ON THE ASYMPTOTIC BEHAVIOR OF DECISION PROCEDURES[1,2]

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0. Summary and introduction. In this paper, the asymptotic behavior of decision procedures will be studied for a particular class of multiple decision problems. The study will throw some light on the desirability of the minimax decision procedure when the number of observations is large, and it will be seen that decision procedures frequently exist which are superior to the minimax decision procedure for large samples. The fact that the minimax decision procedure may be desirable in certain problems for small samples but undesirable for large samples was revealed by Hodges and Lehmann [1] in connection with estimation problems. Robbins [2] suggested the term "asymptotically subminimax" for the type of superior procedures which may then exist. A definition of this term which will be useful for an investigation of the asymptotic behavior of decision procedures, will be given in Section 1. A major part of this paper will be concerned with certain sequences of decision procedures called asymptotically admissible which have desirable properties similar to those of admissible decision procedures for the case of some fixed sample size. These asymptotically admissible decision procedures include a subclass of the asymptotically subminimax procedures, and the sequences of minimax procedures for those problems for which asymptotically subminimax procedures do not exist.

The problems to be considered are those in which a random variable, X, is known to have a distribution function belonging to the distribution space, $\Omega = \{F_i(x)\}$, $i = 1, 2, \cdots, k$, and it is desired to select the true distribution function based on a sample of n independent observations of X. It will be assumed that all $F_i(x)$ are absolutely continuous distribution functions having density functions, $f_i(x)$, and that for every constant K, the set of points for which $f_i(x)/f_j(x) = K$ $(i \neq j)$ is a set of probability measure zero under every F and all possible i and j. A simple loss function, $W(F_i, d_j)$, where d_j is the decision to select F_j, will be used with $W(F_i, d_j) = 1$ if an incorrect decision is made (i.e. $i \neq j$), and $W(F_i, d_j) = 0$ if a correct decision is made (i.e. $i = j$). For such a loss function, the expected loss is simply the probability of making an incorrect decision.

Section 1 will be concerned with asymptotically minimax sequences of decision procedures and Section 2 will be concerned with asymptotic admissibility. It will be seen that the asymptotic behavior of the minimax decision procedure

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depends on the limits of the components in the sequence of least favorable a priori distributions. Theorem 2.2 gives a sufficient condition, in terms of these limit values, for the minimax procedure to be asymptotically admissible.

In Section 3, a detailed study will be made of the class of problems where Ω consists of k univariate normal distributions having the same variance but different means. Let the means be denoted by θ_i $(i = 1, 2, \cdots, k)$ with $\theta_1 < \theta_2 < \cdots < \theta_k$, and let $\min_i(\theta_{i+1} - \theta_i) = \gamma$. Then Theorems 3.1 and 3.6 will show that the minimax procedure is asymptotically admissible when the means can be put into sets, each set containing the same number, $\bar{n} \geqq 2$, of consecutive means, with a difference of γ between any two consecutive means of a set, and a difference greater than γ between any two means not belonging to the same set. Theorems 3.2 and 3.3 will show that in all other cases the minimax procedure is asymptotically inadmissible, and asymptotically subminimax procedures will be constructed for all these cases. Although a complete study of the asymptotic admissibility of asymptotically subminimax procedures will not be made in this paper, Theorem 3.7 will show that a certain asymptotically subminimax procedure is asymptotically admissible for all the cases covered by Theorem 3.3 and for some of the cases covered by Theorem 3.2. On the other hand, for those cases covered by Theorem 3.2 with an Ω consisting of only 3 means, it will be shown that every asymptotically subminimax procedure is asymptotically inadmissible.

1. Asymptotically minimax decision procedures. Consider an Ω consisting of only two distribution functions, $F_1(x)$ and $F_2(x)$, having density functions $f_1(x)$ and $f_2(x)$ respectively, and let the observed values be $x_1, x_2, \cdots, x_n$. Using a simple loss function and denoting the least favorable a priori distribution by $\hat{g}_1^{(n)}$ and $1 - \hat{g}_1^{(n)}$, the minimax decision procedure, $\hat{\delta}_n$, is as follows:

Select $F_1(x)$ if

$$\hat{g}_1^{(n)} \prod_{j=1}^{n} f_1(x_j) > (1 - \hat{g}_1^{(n)}) \prod_{j=1}^{n} f_2(x_j),$$

and select $F_2(x)$ if

$$\hat{g}_1^{(n)} \prod_{j=1}^{n} f_1(x_j) < (1 - \hat{g}_1^{(n)}) \prod_{j=1}^{n} f_2(x_j).$$

Throughout this paper we shall ignore the possibility of equality in the above expressions because by the previous assumption the probability of such an event is zero. The risks associated with the minimax procedure are

$$(1.1)\quad \begin{aligned} r_1(\hat{\delta}_n) &= \Pr\left[\hat{g}_1^{(n)} \prod_{j=1}^{n} f_1(x_j) < (1 - \hat{g}_1^{(n)}) \prod_{j=1}^{n} f_2(x_j) / f_1(x) \text{ is true d. f.}\right] \\ r_2(\hat{\delta}_n) &= \Pr\left[\hat{g}_1^{(n)} \prod_{j=1}^{n} f_1(x_j) > (1 - \hat{g}_1^{(n)}) \prod_{j=1}^{n} f_2(x_j) / f_2(x) \text{ is true d. f.}\right]. \end{aligned}$$

For the minimax procedure, the two risks are equal, and their common value will be denoted by $r(\hat{\delta}_n)$.

Let us suppose that the sequence $\hat{g}_1^{(1)}, \hat{g}_1^{(2)}, \cdots, \hat{g}_1^{(n)}, \cdots$ is a null sequence, and consider the sequence of Bayes procedures, $\{\delta_n\}$, corresponding to a sequence of a priori distributions, $\{g_1^{(n)}, 1 - g_1^{(n)}\}$, where the $g_1^{(n)}$ satisfy the following two conditions:

(A) $g_1^{(n)} \geqq \hat{g}_1^{(n)}$ except for a finite number of values of n.

(B) The sequence, $\{g_1^{(n)}\}$, is a null sequence.

It then follows from (1.1) and (A) that there exists an integer, N, such that if $n \geqq N$, the risks, $r_i(\delta_n)$, satisfy

$$r_1(\delta_n) \leqq r(\hat{\delta}_n) \leqq r_2(\delta_n). \tag{1.2}$$

Since a Bayes procedure minimizes the average risk for the corresponding a priori distribution, we have

$$g_1^{(n)} r_1(\delta_n) + (1 - g_1^{(n)}) r_2(\delta_n) \leqq r(\hat{\delta}_n). \tag{1.3}$$

By dividing (1.3) by $r(\hat{\delta}_n)$ and using (1.2) and (B), we obtain

$$\lim_{n\to\infty} \frac{r_2(\delta_n)}{r(\hat{\delta}_n)} = 1. \tag{1.4}$$

Thus, $\{\delta_n\}$ is a sequence of Bayes procedures with the ratio of its maximum risk to the common minimax risk approaching 1 as n approaches infinity. Clearly, for large n, $\{\delta_n\}$, can not be much worse than the minimax procedure when $F_2(x)$ is the true distribution, but when $F_1(x)$ is the true distribution $\{\delta_n\}$ may conceivably be much better than the minimax procedure. For this reason it seemed desirable to study the class of decision procedures having such properties in the more general case when Ω contains k distributions.

We start out with the following definition:

A sequence of decision procedures, $\{\delta_n\}$, where n corresponds to the number of observations, will be said to be *asymptotically minimax* if

$$\lim_{n\to\infty} \frac{\max_i r_i(\delta_n)}{r(\hat{\delta}_n)} = 1, \tag{1.5}$$

where $r_i(\delta_n)$ denotes the risk associated with δ_n when $F_i(x)$ is the true distribution function, and $r(\hat{\delta}_n)$ denotes the minimax risk.

In Section 3, examples will be given for which asymptotically minimax decision procedures exist with the ratio of one of its risks to the common minimax risk approaching zero! For sufficiently large n, such a procedure would be more desirable than the minimax procedure for most problems arising in practice.

A lemma will now be given which will be used in the proof of Theorem 1.1.

LEMMA 1.1. *If δ is a Bayes procedure relative to an a priori distribution, g, and $\hat{\delta}$ is the minimax procedure, then*

$$\min_i r_i(\delta) \leqq r(\hat{\delta}) \leqq \max_i r_i(\delta). \tag{1.6}$$

PROOF. From the definition of a minimax procedure, we have that $r(\hat{\delta}) \leqq$

$\max_i r_i(\delta)$. To prove the other inequality assume $\min_i r_i(\delta) > r(\hat{\delta})$; it then follows that

$$\sum_{i=1}^{k} g_i r_i(\delta) > \sum_{i=1}^{k} g_i r_i(\hat{\delta}). \tag{1.7}$$

Since the Bayes procedure, δ, minimizes the average risk when g is the a priori distribution, (1.7) is impossible. This contradiction completes the proof.

THEOREM 1.1. *A sufficient condition for a sequence of Bayes procedures,* $\{\delta_n\}$, *to be asymptotically minimax is*

$$\lim_{n\to\infty} \frac{r_i(\delta_n)}{r_j(\delta_n)} = 1 \qquad \textit{for all } i, j = 1, 2, \cdots, k. \tag{1.8}$$

PROOF. Equation (1.8) implies

$$\lim_{n\to\infty} \frac{\max_i r_i(\delta_n)}{\min_i r_i(\delta_n)} = 1. \tag{1.9}$$

From Lemma 1.1 we have for all n

$$1 \leqq \frac{\max_i r_i(\delta_n)}{r(\hat{\delta}_n)} \leqq \frac{\max_i r_i(\delta_n)}{\min_i r_i(\delta_n)}. \tag{1.10}$$

Hence

$$\lim_{n\to\infty} \frac{\max_i r_i(\delta_n)}{r(\hat{\delta}_n)} = 1, \tag{1.11}$$

which completes the proof.

THEOREM 1.2. *A necessary condition for a sequence of Bayes procedures,* $\{\delta_n\}$, *corresponding to the a priori distributions,* $\{g^{(n)}\}$, *to be asymptotically minimax is*

$$\lim_{n\to\infty} \frac{r_\alpha(\delta_n)}{r_\beta(\delta_n)} = 1 \tag{1.12}$$

for all α *and* β *belonging to* Γ, *where* Γ *is the set of integers,* j, *for which* $\liminf_{n\to\infty} \hat{g}_j^{(n)} > 0$.

PROOF. Since the minimax procedure, $\hat{\delta}_n$, is the Bayes procedure corresponding to $\hat{g}^{(n)}$, we have

$$\sum_{i=1}^{k} \hat{g}_i^{(n)} r_i(\delta_n) \geqq \sum_{i=1}^{k} \hat{g}_i^{(n)} r(\hat{\delta}_n), \tag{1.13}$$

and consequently

$$\sum_{i=1}^{k} \hat{g}_i^{(n)} \frac{r_i(\delta_n)}{r(\hat{\delta}_n)} \geqq 1. \tag{1.14}$$

Since for any $j \varepsilon \Gamma$, $\lim \inf_{n\to\infty} \hat{g}_j^{(n)} > 0$, there exists a $\delta > 0$ and an integer, N, such that $\hat{g}_j^{(n)} > \delta$ for all $n > N$. Now for $\{\delta_n\}$ to be asymptotically minimax, for any given $\epsilon > 0$ there must exist an integer, M, such that for all i and all $n > M$,

$$\frac{r_i(\delta_n)}{r(\hat{\delta}_n)} < 1 + \epsilon\delta. \tag{1.15}$$

Hence for $n > M$,

$$\sum_{i \neq j} \hat{g}_i^{(n)}(1 + \epsilon\delta) + \hat{g}_j^{(n)} \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} > 1, \tag{1.16}$$

from which it follows that

$$1 - \hat{g}_j^{(n)} + \epsilon\delta + \hat{g}_j^{(n)} \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} > 1. \tag{1.17}$$

Thus, for $n > \max(N, M)$

$$\frac{r_j(\delta_n)}{r(\hat{\delta}_n)} > \frac{\hat{g}_j^{(n)} - \epsilon\delta}{\hat{g}_j^{(n)}} > 1 - \epsilon. \tag{1.18}$$

But since $\delta < 1$, from (1.15) we have

$$\frac{r_j(\delta_n)}{r(\hat{\delta}_n)} < 1 + \epsilon, \tag{1.19}$$

and therefore

$$\lim_{n\to\infty} \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} = 1. \tag{1.20}$$

It then follows that if both α and β belong to Γ, we have

$$\lim_{n\to\infty} \frac{r_\alpha(\delta_n)}{r_\beta(\delta_n)} = 1. \tag{1.21}$$

Theorem 1.2 is useful in proving that certain sequences of Bayes procedures can not be asymptotically minimax. For example, suppose Ω consists of k univariate normal distributions $N(\theta_i, \sigma^2)$, $i = 1, 2, \cdots, k$, with $\theta_i = \theta_1 + (i - 1)\gamma$, for some $\gamma > 0$. In Section 3 it will be seen that for such a class of distribution functions, $\lim \inf_{n\to\infty} \hat{g}_i^{(n)} > 0$ for all i. Now consider the sequence of Bayes procedures, $\{\delta_n\}$, where δ_n corresponds to the a priori distribution $g_i^{(n)} = 1/k$ for all i. According to the decision procedure δ_n one selects

$$\begin{aligned}
&N(\theta_1, \sigma^2) \quad &&\text{if } \bar{X} < \theta_1 + \tfrac{1}{2}\gamma, \\
&N(\theta_i, \sigma^2) \quad &&\text{if } \theta_i - \tfrac{1}{2}\gamma < \bar{X} < \theta_i + \tfrac{1}{2}\gamma \quad i = 2, 3, \cdots, k - 1, \\
&N(\theta_k, \sigma^2) \quad &&\text{if } \bar{X} > \theta_k - \tfrac{1}{2}\gamma,
\end{aligned}$$

where $\bar{X} = \sum_{i=1}^n x_i/n$. For $k > 2$ it is easily seen that $r_2(\delta_n)/r_1(\delta_n) = 2$ for all n. Therefore $\{\delta_n\}$ can not be asymptotically minimax.

The following theorem is of interest in connection with the construction of asymptotically minimax procedures. It includes the special case that if only one component of $\hat{g}^{(n)}$ approaches zero, say $\hat{g}_\alpha^{(n)} \to 0$, and none of the sequences of the other components has zero as a limit, then given any null sequence, $\{h^{(n)}\}$, such that $h^{(n)} \geqq \hat{g}_\alpha^{(n)}$ from some n on, it is always possible to construct an asymptotically minimax sequence of Bayes procedures with $h^{(n)}$ as the αth component of $g^{(n)}$.

THEOREM 1.3. *If $\{\hat{g}_\alpha^{(n)}\}$ is a null sequence and $\{\hat{g}_i^{(n)}/\hat{g}_\alpha^{(n)}\}$ is bounded for all $i \varepsilon \tau$, where τ is the set of integers, t, for which $\{\hat{g}_i^{(n)}\}$ is a null sequence, and if* $\liminf_{n\to\infty} \hat{g}_i^{(n)} > 0$ *for all $i \not\varepsilon \tau$, then an asymptotically minimax sequence of Bayes procedures can be found with any given sequence, $\{h^{(n)}\}$, as the αth components of $\{g^{(n)}\}$, provided $\{h^{(n)}\}$ is a null sequence and $h^{(n)} \geqq \hat{g}_\alpha^{(n)}$ except for at most a finite number of values of n.*

PROOF. Let $\{\delta_n\}$ be the sequence of Bayes procedures relative to the sequence of a priori distributions, $\{g^{(n)}\}$, given by

$$g_i^{(n)} = \frac{h^{(n)}}{\hat{g}_\alpha^{(n)}} \hat{g}_i^{(n)} \qquad \text{for } i \varepsilon \tau \tag{1.22}$$
$$g_i^{(n)} = \frac{1 - \frac{h^{(n)}}{\hat{g}_\alpha^{(n)}} \sum_{j\varepsilon\tau} \hat{g}_j^{(n)}}{\sum_{j \not\varepsilon \tau} \hat{g}_j^{(n)}} \hat{g}_i^{(n)} \qquad \text{for } i \not\varepsilon \tau.$$

It can easily be verified that from some n on, each $g^{(n)}$ defined by (1.22) is a probability distribution, and its αth component is $h^{(n)}$. Also, for all j

$$g_i^{(n)}/g_j^{(n)} \geqq \hat{g}_i^{(n)}/\hat{g}_j^{(n)} \qquad \text{for } i \varepsilon \tau \tag{1.23}$$
$$g_i^{(n)}/g_j^{(n)} \leqq \hat{g}_i^{(n)}/\hat{g}_j^{(n)} \qquad \text{for i } \not\varepsilon \tau.$$

Hence

$$r_i(\delta_n) \leqq r(\hat{\delta}_n) \qquad \text{for } i \varepsilon \tau \tag{1.24}$$
$$r_i(\delta_n) \geqq r(\hat{\delta}_n) \qquad \text{for } i \not\varepsilon \tau.$$

Therefore we need only prove that

$$\lim_{n\to\infty} \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} = 1$$

for all $j \not\varepsilon \tau$.

Since $\hat{g}^{(n)}$ maximizes the average risk, we have

$$\sum_{i=1}^{k} g_i^{(n)} r_i(\delta_n) \leqq \sum_{i=1}^{k} \hat{g}_i^{(n)} r(\hat{\delta}_n) = r(\hat{\delta}_n). \tag{1.25}$$

Therefore

$$\sum_{i \not\varepsilon \tau} g_i^{(n)} \frac{r_i(\delta_n)}{r(\hat{\delta}_n)} \leqq 1. \tag{1.26}$$

Consider a particular $j \not\in \tau$. In view of (1.24), we can replace $r_i(\delta_n)/r(\hat{\delta}_n)$ in (1.26) by 1 for all $i \neq j$, yielding

$$1 \leqq \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} \leqq \frac{1 - \sum_{\substack{i \not\in \tau \\ i \neq j}} g_i^{(n)}}{g_j^{(n)}} = \frac{g_j^{(n)} + \sum_{i \in \tau} g_i^{(n)}}{g_j^{(n)}}. \tag{1.27}$$

But since $g_i^{(n)}/g_j^{(n)} \to 0$ for all $i \in \tau$, we have the desired result.

We now define a sequence of decision procedures, $\{\delta_n\}$, to be *asymptotically subminimax*, if $\{\delta_n\}$ is asymptotically minimax and satisfies

$$\limsup_{n\to\infty} \frac{r_j(\delta_n)}{r(\hat{\delta}_n)} < 1 \tag{1.28}$$

for at least one value of j.

2. Asymptotically admissible decision procedures. In view of the existence of sequences of decision procedures which are asymptotically subminimax, it is desirable to set up some criterion for distinguishing the more desirable asymptotically subminimax procedures. In this connection, a sequence of decision procedures, $\{\delta_n\}$, will be said to be *asymptotically admissible* if there does not exist another sequence of decision procedures, $\{\delta_n'\}$, such that

$$\limsup_{n\to\infty} \frac{r_i(\delta_n')}{r_i(\delta_n)} \leqq 1 \tag{2.1}$$

for all i, and the strict inequality holds for at least one value of i. When such a $\{\delta_n'\}$ exists, $\{\delta_n\}$ will be said to be *asymptotically inadmissible.*

As a consequence of the above definition, if no asymptotically subminimax procedure exists, then the minimax procedure and all asymptotically minimax procedures are asymptotically admissible. On the other hand, when an asymptotically subminimax procedure does exist, the minimax procedure is asymptotically inadmissible. An asymptotically subminimax procedure may or may not be asymptotically admissible.

For large values of n it seems reasonable to require that the procedure selected should be asymptotically admissible when such a procedure exists. For this reason, it is of interest to know when the minimax procedure is asymptotically admissible.

The following theorem will be helpful in determining the asymptotic behavior of sequences of decision procedures.

THEOREM 2.1. *If the sequence of Bayes decision procedures,* $\{\delta_n\}$, *is asymptotically minimax and* $\liminf_{n\to\infty} \hat{g}_s^{(n)} > 0$, *where* $\hat{g}_s^{(n)}$ *is the* sth *component of the least favorable a priori distribution for* n *observations, then*

$$\lim_{n\to\infty} \frac{r_s(\delta_n)}{r(\hat{\delta}_n)} = 1. \tag{2.2}$$

PROOF. The integer s belongs to the set Γ defined in Theorem 1.2, and the proof of that theorem up to (1.20) proves (2.2).

From Theorem 2.1 it is seen that if $\lim\inf_{n\to\infty} \hat{g}_s^{(n)} > 0$, then it is impossible to construct an asymptotically subminimax procedure with the strict inequality holding for the sth risk. Hence a necessary condition for the existence of an asymptotically subminimax procedure is that $\lim\inf_{n\to\infty} \hat{g}_i^{(n)} = 0$ for at least one value of i. Thus we have the following sufficient condition for the minimax procedure to be asymptotically admissible.

THEOREM 2.2. *If* $\lim\inf_{n\to\infty} \hat{g}_i^{(n)} > 0$ *for all* i, *then the minimax procedure is asymptotically admissible.*

In order to determine the asymptotic behavior of the minimax procedure, it becomes essential to know whether any of the $\hat{g}_i^{(n)}$ have zero as a lower limit in a given problem. If none of them has a zero limit, then the minimax procedure is asymptotically admissible and would appear to be a good decision procedure even for large values of n. When some of the $\hat{g}_i^{(n)}$ do have a zero limit, we would like to know which ones, and whether the minimax procedure is then asymptotically inadmissible. If this should be the case, we would then like to know if there is an asymptotically admissible asymptotically subminimax procedure, and how to find it.

In the next section, a detailed study will be made of the limits of the $\hat{g}_i^{(n)}$ when Ω consists of k univariate normal distributions, all having the same variance but different means. The questions raised in the above paragraph on the lower limits of the $\hat{g}_i^{(n)}$, and on the asymptotic admissibility of the minimax procedure are completely resolved for the decision problems under consideration. Results are also obtained on the construction of asymptotically subminimax procedures and on their asymptotic admissibility.

3. Asymptotic theory for normal distributions. Throughout this section it will be assumed that Ω consists of k univariate normal distributions, $N(\theta_i, \sigma^2)$, $i = 1, 2, \cdots, k$, all having the same known variance. Without loss in generality it will be assumed that the k means are labeled so that $\theta_1 < \theta_2 < \cdots < \theta_k$. Wald [3] showed that the minimax decision procedure for selecting the true mean for any fixed sample size, n, can then be obtained by determining $k - 1$ points, $t_1 < t_2 < \cdots < t_{k-1}$ which divide the sample space of $\bar{X} = \sum_{i=1}^{n} x_i/n$, where x_i is the ith observation, into k intervals in such a way that the minimax decision rule is given by selecting θ_i if $\bar{X}$ lies in (t_{i-1}, t_i), where $t_0 = -\infty$ and $t_k = +\infty$. These $k - 1$ points can be found from the system of equations:

$$
\begin{aligned}
&\int_{-\infty}^{t_1} p_1(y)\,dy = \lambda \\
&\int_{t_1}^{t_2} p_2(y)\,dy = \lambda \\
&\qquad\vdots \\
&\int_{t_{k-2}}^{t_{k-1}} p_{k-1}(y)\,dy = \lambda \\
&\int_{t_{k-1}}^{\infty} p_k(y)\,dy = \lambda,
\end{aligned}
\tag{3.1}
$$

where

$$p_i(y) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-n(y-\theta_i)^2/2\sigma^2}$$

The value of λ obtained from (3.1) is the probability of making a correct decision.

If we let

$$G(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\, dw, \tag{3.2}$$

then (3.1) can be written as

$$\begin{aligned} G[\sqrt{n}(t_1 - \theta_1)/\sigma] &= \lambda \\ G[\sqrt{n}\,(t_2 - \theta_2)/\sigma] - G[\sqrt{n}(t_1 - \theta_2)/\sigma] &= \lambda \\ &\vdots \\ G[\sqrt{n}(t_{k-1} - \theta_{k-1})/\sigma] - G[\sqrt{n}(t_{k-2} - \theta_{k-1})/\sigma] &= \lambda \\ 1 - G[\sqrt{n}(t_{k-1} - \theta_k)/\sigma] &= \lambda. \end{aligned} \tag{3.3}$$

The least favorable a priori distribution, $\hat{g}^{(n)}$, can then be obtained from

$$\log \frac{\hat{g}_i^{(n)}}{\hat{g}_{i+1}^{(n)}} = \frac{n}{2\sigma^2}(\theta_{i+1} - \theta_i)(2t_i - \theta_{i+1} - \theta_i) \tag{3.4}$$

and

$$\sum_{i=1}^{k} \hat{g}_i^{(n)} = 1. \tag{3.5}$$

If the transformations, $x_i' = x_i/\sigma$ and $\varphi_i = \theta_i/\sigma$ are applied to the observations and distribution functions respectively, the minimax procedure would yield decision intervals having end points $\hat{c}_i = t_i/\sigma$, with the same minimax risk and least favorable a priori distribution as in the original problem. Thus, we need only investigate the behavior of the minimax decision procedure for $\Omega = \{N(\varphi_i, 1)\}$, and then interpret the results obtained in terms of the φ's and $\hat{c}$'s into equivalent results in terms of the θ's and t's. For this reason we shall from now on consider the distribution space to be $\{N(\varphi_i, 1)\}$, without any loss in generality.

Since we shall be interested in the behavior of the $\hat{c}_i$ with increasing n, we shall usually write $\hat{c}_i^{(n)}$ instead of $\hat{c}_i$. Then, for the distribution space $\{N(\varphi_i, 1)\}$, (3.3) becomes

$$\begin{aligned} G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_1)] &= \lambda \\ G[\sqrt{n}(\hat{c}_2^{(n)} - \varphi_2)] - G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)] &= \lambda \\ &\vdots \\ G[\sqrt{n}(\hat{c}_{k-1}^{(n)} - \varphi_{k-1})] - G[\sqrt{n}(\hat{c}_{k-2}^{(n)} - \varphi_{k-1})] &= \lambda \\ 1 - G[\sqrt{n}(\hat{c}_{k-1}^{(n)} - \varphi_k)] &= \lambda, \end{aligned} \tag{3.6}$$

and (3.4) becomes

$$\log \frac{\hat{g}_i^{(n)}}{\hat{g}_{i+1}^{(n)}} = \tfrac{1}{2}n(\varphi_{i+1} - \varphi_i)(2\hat{c}_i^{(n)} - \varphi_{i+1} - \varphi_i). \tag{3.7}$$

The solution of (3.6) for any given n can be accomplished by various iterative procedures. We shall not consider these methods in this paper because we shall be concerned only with the limits of the sequences of the $\hat{c}_i^{(n)}$ and of the $\hat{g}_i^{(n)}$

It will be convenient first to prove several lemmas from which the theorems concerning the asymptotic behavior of the minimax procedure for all possible sets of values for the φ_i will follow easily.

LEMMA 3.1. *If* $\{A_n\}$ *and* $\{B_n\}$ *are sequences of real numbers such that* $\lim_{n\to\infty} A_n < \lim_{n\to\infty} B_n \leqq 0$, *then*

$$\lim_{n\to\infty} \frac{G(\sqrt{n}A_n)}{G(\sqrt{n}B_n)} = 0. \tag{3.8}$$

PROOF. The conclusion is obvious except when there exists an infinite subsequence of B_n for which $\sqrt{n}B_n \to -\infty$. In that case, by applying the first term of the asymptotic expansion for $G(t)$ for negative values of t, namely

$$G(t) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}t^2}}{-t}, \tag{3.9}$$

to the left member of (3.8), it becomes

$$\lim_{n\to\infty} \frac{B_n}{A_n} \exp \{-\tfrac{1}{2}n(A_n^2 - B_n^2)\}. \tag{3.10}$$

But since

$$0 \leqq \lim_{n\to\infty} \frac{B_n}{A_n} < 1$$

and

$$\lim_{n\to\infty} (A_n^2 - B_n^2) > 0,$$

the value of (3.10) is zero.

LEMMA 3.2. *If we denote by* γ *the value of* $\min_i (\varphi_{i+1} - \varphi_i)$, *then the* $\hat{c}_i^{(n)}$ *which determine the minimax decision intervals, satisfy*

$$\liminf_{n\to\infty} (\varphi_i - \hat{c}_{i-1}^{(n)}) \geqq \tfrac{1}{2}\gamma \tag{3.11}$$

and

$$\liminf_{n\to\infty} (\hat{c}_i^{(n)} - \varphi_i) \geqq \tfrac{1}{2}\gamma, \tag{3.12}$$

and at least one of the above holds as an equality.

PROOF. Let j be the smallest integer for which $\gamma = \varphi_{j+1} - \varphi_j$. Now consider the decision procedure determined by

$$(3.13) \qquad \begin{aligned} c_i &= \varphi_i + \tfrac{1}{2}\gamma && \text{for } i \leqq j \\ c_i &= \varphi_{i+1} - \tfrac{1}{2}\gamma && \text{for } i > j, \end{aligned}$$

where we select φ_1 if $\bar{X} < c_1$, φ_i if $c_{i-1} < \bar{X} < c_i$ for $1 < i < k$, and φ_k if $\bar{X} > c_{k-1}$. This is the Bayes procedure corresponding to the a priori distribution $g_1^{(n)}, g_2^{(n)}, \cdots, g_k^{(n)}$ given by

$$(3.14) \qquad \begin{aligned} &\log \frac{g_i^{(n)}}{g_{i+1}^{(n)}} = \tfrac{1}{2}n(\varphi_{i+1} - \varphi_i)(2c_i - \varphi_{i+1} - \varphi_i), \quad i = 1, 2, \cdots, k-1 \\ &\sum_{i=1}^{k} g_i^{(n)} = 1. \end{aligned}$$

For any n, the minimum risk is $G(-\sqrt{n}\,\gamma/2)$ and the maximum risk does not exceed $2G(-\sqrt{n}\,\gamma/2)$. By Lemma 1.1 the minimax risk must lie between these two values. Hence

$$(3.15) \qquad 1 \leqq \frac{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)] + G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G(-\sqrt{n}\,\gamma/2)} \leqq 2.$$

If (3.11) or (3.12) were false, then, by Lemma 3.1, for a sufficiently large n the fraction in (3.15) would become greater than 2, contradicting (3.15). If neither (3.11) nor (3.12) held as an equality, the fraction would become smaller than 1, again contradicting (3.15).

LEMMA 3.3. *The minimax decision interval end points, $\hat{c}_i^{(n)}$, and the components of the least favorable a priori distributions, $\hat{g}_i^{(n)}$, satisfy*

$$(3.16) \qquad \liminf_{n\to\infty} \frac{\hat{g}_{i+1}^{(n)}}{\hat{g}_i^{(n)}} = \liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]},$$

and (3.16) *also holds if the lower limits are replaced by upper limits.*

PROOF. For any given i and n, the $\hat{c}_i^{(n)}, \hat{g}_i^{(n)}, \hat{g}_{i+1}^{(n)}$ satisfy

$$(3.17) \qquad \frac{\hat{g}_{i+1}^{(n)}}{\hat{g}_i^{(n)}} = \exp\{-\tfrac{1}{2}n[(\hat{c}_i^{(n)} - \varphi_i)^2 - (\hat{c}_i^{(n)} - \varphi_{i+1})^2]\}.$$

Since the minimax risk approaches zero as n approaches infinity, the $\hat{c}_i^{(n)}$ must eventually satisfy $\varphi_i < \hat{c}_i^{(n)} < \varphi_{i+1}$.

Now by using (3.9) the right member of (3.16) becomes

$$(3.18) \qquad \liminf_{n\to\infty} \exp\{-\tfrac{1}{2}n[(\varphi_i - \hat{c}_i^{(n)})^2 - (\hat{c}_i^{(n)} - \varphi_{i+1})^2]\} \cdot \frac{\hat{c}_i^{(n)} - \varphi_{i+1}}{\varphi_i - \hat{c}_i^{(n)}}.$$

Since both factors in (3.18) approach their lower limits through the same subsequences, it can be written as

$$(3.19) \qquad \liminf_{n\to\infty} \frac{\hat{g}_{i+1}^{(n)}}{\hat{g}_i^{(n)}} \cdot L,$$

where

$$L = \liminf_{n\to\infty} \frac{\hat{c}_i^{(n)} - \varphi_{i+1}}{\varphi_i - \hat{c}_i^{(n)}}. \tag{3.20}$$

Now if $L = 1$, then we have (3.16). If $L < 1$, then from (3.18) we see that the right member of (3.16) is zero, and by (3.17) the left member of (3.16) is also zero. If $L > 1$, then from (3.17) and (3.18) it follows that both members of (3.16) become infinite. The remainder of the theorem is proved in exactly the same manner by replacing lower limits by upper limits throughout the above.

In several of the lemmas and theorems to follow, it will be convenient to refer to those φ_i which belong to a set of means of Ω satisfying Condition A given below. In stating the condition and in the remainder of this section except when otherwise stated, the value of $\min_i (\varphi_{i+1} - \varphi_i)$ will be denoted by γ.

Condition A. A set of means of Ω with consecutive subscripts, say $\varphi_s, \varphi_{s+1}, \cdots, \varphi_{s+t}$ will be said to satisfy Condition A if $\varphi_{i+1} - \varphi_i = \gamma$ for $i = s, s + 1, \cdots, s + t - 1$.

LEMMA 3.4. *If φ_i belongs to a set of means satisfying Condition* A, *then*

$$\lim_{n\to\infty} (\hat{c}_i^{(n)} - \varphi_i) = \tfrac{1}{2}\gamma \qquad s \leqq i < s + t \tag{3.21}$$

and

$$\lim_{n\to\infty} (\varphi_i - \hat{c}_{i-1}^{(n)}) = \tfrac{1}{2}\gamma \qquad s < i \leqq s + t. \tag{3.22}$$

PROOF. Suppose that

$$\limsup_{n\to\infty} (\hat{c}_\alpha^{(n)} - \varphi_\alpha) > \tfrac{1}{2}\gamma \tag{3.23}$$

for some α satisfying $s \leqq \alpha < s + t$. Then from

$$(\varphi_{\alpha+1} - \hat{c}_\alpha^{(n)}) + (\hat{c}_\alpha^{(n)} - \varphi_\alpha) = \gamma \tag{3.24}$$

we get

$$\liminf_{n\to\infty} (\varphi_{\alpha+1} - \hat{c}_\alpha^{(n)}) < \tfrac{1}{2}\gamma. \tag{3.25}$$

Since (3.25) contradicts Lemma 3.2, we have

$$\limsup_{n\to\infty} (\hat{c}_\alpha^{(n)} - \varphi_\alpha) \leqq \tfrac{1}{2}\gamma. \tag{3.26}$$

By Lemma 3.2 we have also

$$\liminf_{n\to\infty} (\hat{c}_\alpha^{(n)} - \varphi_\alpha) \geqq \tfrac{1}{2}\gamma, \tag{3.27}$$

and (3.21) follows. Then from (3.21) and (3.24) we get (3.22).

LEMMA 3.5. *If φ_i and φ_{i+1} belong to a set of means satisfying Condition* A, *then*

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]} \leqq 4, \tag{3.28}$$

and

$$\liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]} \geqq \tfrac{1}{4}. \tag{3.29}$$

PROOF. As in the proof of Lemma 3.2, we have

$$\frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})] + G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]}{G(-\sqrt{n}\,\gamma/2)} \leqq 2. \tag{3.30}$$

Therefore

$$\frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G(-\sqrt{n}\,\gamma/2)} \leqq 2. \tag{3.31}$$

By applying (3.9) to (3.31) we obtain

$$\limsup_{n\to\infty} \exp\left\{-\tfrac{1}{2}n\left[(\varphi_i - \hat{c}_i^{(n)})^2 - \frac{\gamma^2}{4}\right]\right\} \cdot \frac{-\frac{1}{2}\gamma}{\varphi_i - \hat{c}_i^{(n)}} \leqq 2. \tag{3.32}$$

For any i satisfying $s \leqq i < s + t$, from Lemma 3.4 we have that $\hat{c}_i^{(n)} - \varphi_i \to \frac{1}{2}\gamma$. Therefore

$$\limsup_{n\to\infty} \exp\{\tfrac{1}{2}n\gamma(\varphi_i - \hat{c}_i^{(n)} + \tfrac{1}{2}\gamma)\} \leqq 2, \tag{3.33}$$

and finally

$$\limsup_{n\to\infty} \exp\left\{\tfrac{1}{2}n\gamma\left(-\hat{c}_i^{(n)} + \frac{\varphi_i + \varphi_{i+1}}{2}\right)\right\} \leqq 2. \tag{3.34}$$

Again from (3.9) the left hand member of (3.28) can be written as

$$\limsup_{n\to\infty} \exp\{-\tfrac{1}{2}n[(\varphi_i - \hat{c}_i^{(n)})^2 - (\hat{c}_i^{(n)} - \varphi_{i+1})^2]\} \cdot \frac{\hat{c}_i^{(n)} - \varphi_{i+1}}{\varphi_i - \hat{c}_i^{(n)}}, \tag{3.35}$$

and by Lemma 3.4, (3.35) becomes

$$\limsup_{n\to\infty} \exp\left\{n\gamma\left(-\hat{c}_i^{(n)} + \frac{\varphi_i + \varphi_{i+1}}{2}\right)\right\}. \tag{3.36}$$

Then from (3.34) we get

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]} \leqq 4. \tag{3.37}$$

From (3.30) we also have

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]}{G(-\sqrt{n}\,\gamma/2)} \leqq 2, \tag{3.38}$$

and in exactly the same way we obtain

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]}{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]} \leqq 4, \tag{3.39}$$

from which (3.29) follows.

LEMMA 3.6. *If φ_i belongs to a set of means satisfying Condition* A, *and $s < i < s + t$, then*

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]} \leqq 4, \tag{3.40}$$

and

$$\liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]} \geqq \tfrac{1}{4}. \tag{3.41}$$

PROOF. Since for any given $T < 0$, the function $G(y) + G(T - y)$ takes on its minimum value when $y = T/2$, we have

$$2G(-\sqrt{n}\,\gamma/2) \leqq G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})] + G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})] \tag{3.42}$$

for $s \leqq i < s + t$. Now as in the proof of Lemma 3.2, we have that $r(\hat{\delta}_n) \leqq 2G(-\sqrt{n}\,\gamma/2)$. Hence

$$\liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]} \geqq \liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_i^{(n)} - \varphi_{i+1})]}. \tag{3.43}$$

Then by Lemma 3.5, the last expression is equal to or greater than $\frac{1}{4}$. Similarly we obtain

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \hat{c}_i^{(n)})]}{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]} \leqq \limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{i-1} - \hat{c}_{i-1}^{(n)})]}{G[\sqrt{n}(\hat{c}_{i-1}^{(n)} - \varphi_i)]} \leqq 4. \tag{3.44}$$

LEMMA 3.7. *If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers such that $\lim_{n\to\infty} a_n = -\gamma/2$, and*

$$\lim_{n\to\infty} \frac{G(\sqrt{n}a_n)}{G(\sqrt{n}b_n)} = 1, \tag{3.45}$$

then

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(-a_n - \gamma)]}{G[\sqrt{n}(-b_n - \gamma)]} = 1. \tag{3.46}$$

PROOF. By applying (3.9) to the left member of (3.45), we get

$$\lim_{n\to\infty} \exp\{-\tfrac{1}{2}n(a_n^2 - b_n^2)\} \cdot \frac{b_n}{a_n} = 1.$$

This implies that

$$\lim_{n\to\infty} n(a_n^2 - b_n^2) = 0,$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = -\tfrac{1}{2}\gamma, \tag{3.47}$$

from which we get

$$\lim_{n\to\infty} n(a_n - b_n) = 0. \tag{3.48}$$

By applying (3.9) to (3.46) we must show that

$$\lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[(a_n+\gamma)^2 - (b_n+\gamma)^2]\} \cdot \frac{b_n+\gamma}{a_n+\gamma} = 1, \tag{3.49}$$

which can be written as

$$\lim_{n\to\infty} \exp\{-\tfrac{1}{2}n(a_n - b_n)(a_n + b_n + 2\gamma)\} \cdot \frac{b_n+\gamma}{a_n+\gamma} = 1. \tag{3.50}$$

But this last line follows from (3.47) and (3.48).

THEOREM 3.1. *If all the means are equally spaced, that is, if* $\varphi_{i+1} - \varphi_i = \gamma$ *for* $i = 1, 2, \cdots, k-1$, *then* $\liminf_{n\to\infty} \hat{g}_i^{(n)} > 0$ *for all i, and the minimax procedure is asymptotically admissible.*

PROOF. By Lemmas 3.3 and 3.5 we get that

$$\tfrac{1}{4} \leqq \liminf_{n\to\infty} \frac{\hat{g}_{i+1}^{(n)}}{\hat{g}_i^{(n)}} \leqq 4, \qquad i = 1, 2, \cdots, k-1. \tag{3.51}$$

Since $\sum_{i=1}^{k} \hat{g}_i^{(n)} = 1$, it follows that $\liminf_{n\to\infty} \hat{g}_i^{(n)} > 0$ for all values of i. Then by Theorem 2.2 we have that the minimax procedure is asymptotically admissible.

THEOREM 3.2. *If* Ω *has a mean,* φ_α, *such that* $\min\{(\varphi_\alpha - \varphi_{\alpha-1}), (\varphi_{\alpha+1} - \varphi_\alpha)\} > \gamma$, *then* $\liminf_{n\to\infty} \hat{g}_\alpha^{(n)} = 0$, *and the minimax procedure is asymptotically inadmissible. (For* $\alpha = 1$ *take* $\varphi_0 = -\infty$ *and for* $\alpha = k$ *take* $\varphi_{k+1} = +\infty$).

PROOF. By Lemma 3.2, we have that

$$\liminf_{n\to\infty} (\varphi_\alpha - \hat{c}_{\alpha-1}^{(n)}) = \tfrac{1}{2}\gamma \tag{3.52}$$

or

$$\liminf_{n\to\infty} (\hat{c}_\alpha^{(n)} - \varphi_\alpha) = \tfrac{1}{2}\gamma. \tag{3.53}$$

If (3.52) holds, then since $(\varphi_\alpha - \varphi_{\alpha-1}) > \gamma$, by Lemmas 3.1 and 3.3, we have $\liminf_{n\to\infty} \hat{g}_\alpha^{(n)}/\hat{g}_{\alpha-1}^{(n)} = 0$. Hence $\liminf_{n\to\infty} \hat{g}_\alpha^{(n)} = 0$. Similarly if (3.53) holds, we get that $\limsup_{n\to\infty} \hat{g}_{\alpha+1}^{(n)}/\hat{g}_\alpha^{(n)} = \infty$, and therefore $\liminf_{n\to\infty} \hat{g}_\alpha^{(n)} = 0$.

To show that the minimax procedure is asymptotically inadmissible, consider the decision procedure, $\bar{\delta}_n$, having decision interval end points $\bar{c}_i^{(n)}$, where

$$\begin{aligned} \bar{c}_i^{(n)} &= \hat{c}_i^{(n)} \quad \text{for } i = 1, 2, \cdots, \alpha-2, \alpha+1, \cdots, k-1 \\ \bar{c}_i^{(n)} &= \tfrac{1}{2}(\varphi_i + \varphi_{i+1}) \qquad \text{for } i = \alpha-1, \alpha. \end{aligned} \tag{3.54}$$

Clearly $r_i(\bar{\delta}_n) = r(\hat{\delta}_n)$ for $i < \alpha - 1$ and for $i > \alpha + 1$. Since $r(\hat{\delta}_n) \geqq G(-\sqrt{n}\,\gamma/2)$ by Lemma 3.1 we get

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_i - \bar{c}_i^{(n)})]}{r(\hat{\delta}_n)} = 0, \qquad i = \alpha-1, \alpha \tag{3.55}$$

and

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_i^{(n)} - \varphi_{i+1})]}{r(\hat{\delta}_n)} = 0, \qquad i = \alpha - 1, \alpha. \tag{3.56}$$

Hence

$$\limsup_{n\to\infty} \frac{r_i(\bar{\delta}_n)}{r(\hat{\delta}_n)} \leq 1, \qquad i = \alpha - 1, \alpha + 1$$

and

$$\lim_{n\to\infty} \frac{r_\alpha(\bar{\delta}_n)}{r(\hat{\delta}_n)} = 0, \tag{3.57}$$

which completes the proof.

Theorems 3.1 and 3.2 cover all possible positions of the k means except when they can be put into two or more sets, such that, each set contains two or more consecutive means, the consecutive means in each set differ by γ, and all the differences between consecutive means not belonging to the same set are greater than γ. Throughout the remainder of this section, except when otherwise stated, only those cases when the means of Ω fall into such sets will be considered. The number of such sets will be denoted by r and the number of means in the ith set will be denoted by n_i. Thus $\sum_{i=1}^{r} n_i = k$. The jth mean in the ith set will be denoted by $\varphi_{i,j}$ and the component of the least favorable a priori distribution corresponding to this mean for samples of n will be denoted by $\hat{g}_{i,j}^{(n)}$. The right end point of the minimax decision interval for selecting $\varphi_{i,j}$ for samples of n will be denoted by $\hat{c}_{i,j}^{(n)}$.

THEOREM 3.3. *If $r \geq 2$ and all the sets do not contain the same number of means, say $n_j < \max_i(n_i)$, then $\liminf_{n\to\infty} \hat{g}_{j,t}^{(n)} = 0$ for all t, and the minimax procedure is asymptotically inadmissible.*

PROOF. Suppose

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_{j-1,n_{j-1}}^{(n)} - \varphi_{j,1})]}{r(\hat{\delta}_n)} > 0, \tag{3.58}$$

then since $(\varphi_{j,1} - \varphi_{j-1,n_{j-1}}) > \gamma$, by Lemmas 3.1 and 3.3 we get that $\liminf_{n\to\infty} \hat{g}_{j,1}^{(n)} = 0$, and by Lemmas 3.3 and 3.5 we get that $\liminf_{n\to\infty} \hat{g}_{j,t}^{(n)} = 0$ for all t.

If $j = 1$, or if the upper limit in (3.58) is zero, then

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,1} - \hat{c}_{q,1}^{(n)})]}{G[\sqrt{n}(\varphi_{j,1} - \hat{c}_{j,1}^{(n)})]} \leq 1, \tag{3.59}$$

for any q such that $n_q = \max_i(n_i)$. Since $\varphi_{q,2} - \varphi_{q,1} = \varphi_{j,2} - \varphi_{j,1} = \gamma$, with the use of Lemmas 3.4 and 3.7, we get

$$\liminf_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_{q,1}^{(n)} - \varphi_{q,2})]}{G[\sqrt{n}(\hat{c}_{j,1}^{(n)} - \varphi_{j,2})]} \geq 1. \tag{3.60}$$

Then since

$$(3.61)\qquad \begin{aligned} r(\hat{\delta}_n) &= G[\sqrt{n}(\hat{c}_{q,1}^{(n)} - \varphi_{q,2})] + G[\sqrt{n}(\varphi_{q,2} - \hat{c}_{q,2}^{(n)})] \\ &= G[\sqrt{n}(\hat{c}_{j,1}^{(n)} - \varphi_{j,2})] + G[\sqrt{n}(\varphi_{j,2} - \hat{c}_{j,2}^{(n)})], \end{aligned}$$

it follows that

$$(3.62)\qquad \limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,2} - \hat{c}_{q,2}^{(n)})]}{G[\sqrt{n}(\varphi_{j,2} - \hat{c}_{j,2}^{(n)})]} \leqq 1.$$

In the case when $j < r$, by continuing in this manner we finally reach

$$(3.63)\qquad \limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,n_j} - \hat{c}_{q,n_j}^{(n)})]}{G[\sqrt{n}(\varphi_{j,n_j} - \hat{c}_{j,n_j}^{(n)})]} \leqq 1.$$

But by Lemma 3.6

$$(3.64)\qquad \liminf_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,n_j} - \hat{c}_{q,n_j}^{(n)})]}{r(\hat{\delta}_n)} > 0.$$

Hence

$$(3.65)\qquad \limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{j,n_j} - \hat{c}_{j,n_j}^{(n)})]}{r(\hat{\delta}_n)} > 0.$$

Then as in the earlier part of this proof, we get that $\liminf_{n\to\infty} \hat{g}_{j,t}^{(n)} = 0$ for all t.

If $j = r$, then when we reach

$$(3.66)\qquad \liminf_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_{q,n_j-1}^{(n)} - \varphi_{q,n_j})]}{G[\sqrt{n}(\hat{c}_{j,n_j-1}^{(n)} - \varphi_{j,n_j})]} \geqq 1,$$

it follows that

$$(3.67)\qquad \lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,n_j} - \hat{c}_{q,n_j}^{(n)})]}{G[\sqrt{n}(\hat{c}_{q,n_j-1}^{(n)} - \varphi_{q,n_j})]} = 0,$$

contrary to Lemma 3.6. Hence when $j = r$, we must have the previous case with (3.58) being satisfied.

To prove that the minimax procedure is asymptotically inadmissible, first consider the problems $P_1, P_2, \cdots, P_r$, where Ω_i for P_i consists only of the distribution functions corresponding to the means $\varphi_{i,j}$ of the ith set. Now define the decision procedure $\bar{\delta}_n$ for the original problem by taking for $\bar{c}_{i,j}^{(n)}$ $(1 \leqq j < n_i)$, the right end point of the minimax decision interval for selecting $\varphi_{i,j}$ for P_i and taking $\bar{c}_{i,n_i}^{(n)} = \frac{1}{2}(\varphi_{i,n_i} + \varphi_{i+1,1})$ for $i = 1, 2, \cdots, r-1$. When some of the means of Ω are deleted, the minimax risk for such a sub-problem is clearly smaller than the minimax risk for the original problem. Therefore $r_{i,j}(\bar{\delta}_n) < r(\hat{\delta}_n)$ for all i and $1 < j < n_i$. But for all i,

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{i,n_i} - \bar{c}_{i,n_i}^{(n)})]}{r(\hat{\delta}_n)} = 0,$$

and

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_{i,n_i}^{(n)} - \varphi_{i+1,1})]}{r(\hat{\delta}_n)} = 0.$$

Hence

$$\limsup_{n\to\infty} \frac{r_{i,j}(\bar{\delta}_n)}{r(\hat{\delta}_n)} \leqq 1$$

for all i, j.

Now since $n_j < n_q$, the minimax risk for P_j is less than the minimax risk for P_q, and we have

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{j,1} - \bar{c}_{j,1}^{(n)})]}{G[\sqrt{n}(\varphi_{q,1} - \bar{c}_{q,1}^{(n)})]} \leqq 1. \tag{3.68}$$

If this upper limit is 1, then there exists an infinite subsequence of these ratios which converges to this limit. Let the sequence of values of n corresponding to this subsequence be denoted by $\{t\}$. Then by Lemmas 3.4 and 3.7, we have

$$\lim_{t\to\infty} \frac{G[\sqrt{t}(\bar{c}_{j,1}^{(t)} - \varphi_{j,2})]}{G[\sqrt{t}(\bar{c}_{q,1}^{(t)} - \varphi_{q,2})]} = 1. \tag{3.69}$$

Then since

$$\lim_{t\to\infty} \frac{G[\sqrt{t}(\varphi_{j,1} - \bar{c}_{j,1}^{(t)})]}{G[\sqrt{t}(\varphi_{q,1} - \bar{c}_{q,1}^{(t)})]} = \lim_{t\to\infty} \frac{G[\sqrt{t}(\bar{c}_{j,1}^{(t)} - \varphi_{j,2})] + G[\sqrt{t}(\varphi_{j,2} - \bar{c}_{j,2}^{(t)})]}{G[\sqrt{t}(\bar{c}_{q,1}^{(t)} - \varphi_{q,2})] + G[\sqrt{t}(\varphi_{q,2} - \bar{c}_{q,2}^{(t)})]}, \tag{3.70}$$

by use of Lemma 3.6, we have

$$\lim_{t\to\infty} \frac{G[\sqrt{t}(\varphi_{j,2} - \bar{c}_{j,2}^{(t)})]}{G[\sqrt{t}(\varphi_{q,2} - \bar{c}_{q,2}^{(t)})]} = 1. \tag{3.71}$$

Continuing in this manner we finally reach

$$\lim_{t\to\infty} \frac{G[\sqrt{t}(\varphi_{j,n_j} - \bar{c}_{j,n_j}^{(t)})]}{G[\sqrt{t}(\varphi_{q,n_j} - \bar{c}_{q,n_j}^{(t)})]} = 1. \tag{3.72}$$

But since $(\varphi_{q,n_j} - \bar{c}_{q,n_j}^{(t)}) \to -\frac{1}{2}\gamma$ and for all t,

$$(\varphi_{j,n_j} - \bar{c}_{j,n_j}^{(t)}) = \tfrac{1}{2}(\varphi_{j,n_j} - \varphi_{j+1,1}) < -\tfrac{1}{2}\gamma,$$

(3.72) contradicts Lemma 3.1. In the case when $j = r$, the contradiction is obtained by replacing $\bar{c}_{r,n_r}^{(t)}$ by ∞. Thus we have shown that

$$\limsup_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{j,1} - \bar{c}_{j,1}^{(n)})]}{G[\sqrt{n}(\varphi_{q,1} - \bar{c}_{q,1}^{(n)})]} < 1. \tag{3.73}$$

But since the minimax risk for P_q is the maximum of the minimax risks for the sub-problems, we have

$$(3.74)\qquad \lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{q,1} - \bar{c}_{q,1}^{(n)})]}{r(\hat{\delta}_n)} = 1.$$

Now, in view of

$$(3.75)\qquad \lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_{j-1,n_{j-1}}^{(n)} - \varphi_{j,1})]}{r(\hat{\delta}_n)} = 0,$$

we get

$$(3.76)\qquad \limsup_{n\to\infty} \frac{r_{j,1}(\bar{\delta}_n)}{r(\hat{\delta}_n)} < 1.$$

The existence of this asymptotically subminimax decision procedure proves that the minimax procedure is asymptotically inadmissible.

It will be seen in the proof of Theorem 3.7 that $\{\bar{\delta}_n\}$ is also asymptotically admissible.

THEOREM 3.4. *If there are only two sets of means with the same number of means in each set, then all the components of the least favorable a priori distribution have positive lower limits, and the minimax procedure is asymptotically admissible.*

PROOF. Let $n_1 = n_2 = \bar{n}$. Since the minimax risks are equal, we have for all n,

$$(3.77)\qquad \varphi_{1,i} - \hat{c}_{1,i}^{(n)} = \hat{c}_{2,\bar{n}-i}^{(n)} - \varphi_{2,\bar{n}-i+1}, \qquad i \leqq \bar{n} - 1,$$

and finally

$$(3.78)\qquad \varphi_{1,\bar{n}} - \hat{c}_{1,\bar{n}}^{(n)} = \hat{c}_{1,\bar{n}}^{(n)} - \varphi_{2,1}.$$

Hence

$$(3.79)\qquad \hat{c}_{1,\bar{n}}^{(n)} = \tfrac{1}{2}(\varphi_{1,\bar{n}} + \varphi_{2,1}),$$

and from (3.7) we get that $\hat{g}_{1,\bar{n}}^{(n)} = \hat{g}_{2,1}^{(n)}$. It then follows from Lemmas 3.3 and 3.5 that all the components of $\hat{g}^{(n)}$ have lower limits greater than zero. Then by Theorem 2.2 the minimax procedure is asymptotically admissible.

THEOREM 3.5. *If all the sets have the same number of means but the means are not symmetrical with respect to the point, $\frac{1}{2}(\varphi_{1,1} + \varphi_{r,n_r})$, then all the $\hat{g}_{i,j}^{(n)}$ of at least one set have zero as their lower limits.*

PROOF. Let the number of means in each set be $\bar{n}$ and let $\gamma_i = \varphi_{i+1,1} - \varphi_{i,n_i}$. Let j be the smallest integer for which $\gamma_j \neq \gamma_{r-j}$. The hypothesis assures the existence of such a j. Since all the minimax risks are equal, we have

$$(3.80)\qquad \begin{aligned} \varphi_{1,1} - \hat{c}_{1,1}^{(n)} &= \hat{c}_{r,\bar{n}-1}^{(n)} - \varphi_{r,\bar{n}} \\ \varphi_{1,2} - \hat{c}_{1,2}^{(n)} &= \hat{c}_{r,\bar{n}-2}^{(n)} - \varphi_{r,\bar{n}-1} \\ &\vdots \\ \varphi_{j,\bar{n}} - \hat{c}_{j,\bar{n}}^{(n)} &= \hat{c}_{r-j,\bar{n}}^{(n)} - \varphi_{r-j+1,1}. \end{aligned}$$

Now in order for $\hat{g}_{j,\bar{n}}^{(n)}$, $\hat{g}_{j+1,1}^{(n)}$, $\hat{g}_{r-j,\bar{n}}^{(n)}$, and $\hat{g}_{r-j+1,1}^{(n)}$ to have lower limits greater than zero, Lemmas 3.1 and 3.3 require that

$$\lim_{n\to\infty} (\varphi_{j,\bar{n}} - \hat{c}_{j,\bar{n}}^{(n)}) = -\tfrac{1}{2}\gamma_j \tag{3.81}$$

and

$$\lim_{n\to\infty} (\hat{c}_{r-j,\bar{n}}^{(n)} - \varphi_{r-j+1,1}) = -\tfrac{1}{2}\gamma_{r-j}\,. \tag{3.82}$$

In view of the last line of (3.80), and since $\gamma_j \neq \gamma_{r-j}$, (3.81) and (3.82) can not both hold. Finally, since at least one component of $\hat{g}^{(n)}$, say $\hat{g}_{a,b}^{(n)}$ has zero for its lower limit, all the components of $\hat{g}^{(n)}$ corresponding to means of the ath set also have zero for their lower limits.

Although under the conditions of Theorem 3.5 the lower limits of some of the components of $\hat{g}^{(n)}$ are zero, nevertheless the following theorem shows that the minimax procedure is asymptotically admissible.

THEOREM 3.6. *If all the sets contain the same number of means, $\bar{n}$, then the minimax procedure is asymptotically admissible.*

PROOF. Consider the sequence of decision procedures, $\{\bar{\delta}_n\}$, defined as in the proof of Theorem 3.3. Since all the sets contain the same number of means, the $r_{i,j}(\bar{\delta}_n)$, $(1 \leqq i \leqq r,\ 1 < j < \bar{n})$ are equal for all n. Denote its value by $r(\bar{\delta}_n)$. We have

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{i,\bar{n}} - \bar{c}_{i,\bar{n}}^{(n)})]}{r(\bar{\delta}_n)} = 0, \tag{3.83}$$

and

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_{i,\bar{n}}^{(n)} - \varphi_{i+1,1})]}{r(\bar{\delta}_n)} = 0. \tag{3.84}$$

Hence

$$\lim_{n\to\infty} \frac{r_{i,j}(\bar{\delta}_n)}{r(\bar{\delta}_n)} = 1 \tag{3.85}$$

for all i, j, and consequently

$$\lim_{n\to\infty} \frac{r(\hat{\delta}_n)}{r(\bar{\delta}_n)} = 1. \tag{3.86}$$

Now suppose $\{\delta_n\}$ is an asymptotically subminimax procedure with

$$\limsup_{n\to\infty} \frac{r_{\alpha,\beta}(\delta_n)}{r(\hat{\delta}_n)} < 1, \tag{3.87}$$

where $r_{\alpha,\beta}(\delta_n)$ is the risk associated with the βth mean of the αth set. Consider the sub-problem, P_α, with $\Omega_\alpha = \{N(\varphi_{\alpha,j}, 1)\}$, $(j = 1, 2, \cdots, \bar{n})$. The minimax risk for P_α is $r(\bar{\delta}_n)$. In view of (3.86), the procedure derived from δ_n by using only the end points $c_{\alpha,j}^{(n)}$ $(j = 1, 2, \cdots, \bar{n} - 1)$ would be an asymptotically subminimax procedure for P_α. This is in contradiction to Theorem 3.1 which completes the proof.

By Theorems 3.2 and 3.3 we see that the minimax procedure is undesirable for large values of n when Ω consists of k normal distributions with a known common variance and different means, except for the special cases covered by Theorems 3.1 and 3.6.

In order to compare the minimax procedure with an asymptotically subminimax procedure for some specific values of n, consider the decision problem where $\Omega = \{N(0, 1), N(.2, 1), N(1, 1)\}$. It is easily seen that for $n = 100$, the minimax risk satisfies the inequality

$$G(-1) < r(\hat{\delta}_{100}) < G(-1) + G(-7). \tag{3.88}$$

Now consider the asymptotically subminimax procedure, $\{\delta_n\}$, given by $c_1^{(n)} = \hat{c}_1^{(n)}$, $c_2^{(n)} = \frac{1}{2}(\varphi_2 + \varphi_3) = .6$. For $n = 100$, we get

$$\begin{aligned} r_1(\delta_{100}) &= r(\hat{\delta}_{100}) \\ r_2(\delta_{100}) &< r(\hat{\delta}_{100}) + G(-4) \\ r_3(\delta_{100}) &= G(-4). \end{aligned} \tag{3.89}$$

Hence

$$\begin{aligned} r_1(\delta_{100})/r(\hat{\delta}_{100}) &= 1 \\ r_2(\delta_{100})/r(\hat{\delta}_{100}) &< 1 + G(-4)/G(-1) < 1.0002 \\ r_3(\delta_{100})/r(\hat{\delta}_{100}) &< G(-4)/G(-1) < .0002. \end{aligned} \tag{3.90}$$

Clearly δ_{100} is a more desirable decision procedure than the minimax procedure. Even for n as small as 25 it is similarly found that

$$\begin{aligned} r_1(\delta_{25})/r(\hat{\delta}_{25}) &= 1 \\ r_2(\delta_{25})/r(\hat{\delta}_{25}) &< 1.08 \\ r_3(\delta_{25})/r(\hat{\delta}_{25}) &< .08, \end{aligned} \tag{3.91}$$

which shows that the minimax procedure might be undesirable even for moderately small values of n.

Now that we have determined the asymptotic behavior of the minimax procedure for the class of problems for which $\Omega = \{N(\varphi_i, 1)\}$, and since we have seen that the minimax procedure is most frequently asymptotically inadmissible, the question naturally arises whether asymptotically admissible asymptotically subminimax procedures exist in those cases. It will be seen that such procedures do exist for some problems but not for others. For example, consider the problem where $\Omega = \{N(\varphi_1, 1), N(\varphi_1 + \gamma, 1), N(\varphi_1 + \gamma + \gamma_1, 1)\}$ with $\gamma_1 > \gamma > 0$. For this problem we know from Theorem 3.2 that the minimax procedure is asymptotically inadmissible, and if $\{\delta_n\}$ given by $\{c_1^{(n)}, c_2^{(n)}\}$ is an asymptotically subminimax procedure, then

$$\lim_{n\to\infty} \frac{r_1(\delta_n)}{r(\hat{\delta}_n)} = \lim_{n\to\infty} \frac{r_2(\delta_n)}{r(\hat{\delta}_n)} = 1, \tag{3.92}$$

and

$$\limsup_{n\to\infty} \frac{r_3(\delta_n)}{r(\hat{\delta}_n)} < 1. \tag{3.93}$$

It will now be shown that it is always possible to find another asymptotically subminimax procedure, $\{\bar{\delta}_n\}$, given by $\{\bar{c}_1^{(n)}, \bar{c}_2^{(n)}\}$ with

$$\lim_{n\to\infty} \frac{r_3(\bar{\delta}_n)}{r_3(\delta_n)} = 0. \tag{3.94}$$

Hence no asymptotically admissible asymptotically subminimax procedure exists for this problem.

For the procedure, δ_n, we have

$$\lim_{n\to\infty} \frac{r_1(\delta_n)}{r(\hat{\delta}_n)} = \lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_1 - c_1^{(n)})]}{G[\sqrt{n}(\varphi_1 - \hat{c}_1^{(n)})]} = 1. \tag{3.95}$$

Consequently, by Lemma 3.7, we have

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(c_1^{(n)} - \varphi_2)]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)]} = 1. \tag{3.96}$$

Also

$$\lim_{n\to\infty} \frac{r_2(\delta_n)}{r(\hat{\delta}_n)} = \lim_{n\to\infty} \frac{G[\sqrt{n}(c_1^{(n)} - \varphi_2)] + G[\sqrt{n}(\varphi_2 - c_2^{(n)})]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)] + G[\sqrt{n}(\varphi_2 - \hat{c}_2^{(n)})]} = 1, \tag{3.97}$$

and since $\hat{c}_1^{(n)} - \varphi_2 \to -\frac{1}{2}\gamma$ and $\varphi_2 - \hat{c}_2^{(n)} \to -\gamma_1 + \frac{1}{2}\gamma < -\frac{1}{2}\gamma$, we have

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_2 - \hat{c}_2^{(n)})]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)]} = 0. \tag{3.98}$$

By use of (3.96), (3.97), and (3.98), we get

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_2 - c_2^{(n)})]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)]} = 0. \tag{3.99}$$

Denoting $(\hat{c}_1^{(n)} - \varphi_2) - (\varphi_2 - c_2^{(n)})$ by d_n, (3.99) becomes

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2 - d_n)]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)]} = 0. \tag{3.100}$$

Now by use of (3.9) we have

$$\lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[(\hat{c}_1^{(n)} - \varphi_2 - d_n)^2 - (\hat{c}_1^{(n)} - \varphi_2)^2]\} \cdot \frac{\hat{c}_1^{(n)} - \varphi_2}{\hat{c}_1^{(n)} - \varphi_2 - d_n} = 0. \tag{3.101}$$

Since

$$\liminf_{n\to\infty} \frac{\hat{c}_1^{(n)} - \varphi_2}{\hat{c}_1^{(n)} - \varphi_2 - d_n} > 0, \tag{3.102}$$

we get finally

$$\lim_{n\to\infty} \exp\{-\tfrac{1}{2}n(-d_n)[2(\hat{c}_1^{(n)} - \varphi_2) - d_n]\} = 0. \tag{3.103}$$

But

$$\lim_{n\to\infty} (\hat{c}_1^{(n)} - \varphi_2) = -\tfrac{1}{2}\gamma \tag{3.104}$$

and

$$\liminf_{n\to\infty} d_n \geqq 0. \tag{3.105}$$

From (3.103) we then get

$$\lim_{n\to\infty} nd_n = \infty. \tag{3.106}$$

Now consider the sequence of procedures, $\{\bar{\delta}_n\}$, defined by

$$\bar{c}_1^{(n)} = c_1^{(n)}, \qquad \bar{c}_2^{(n)} = c_2^{(n)} - \tfrac{1}{2}d_n . \tag{3.107}$$

Clearly $r_1(\bar{\delta}_n) = r_1(\delta_n)$. Also

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_2(\bar{\delta}_n)}{r_2(\delta_n)} &= \lim_{n\to\infty} \frac{G[\sqrt{n}(c_1^{(n)} - \varphi_2)] + G[\sqrt{n}(\varphi_2 - c_2^{(n)} + \tfrac{1}{2}d_n)]}{G[\sqrt{n}(c_1^{(n)} - \varphi_2)] + G[\sqrt{n}(\varphi_2 - c_2^{(n)})]} \\
&= 1 + \lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_2 - c_2^{(n)} + \tfrac{1}{2}d_n)]}{G[\sqrt{n}(c_1^{(n)} - \varphi_2)]} = 1 + \lim_{n\to\infty} \frac{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2 - \tfrac{1}{2}d_n)]}{G[\sqrt{n}(\hat{c}_1^{(n)} - \varphi_2)]} \\
&= 1 + \lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[(\hat{c}_1^{(n)} - \varphi_2 - \tfrac{1}{2}d_n)^2 - (\hat{c}_1^{(n)} - \varphi_2)^2]\} \cdot \frac{\hat{c}_1^{(n)} - \varphi_2}{\hat{c}_1^{(n)} - \varphi_2 - \tfrac{1}{2}d_n} \\
&= 1 + \lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[-\tfrac{1}{2}d_n][2(\hat{c}_1^{(n)} - \varphi_2) - \tfrac{1}{2}d_n]\} \cdot \frac{\hat{c}_1^{(n)} - \varphi_2}{\hat{c}_1^{(n)} - \varphi_2 - \tfrac{1}{2}d_n} = 1,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_3(\bar{\delta}_n)}{r_3(\delta_n)} &= \lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_2^{(n)} - \varphi_3)]}{G[\sqrt{n}(c_2^{(n)} - \varphi_3)]} = \lim_{n\to\infty} \frac{G[\sqrt{n}(c_2^{(n)} - \varphi_3 - \tfrac{1}{2}d_n)]}{G[\sqrt{n}(c_2^{(n)} - \varphi_3)]} \\
&= \lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[(c_2^{(n)} - \varphi_3 - \tfrac{1}{2}d_n)^2 - (c_2^{(n)} - \varphi_3)^2]\} \cdot \frac{c_2^{(n)} - \varphi_3}{c_2^{(n)} - \varphi_3 - \tfrac{1}{2}d_n} \\
&= \lim_{n\to\infty} \exp\{-\tfrac{1}{2}n[-\tfrac{1}{2}d_n][2(c_2^{(n)} - \varphi_3) - \tfrac{1}{2}d_n]\} \cdot \frac{c_2^{(n)} - \varphi_3}{c_2^{(n)} - \varphi_3 - \tfrac{1}{2}d_n} = 0.
\end{aligned}$$

The existence of such a $\{\bar{\delta}_n\}$ shows that every asymptotically subminimax procedure for this problem is asymptotically inadmissible.

However, asymptotically admissible asymptotically subminimax procedures do exist for some problems. This will be brought out by the following theorem:

THEOREM 3.7. *If the means can be separated into two or more sets, each containing two or more consecutive means, with a common difference,* γ_i, *between the consecutive means of the* ith *set with* $\varphi_{i+1,1} - \varphi_{i,n_i} > \frac{1}{2}(\gamma_i + \gamma_{i+1})$ *for all* i, *and*

(a) *the* γ_i *are not all equal;*

or

(b) *all the* γ_i *are equal but all the sets do not contain the same number of means; then there exists an asymptotically subminimax procedure which is asymptotically admissible.*

PROOF. Consider the problems, $P_1, P_2, \cdots, P_r$ defined as in the proof of Theorem 3.3, and the decision procedure $\bar{\delta}_n$ defined there, except we shall now take

$$\bar{c}_{i,n_i}^{(n)} = \tfrac{1}{2}(\varphi_{i,n_i} + \varphi_{i+1,1}) + \tfrac{1}{4}(\gamma_i - \gamma_{i+1}), \tag{3.108}$$

for $i = 1, 2, \cdots, r - 1$.

Suppose (a) is satisfied, then since

$$\begin{aligned} \varphi_{i,n_i} - \bar{c}_{i,n_i}^{(n)} &< -\tfrac{1}{2}\gamma_i \\ \bar{c}_{i,n_i}^{(n)} - \varphi_{i+1,1} &< -\tfrac{1}{2}\gamma_{i+1}, \end{aligned} \tag{3.109}$$

we have for all i,

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{i,n_i} - \bar{c}_{i,n_i}^{(n)})]}{G[\sqrt{n}(\bar{c}_{i,n_i-1}^{(n)} - \varphi_{i,n_i})]} = 0 \tag{3.110}$$

and

$$\lim_{n\to\infty} \frac{G[\sqrt{n}(\bar{c}_{i,n_i}^{(n)} - \varphi_{i+1,1})]}{G[\sqrt{n}(\varphi_{i+1,1} - \bar{c}_{i+1,1}^{(n)})]} = 0. \tag{3.111}$$

Hence if we denote the maximum of the minimax risks for the r sub-problems by $r(\bar{\delta}_n)$, we have

$$\limsup_{n\to\infty} \frac{r_{i,j}(\bar{\delta}_n)}{r(\bar{\delta}_n)} \leqq 1. \tag{3.112}$$

But since $r(\bar{\delta}_n) < r(\hat{\delta}_n)$, we have

$$\limsup_{n\to\infty} \frac{r_{i,j}(\bar{\delta}_n)}{r(\hat{\delta}_n)} \leqq 1 \tag{3.113}$$

for all i, j.

Now denote $\min_i(\gamma_i)$ by γ_b. In case (a) there must exist a $\gamma_a > \gamma_b$, and by (3.111) we have

$$\lim_{n\to\infty} \frac{r_{a,1}(\bar{\delta}_n)}{r_{b,1}(\bar{\delta}_n)} = \lim_{n\to\infty} \frac{G[\sqrt{n}(\varphi_{a,1} - \bar{c}_{a,1}^{(n)})]}{G[\sqrt{n}(\varphi_{b,1} - \bar{c}_{b,1}^{(n)})]}. \tag{3.114}$$

But by Lemmas 3.1 and 3.4 it follows that the right member of (3.114) is zero which proves that $\{\bar{\delta}_n\}$ is asymptotically subminimax.

In the case when (b) is satisfied, $\{\bar{\delta}_n\}$ was shown to be asymptotically subminimax in the proof of Theorem 3.3.

In either case, suppose there exists another sequence of decision procedures, $\{\bar{\bar{\delta}}_n\}$, given by $\{\bar{\bar{c}}_{i,j}^{(n)}\}$ such that

$$\limsup_{n\to\infty} \frac{r_{i,j}(\bar{\bar{\delta}}_n)}{r_{i,j}(\bar{\delta}_n)} \leqq 1 \tag{3.115}$$

for all i, j, and the strict inequality holds for at least one set of values of i, j, say α, β. Then the procedure derived from $\bar{\bar{\delta}}_n$ by using only the end points $\bar{\bar{c}}_{\alpha,j}^{(n)}$, $(j = 1, 2, \cdots, n_\alpha - 1)$, would be an asymptotically subminimax procedure for P_α, contrary to Theorem 3.1.

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