \left(\frac{1}{f^2} - 1\right)\left(\frac{q^i}{1 - q^i}\right)^2 + \cdots + \left(\frac{1}{f^k} - 1\right)\left(\frac{q^i}{1 - q^i}\right)^k}{1 + \frac{1}{f} \frac{q^i}{1 - q^i} + \frac{1}{f^2}\left(\frac{q^i}{1 - q^i}\right)^2 + \cdots + \frac{1}{f^k}\left(\frac{q^i}{1 - q^i}\right)^k} \right\}. \tag{17}$$

Let $z = (1/f)(q^i/1 - q^i)$, then

$$\text{(17a)} \quad \text{AOQ} = p\left\{\frac{[z + z^2 + \cdots + z^k] - [(fz) + (fz)^2 + \cdots + (fz)^k]}{1 + z + z^2 + \cdots + z^k}\right\}$$

or

$$\text{(17b)} \qquad \text{AOQ} = pz\left\{\frac{1 - z^k}{1 - z^{k+1}} - \left(\frac{f - fz}{1 - fz}\right)\left(\frac{[1 - (fz)^k]}{1 - z^{k+1}}\right)\right\}.$$

4. Monotonicity of the AOQ and the AOQL.[3] It is intuitively apparent that in MLP, the average fraction inspected (AFI), for a fixed process average, should decrease as k increases or equivalently that the AOQ, for a fixed process average increase since AOQ $= p(1 - F)$ where F is the AFI. It is also apparent that this result holds more generally than for MLP.

Let us return to the general model of Section 3. A particular sampling plan attaches an integer from the sequence $1, f_1^{-1}, f_2^{-1}, \cdots, f_k^{-1}(1 < f_j^{-1} < f_{j+1}^{-1})$ to each member of a given sample point or sequence, $(x_1, x_2, \cdots)$. If we can show that for every point in sample space when M is sufficiently large, $\sum_{m=1}^{M} f_{j_m}^{-1}$ for the k level plan is less than $\sum_{m=1}^{M} f_{j_m}^{-1}$ for the $k + 1$ level plan, then we will have shown that F^{-1} increases monotonely with k in the above sense.

But look at the sequences of f_j's for a given $(x_1, x_2, \cdots)$ in the k and $k + 1$ level plans. The second is the same as the first until the first time f_k^{-1} appears i_k times in succession. Then at the next step, f_k^{-1} changes to f_{k+1}^{-1} in the second sequence. As soon as a defective is observed ($x_m = 1$) for the first subsequent time, instead of f_{k-1}^{-1} appearing (as in the first sequence), we use f_k^{-1} in the second, etc. The important conclusion is that at every step the f_j^{-1} in the second sequence is greater than or equal that of the first sequence and eventually (with probability 1) some strictly greater relationship will appear since the probability of sampling at rate f_{k+1} is greater than zero. Moreover, the AOQL must then monotonically increase with increasing k, and the minimum AFI monotonically decrease with increasing k.

5. Contours of constant AOQL for fixed k.[4] For any fixed k, it is possible to get AOQL contours paralleling those given by Dodge for $k = 1$. However, getting the AOQL as an explicit function of k, f, and i, in order to obtain contours of constant AOQL appears mathematically intractable. Moreover, the use of computational methods is tedious. Even for $k = 2$, it is expeditious to use an electronic digital computer to obtain contours of constant AOQL. Nevertheless, it is possible to obtain contours of constant AOQL for k infinite and we now

---

[3] The authors are indebted to the referee for pointing out this proof for the general Multi-Level Plan. The authors, in the original manuscript, only proved monotonicity for the MLP plan.

[4] The authors' conjecture that MLP guarantees an AOQL (different from the one presented in this paper) whether or not the process is in a state of statistical control. The proof should follow the method presented in [4].

proceed to derive them. While their use in MLP may not at first appear realistic, they will, together with Dodge's contours, at least provide some guides and insight for $k = 2, 3, 4, \ldots$.

6. AOQL contours for an infinite number of sampling levels. Refer to (17) and let k approach infinity. First assume $z < 1$, then certainly $(fz) < 1$. Thus for infinite k we get

$$\text{AOQ} = p(1 - f)\left[\frac{z}{1 - fz}\right], \tag{18}$$

Now take $z > 1$, then for infinite k

$$\text{AOQ} = p. \tag{19}$$

Now $z > 1$ is equivalent to $p < 1 - (f/1 + f)^{1/i}$. Thus when $p < 1 - (f/1 + f)^{1/i}$ the derivative of AOQ with respect to p is positive, in fact it is always 1. If the derivative of AOQ with respect to p for $p \geqq 1 - (f/1 + f)^{1/i}$ is always negative, then the AOQL must occur when $p = 1 - (f/1 + f)^{1/i}$ since the AOQ is a continuous function in this range. Let us look at $d/dp(\text{AOQ})$ for $p \geqq 1 - (f/1 + f)^{1/i}$ or equivalently $z \leqq 1$. For k infinite,

$$z < 1, \qquad \text{AOQ} = \frac{1 - f}{f}\left[\frac{p(1 - p)^i}{1 - 2(1 - p)^i}\right];$$

then

$$\frac{d}{dp}(\text{AOQ}) = \left(\frac{1 - f}{f}\right) \cdot \left\{\frac{[1 - 2(1 - p)^i][(1 - p)^i - pi(1 - p)^{i-1}] - p(1 - p^i)[2i(1 - p)^{i-1}]}{[1 - 2(1 - p)^i]^2}\right\}. \tag{20}$$

Since $(1 - f)/f$ and the denominator inside the braces are always positive we turn our attention to the numerator inside the braces. This reduces to

$$q^{i-1}\{q(1 - 2q^i) - ip\} \tag{21}$$

where the terms in the braces are of interest since q^{i-1} is always positive. Now the range of interest for p can be written as $p = 1 - \epsilon(f/1 + f)^{1/i}$ where ϵ lies between one and zero. Substituting we get

$$\left[\epsilon\left(\frac{f}{1 + f}\right)^{1/i}\right]\left[1 - 2\epsilon^i\left(\frac{f}{1 + f}\right)\right] - i\left[1 - \epsilon\left(\frac{f}{1 + f}\right)^{1/i}\right] \quad \text{or} \tag{22}$$

$$\epsilon\left(\frac{f}{1 + f}\right)^{1/i}\left[\frac{1 - f(2\epsilon^i - 1)}{1 + f} + i\right] - i. \tag{22a}$$

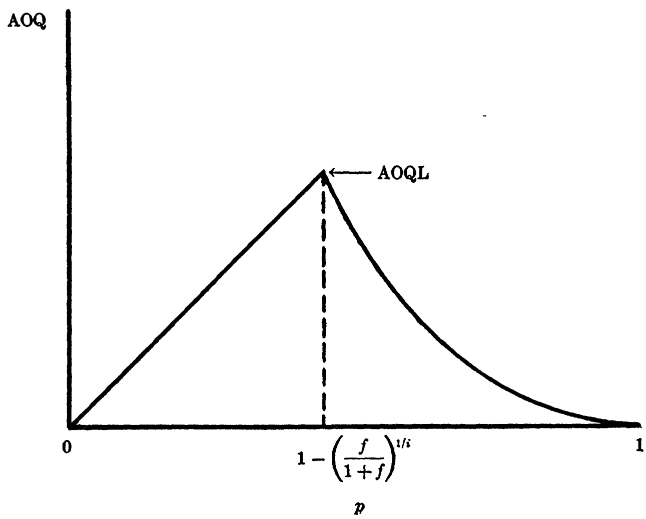
It can now be demonstrated that the maximum value of the first term of (22a) is less than i which shows the derivative is always negative. The greatest value of $\epsilon(f/1 + f)^{1/i}$ is $(\frac{1}{3})^{1/i}$ since $f \leqq \frac{1}{2}(1/f$ is integral); the greatest value of $(1 - f(2\epsilon^i - 1)) / (1 + f)$ is one, thus the first term cannot exceed $(\frac{1}{3})^{1/i}(1 + i)$, but

$$(\tfrac{1}{3})^{1/i}(1 + i) < i$$

since this leads to

$$(1 + 1/i)^i < 3$$

and $(1 + 1/i)^i \leqq e = 2.71828. \ldots$ Moreover, since the derivative of AOQ with respect to p is zero when $p = 1$, the AOQ function for an infinite number of levels may be sketched as follows:



Thus the AOQL occurs at $p = 1 - (f/(1 + \text{f}))^{1/i}$ and is also equal to that value. This yields the explicit relationship

$$f = \frac{(1 - \text{AOQL})^i}{1 - (1 - \text{AOQL})^i}. \tag{23}$$

It is now quite easy to plot contours of constant AOQL for an infinite number of levels and this has been done in Fig. 1 which also contains the same contours for $k = 1$. For $k = 1$, f represents the sampling rate; for k greater than one, f represents the initial sampling rate.

7. AOQL contours for MLP when $k = 2$. Putting $k = 2$ in (17b) we get

$$\text{AOQ} = p(1 - f)\left\{\frac{z + z^2(1 + f)}{1 + z + z^2}\right\} \quad \text{or} \tag{24}$$

$$\text{AOQ} = pq^i(1 - f)\left\{\frac{f + q^i}{f^2 - q^i(2f^2 - f) + q^{2i}(1 - f + f^2)}\right\}. \tag{24a}$$

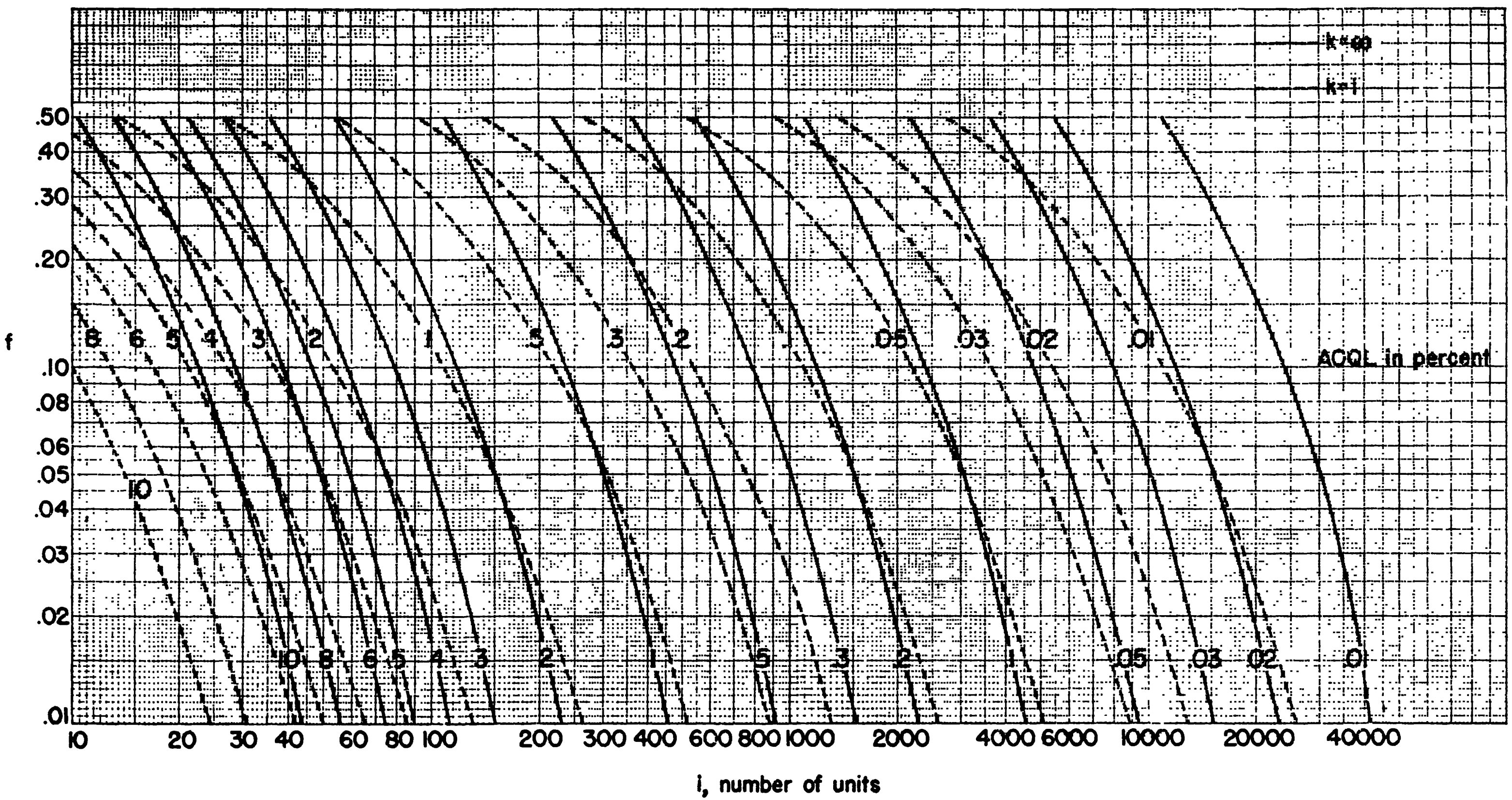


Fig. 1. Curves for Determining Values of f and i for a given Value of AOQL

Taking the derivative of AOQ with respect to p and setting it equal to zero we get after some straightforward manipulation

$$(25)\qquad \begin{aligned} &q^{3i+1}[f^2 - f + 1] + q^{2i+1}[(1 - i)f^3 - f^2(3 + i) + 2f] \\ &\quad + q^{2i}[if^3 + if^2] + q^{i+1}[f^2(2 + 2i) - 2f^3] + q^i[-2if^2] \\ &\quad + q[f^3(1 + i)] - if^3 = 0 \end{aligned}$$

We now desire to obtain the sixteen specified contours of constant AOQL for $k = 2$ already obtained by Dodge for $k = 1$ and just obtained in this paper for k infinite. While (24a) and (25) uniquely determine the specified contours their expeditious computation requires some planning. Any pair (f, i) determines a unique value of q given by (25). When this value of q and pair (f, i) are substituted in (24a), an AOQL is obtained.

However, the problem is to find curves of constant AOQL, e.g., AOQL $= .10$ (ten per cent). A point (f, i) on this curve was found as follows. For any given value of i, four points $[(f^{(r)}, i)\ r = 1, 2, 3, 4]$ were chosen lying between the $k = 1$ and infinite k contours for this AOQL. Each of these points yielded an AOQL value. These points were chosen in such a manner that the desired AOQL $= .10$ was included between the smallest and largest of the four AOQL's. By an Aitken interpolation the pair (f, i) corresponding to the desired AOQL value was obtained. Any number of points on the specified contour can be obtained in this manner. We have lightly passed over the tedious job of evaluating q by (25) and then the AOQL by (24a) for any fixed pair (f, i). Actually, an electronic digital computer was employed for these two steps. The contours of constant AOQL for $k = 2$ were produced in this way and are given in Fig. 2. Also, some of these are contrasted with contours for $k = 1$ and $k = \infty$ in Fig. 3.

8. Contours of constant AOQL when $2 < k < \infty$**.** The computation of contours of constant AOQL for $k > 2$ soon becomes forbidding if the method for obtaining contours of constant AOQL for $k = 2$ is applied. However, it is interesting and fruitful to explore the kind of interpolation necessary to reproduce the $k = 2$ contours from knowledge of the one sampling level and infinite sampling level contours. For $k = 1$ and k infinite we can explicitly write f in terms of i and AOQL, namely

$$(26)\qquad f_1 = \frac{(1 - A)^{i_1}}{(1 - A)^{i_1} + \left(1 + \frac{1}{i_1}\right)^{i_1} (1 + i_1) \frac{A}{1 - A}}$$

and

$$(27)\qquad f_\infty = \frac{(1 - A)^{i_\infty}}{1 - (1 - A)^{i_\infty}}$$

where A is the AOQL and the subscripts on the f's and i's refer to the number of levels used.

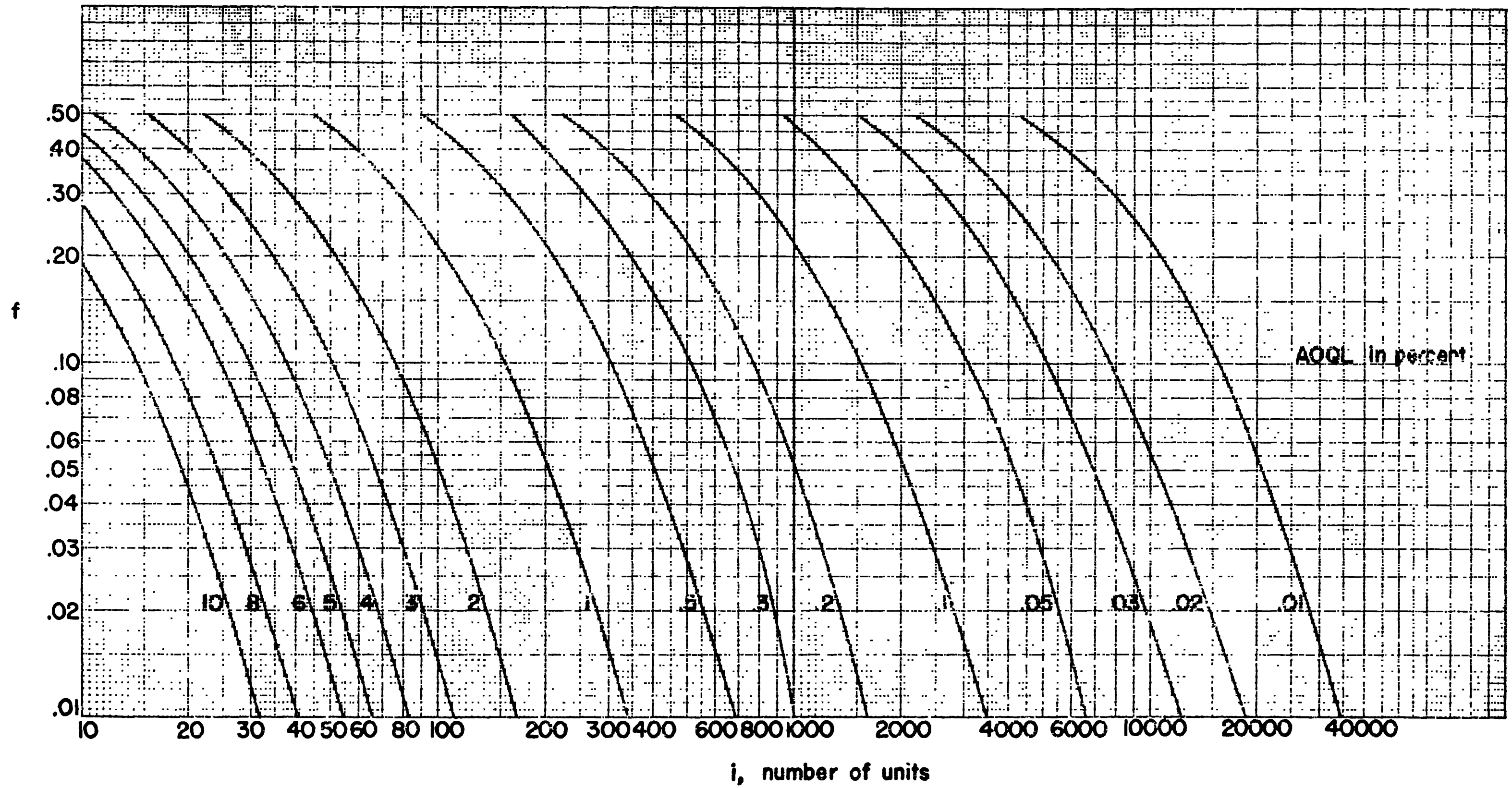


FIG. 2. Curves for Determining Values of f and i for a Given Value of AOQL; $k = 2$

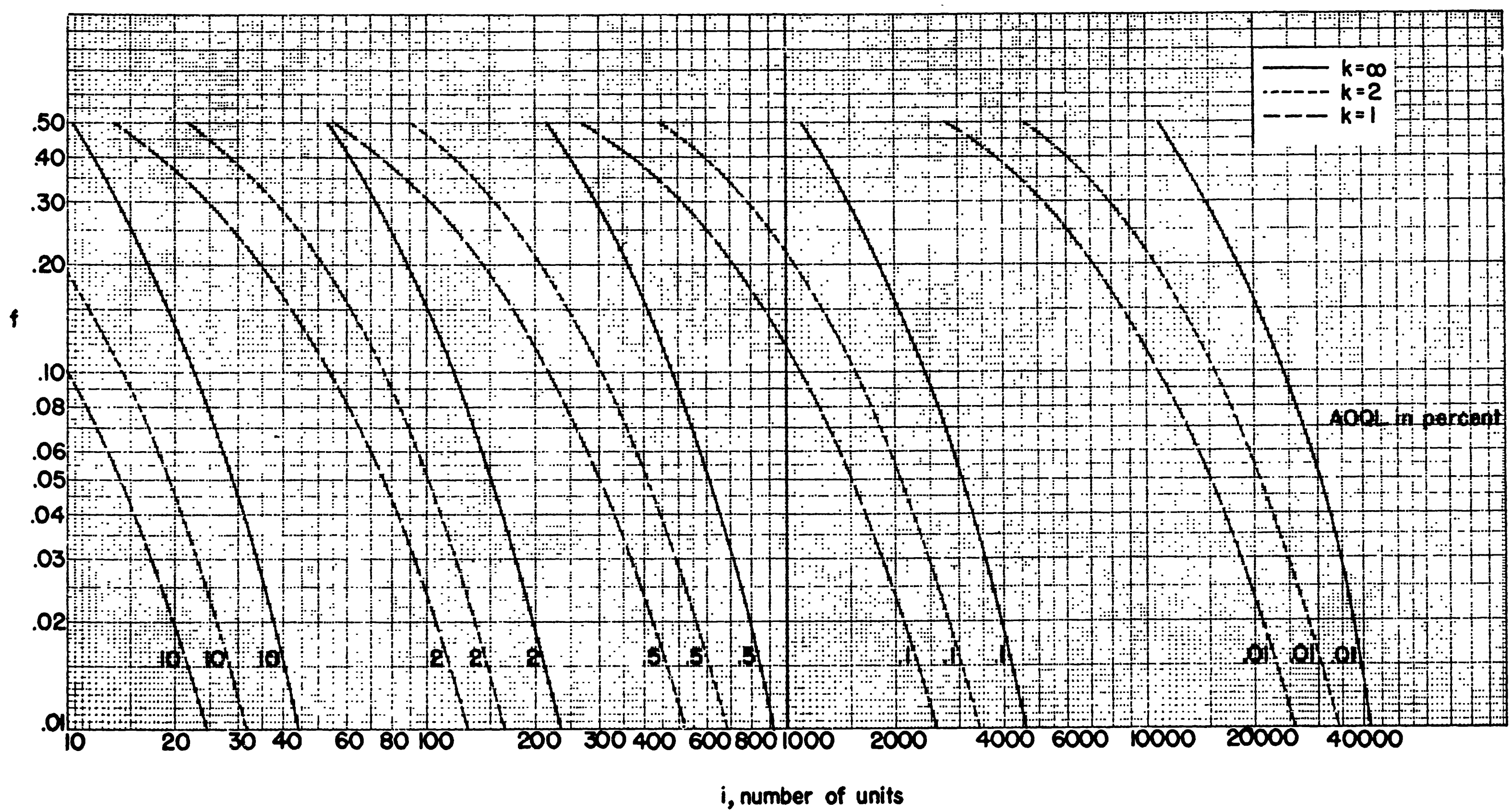


FIG. 3. Graph Showing Effect of Sampling Level on Contours for Selected Values of AOQL

TABLE I

Comparison between exact (f_E) and approximate (f_A) values of f for $k = 2$

AOQL	i	f_E	f_A	$f_E - f_A$
.10	15	.0906	.0869	.0037
	22	.0343	.0339	.0004
	27	.0179	.0183	−.0004
	29	.0138	.0144	−.0006
.08	15	.1453	.1403	.0050
	21	.0725	.0696	.0029
	28	.0340	.0329	.0011
	35	.0170	.0169	.0001
	37	.0140	.0140	.0000
.06	18	.1790	.1745	.0045
	28	.0750	.0717	.0033
	34	.0460	.0445	.0015
	44	.0212	.0212	.0000
.05	19	.2210	.2196	.0014
	32	.0850	.0813	.0037
	45	.0355	.0347	.0008
	56	.0176	.0179	−.0003
.04	18	.3193	.3280	−.0087
	32	.1370	.1310	.0060
	53	.0431	.0416	.0015
	69	.0191	.0191	.0000
.03	38	.1703	.1648	.0055
	60	.0670	.0665	.0005
	90	.0210	.0209	.0001
.02	55	.1830	.1779	.0051
	87	.0740	.0702	.0038
	130	.0238	.0239	−.0001
.01	110	.1850	.1801	.0049
	180	.0690	.0658	.0032
	255	.0270	.0259	.0011
.005	225	.1799	.1749	.0050
	350	.0740	.0708	.0032
	510	.0270	.0316	−.0046

If for fixed $i = i_1 = i_2 = i_\infty$ we write

$$f_2 = f_\infty[1 - (\tfrac{1}{2})^{1/3}] + f_1[(\tfrac{1}{2})^{1/3}], \tag{28}$$

we get a point (f_2, i_2) which falls almost exactly on the contour of constant AOQL for $k = 2$. This is demonstrated in Table I. In other words, harmonic cube root interpolation is appropriate, that is, for fixed AOQL and i, the set f_1, f_2, f_∞ is proportional to 1, $(\frac{1}{2})^{1/3}$, 0. This was of course discovered by trial and error but it also presents a reasonable way for obtaining any of the sixteen specified contours of constant AOQL for any fixed k by using the explicit known values for $k = 1$ and k infinite together with the assumption that f_1, f_k, f_∞ is proportional to 1, $(1/k)^{1/3}$, 0; or

$$f_k = f_\infty\left[1 - \left(\frac{1}{k}\right)^{1/3}\right] + f_1\left[\frac{1}{k}\right]^{1/3}. \tag{29}$$

9. Choice of a MLP plan. Assuming that harmonic cube root interpolation is a satisfactory method for obtaining contours of constant AOQL for any fixed k, there still remains the task of defining valid and reasonable criteria to be employed in the selection of a MLP plan; for, given a process average, an infinite number of such plans exist which can guarantee the attainment of any specified AOQL. Contract specifications, administrative considerations, or psychological grounds can impose a lower bound on the amount of inspection or an upper bound on the number of sampling levels and thus curtail the total number of possible plans. A lower bound on the amount of inspection may also be required to quickly detect the malfunctioning of the production process. Also, it is evident from Figs. 1, 2, and 3 that Dodge in his single level plan considers $f > 1/2$ and $f < 1/100$ as unrealistic, and the authors in MLP consider the same region unrealistic for initial sampling rates and thus, large groups of plans are ignored.

In addition there are cost considerations which our continuous inspection scheme must consider and these will also influence the choice of plans. We will now discuss two types of cost criteria and their effect on the choice of a MLP plan. These criteria are (a) minimum AFI; and (b) local stability which we specifically define as maintained as long as

$$P\{d_N > NA\} \leqq \alpha \tag{30}$$

where d_N is the number of defects remaining in a sequence of N items (N large) which have gone through the inspection process, A is the desired AOQL, and α is the tolerated risk. This definition of local stability is a quantification of the notion expressed in [2]. While only the single sampling level and the infinite sampling plans will be explicitly analyzed, some implications will remain for any MLP plan.

We now turn to the AFI functions for $k = 1$ and k infinite. For simplification write AFI $= F$ and we get for $k = 1$

$$F_1 = \frac{f_1}{f_1 + (1 - f_1)(1 - p)^{i_1}} \tag{31}$$

where f_1 is defined by (26). Thus

$$F_1 = \frac{\left(\frac{1 - A}{1 - p}\right)^{i_1}}{\left(\frac{1 - A}{1 - p}\right)^{i_1} + \left(1 + \frac{1}{i_1}\right)^{i_1}(1 + i_1)\left(\frac{A}{1 - A}\right)}. \tag{32}$$

For k infinite, we get

$$(33) \qquad F_\infty = \begin{cases} 1 - \dfrac{1 - f_\infty}{f_\infty}\left[\dfrac{(1 - p)^{i_\infty}}{1 - 2(1 - p)^{i_\infty}}\right], & \text{for } p > A \\ 0, & \text{for } p \leqq A, \end{cases}$$

where f_∞ is defined by (27). Thus

$$(34) \qquad F_\infty = \begin{cases} \dfrac{\left(\dfrac{1 - A}{1 - p}\right)^{i_\infty} - 1}{\left(\dfrac{1 - A}{1 - p}\right)^{i_\infty} - 2(1 - A)^{i_\infty}}, & \text{for } p > A \\ 0, & \text{for } p \leqq A. \end{cases}$$

When the process average, p, is less than or equal to the desired AOQL, the question of minimum AFI is easily resolved in favor of the infinite sampling level scheme. On the other hand, when p exceeds the AOQL, it is evident from (32) and (34) that F_1 can be made smaller than F_∞ within the ranges of i_1 and i_∞ dictated by a specific choice of A. Table II gives some numerical illustrations.

TABLE II

Minimum values of F_1 and F_∞ for selected values of process average and AOQL ($p > A$)

A	p	i_1	i_∞	f_1	f_∞	F_1	F_∞
.10	.15	16	13	.033	.34	.33	.69
.10	.20	7	11	.16	.43	.50	.88
.02	.03	97	68	.026	.33	.33	.67
.02	.04	47	60	.13	.42	.50	.86
.005	.008	330	269	.041	.35	.38	.72
.0005	.0008	3331	2694	.040	.35	.38	.72

Let us now examine the single sampling level and infinite sampling level plans for local stability. For a sequence of N items (N large, $p(1 - F)$ small), d_N can be approximated by a Poisson distribution with mean $Np(1 - F)$. Thus from (30) and the normal approximation to the Poisson we get

$$(35) \qquad \frac{NA - Np(1 - F)}{[Np(1 - F)]^{1/2}} \geqq K_\alpha$$

where K_α is the $(1 - \alpha)$th percentile of the normal distribution with zero mean and unit variance. Solving the equality in (35) for $p(1 - F) = \text{AOQ}$ we find

$$(36) \qquad p(1 - F) = \frac{2NA + K_\alpha^2 \pm (4NAK_\alpha^2 + K_\alpha^4)^{\frac{1}{2}}}{2N}$$

but since $p(1 - F) \leqq A$, the positive square root must be discarded. The solution to the inequality in (35) is

$$p(1 - F) \leqq A + \frac{K_\alpha^2}{2N} - \left[\frac{K_\alpha^4}{4N^2} + \frac{AK_\alpha^2}{N}\right]^{\frac{1}{2}} = C(A, N, \alpha). \tag{37}$$

For $k = 1$, we obtain

$$\text{AOQ}_1 = p(1 - F_1) = p\left[1 + \left(\frac{1-A}{1-p}\right)^{i_1}\left(\frac{1-A}{A}\right)\left(\frac{1}{1+i_1}\right)\left(1 + \frac{1}{i_1}\right)^{-i_1}\right]^{-1}. \tag{38}$$

When $p \leqq C(A, N, \alpha)$, (37) is satisfied for all values of i_1 and thus local stability is guaranteed by all plans for $k = 1$. For $C(A, N, \alpha) < p \leqq A$ there exists a value i_1^*, given p and A, such that for all $i_1 \leqq i_1^*$ local stability is maintained. When $p > A$, then there exists a value i_1^{**}, given p and A, such that for all $i_1 \geqq i_1^{**}$ local stability is maintained. Thus, given any values for p and A, it is always possible to find plans which yield local stability when $k = 1$; and when quality is exceptionally good all single sampling level plans have this property. This is not surprising since the Dodge plan represents the tightest inspection plan of all MLP plans.

For the infinite sampling level plan, we obtain

$$\text{AOQ}_\infty = \begin{cases} p(1 - F_\infty) = p & \text{when } p < A \\ p(1 - F_\infty) = p\left(\frac{1-p}{1-A}\right)^{i_\infty}\left[\frac{1 - 2(1-A)^{i_\infty}}{1 - 2(1-p)^{i_\infty}}\right] & \text{for } p \geqq A. \end{cases} \tag{39}$$

When $p \leqq C(A, N, \alpha)$, (37) is satisfied for all values of i_∞ and local stability is guaranteed for all infinite sampling level plans. For $A > p > C(A, N, \alpha)$, (37) is never satisfied and local stability is never maintained. This is also true for $p = A$. When $p > A$, there exists a value i_∞^*, given p and A, such that for all $i_\infty \geqq i_\infty^*$ local stability is maintained. Thus, as in the Dodge plan, when quality is exceptionally good, all infinite sampling level plans have the desired property. On the other hand, when quality is good but hovers just short of the AOQL, local stability cannot be maintained. However, when quality exceeds the desired AOQL, it is possible to find some infinite sampling level plans which will maintain local stability.

When quality is exceptionally good, i.e., $p \leqq C(A, N, \alpha)$, then the infinite sampling level plan guarantees local stability and has minimum AFI. It also seems plausible that if a choice between the Dodge plan and, say, $k = 3$ is desired, then the decision should be in favor of $k = 3$ since it will guarantee local stability and a smaller minimum AFI. However, if $C(A, N, \alpha) < p \leqq A$, the choice between the Dodge plan and infinite k is not easily resolved for while the latter has minimum AFI, it does not have local stability. If quality is poor, $p > A$, then the Dodge plan is to be preferred.

REFERENCES

[1] H. F. DODGE, "A sampling inspection plan for continuous production," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 264–279.

[2] A. WALD AND J. WOLFOWITZ, "Sampling inspection plans for continuous production which insure a prescribed limit on the outgoing quality," *Ann. Math. Stat.*, Vol. 16 (1945), pp. 30–49.

[3] H. F. DODGE AND M. N. TORREY, "Additional continuous sampling inspection plans," *Industrial Quality Control*, Vol. 7 (1951), No. 5, pp. 7–12.

[4] G. J. LIEBERMAN, "A note on Dodge's continuous inspection plan," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 480–484.

[5] K. L. CHUNG, "Contributions to the theory of Markov chains," *Trans. Amer. Math. Soc.*, Vol. 77 (1954), pp. 397–419.