CORRECTION TO "ON THE MAXIMUM NUMBER OF CONSTRAINTS OF AN ORTHOGONAL ARRAY"

By Esther Seiden

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The proof of Lemma 2 of the paper mentioned in the above title (*Ann. Math. Stat.* Vol. 26 (1955), pp. 132–135) is incorrect. The number 20 on top of page 134 should be replaced by 15 and hence no contradiction has been reached with $n_{12}^6 = 45$. Fortunately the assertion made in the above mentioned remains valid. The last seven lines of page 133 and the first two lines of page 134 should be deleted and replaced by the following:

This means that every 4-rowed orthogonal subarray must satisfy the equality $n_{14}^4 = 1$, contrary to Lemma 1 of the paper "Further remark on the maximum number of constraints of an orthogonal array" (to appear in the December issue, *Ann. Math. Stat.* Vol. 26 (1955), which asserts that no such array exists.

I wish to thank W. S. Connor for pointing out the mistake in my former proof.

ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the New York meeting of the Institute, December 27–30, 1955*)

1. The Midrange of a Sample as an Estimator of the Population Midrange, Paul R. Rider, Wright-Patterson Air Force Base.

A study is made of the distribution of the midranges of samples from five different symmetric populations of limited range, and of the relative efficiency of midrange and mean in estimating the population midrange, or mean, or median. It is found that the midrange is more efficient than the mean for all of the populations considered, and that this efficiency increases as the standardized fourth moment decreases.

2. Distribution of the Product of Maximum Values in Samples from a Rectangular Population, Paul R. Rider, Wright-Patterson Air Force Base, (By Title).

The distribution of the product of maximum values in samples from a rectangular distribution is derived. Results are obtained for the case of two samples of different sizes and for k samples of the same size.

3. A Note on Non-Recurrent Random Walks, Cyrus Derman, Columbia University, (By Title).

Let $\{X_i\}$, $i = 1, \cdots$, be a sequence of independent and identically distributed random variables with density function $f(x)$ and $EX_i = \lambda > 0$. Let $\{S_n\}$, $n = 1, \cdots$, be the sequence of cumulative sums $S_n = \sum_{i=1}^{n} X_i$, $H(x) = \sum_{n=1}^{\infty} P(S_n < x)$, and $h(x) = H'(x)$. Let A be any Borel set of the positive real numbers and $m(A)$ denote its Lebesgue measure.

Chung and Derman (to appear in the *Pacific J. Math.*) proved that (I) $\Phi(A) = P(S_n \varepsilon A$ infinitely often) $= 0$ if $m(A) < \infty$, and (II) that $\Phi(A) = 1$ if $m(A) = \infty$, provided that as $x \to \infty$, $0 < \liminf h(x) \leqq \limsup h(x) < \infty$. The following theorem was proved: (i) If $\lambda < \infty$ and $\limsup h(x) < \infty$, then $\liminf h(x) > 0$ and consequently (I) and (II) hold. (ii) If $\liminf h(x) > 0$, then (I) holds. (iii) If $\lambda < \infty$, and if there exists a constant $\alpha > 0$ and an interval (a, b) such that $f(x) \geqq \alpha$ for $x \varepsilon (a, b)$, then $\liminf h(x) > 0$.

4. Statistical Spectral Analysis, I: Consistent Asymptotically Normal Estimates of the Covariance Function and Spectral Averages, EMANUEL PARZEN, Columbia University, (By Title).

Let the wide-sense stationary time series $x(t)$ have mean m, covariance $R(v) = E\,x(t)\,x(t + v) - m^2$, spectral distribution function $F(\omega)$ such that $R(v) = \int e^{iv\omega}\,dF(\omega)$, and spectral density function $f(\omega) = F'(\omega)$. The problem of statistical spectral analysis is to estimate these quantities on the basis of an observed sample. We shall be especially concerned with finding consistent and asymptotically normal estimators, for both continuous and discrete parameter processes, under the following *assumptions*: (1) $R(v)$ is absolutely and square summable; (2) the process $y(t) = x(t) - m$ is stationary of order 4; (3) the non-Gaussian part of the fourth moment, or the fourth cumulant, $Q(v_1, v_2, v_3) = E\,y(t)\,y(t + v_1)\,y(t + v_2)\,y(t + v_3) - R(v_1)\,R(v_2 - v_3) - R(v_2)\,R(v_3 - v_1) - R(v_3)\,R(v_1 - v_2)$ is absolutely summable; (4) there is an absolutely integrable function $g(\omega_1, \omega_2, \omega_3)$ such that $Q(v_1, v_2, v_3) = \iiint d\omega_1\,d\omega_2\,d\omega_3 \exp i\,[\omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3]\,g(\omega_1, \omega_2, \omega_3)$. Examples are given of processes satisfying these assumptions; they are examples of multilinear processes. Given $x(t)$, for $0 \leqq t \leqq T$ (or for $t = 1, \cdots, T$ in the discrete case), define the sample mean m_T, the sample covariance $R_T(v)$, the sample spectral density (or periodogram) $f_T(\omega)$, and sample spectral averages $J_T(A)$ by $Tm_T = \int_0^T x(t)\,dt$, $TR_T(v) = \int_0^{T-|v|} [x(t) - m_T]\,[x(t + |v|) - m_T]$, $R_T(v) = \int e^{i\omega v} f_T(\omega)\,d\omega$, $J_T(A) = \int A(\omega)\,f_T(\omega)\,d\omega$ for suitable $A(\omega)$. Expressions are obtained for the limit, as $T \to \infty$, of $TE(m_T - m)^2$, $TE\,|\,R_T(v) - R(v)\,|^2$, and $E\,|\,J_T(A) - J(A)\,|^2$, where $J(A) = \int A(\omega)\,f(\omega)\,d\omega$.

5. Statistical Spectral Analysis, II: Asymptotic Mean Square Error of a Class of Estimates of the Spectral Density, EMANUEL PARZEN, Columbia University.

It is well known that the sample spectral density (or periodogram) $f_T(\omega)$ is not a consistent estimate of the spectral density. A class of consistent estimates may be found in the following way (where we write out the formulas only for the discrete parameter case, noting that similar statements hold for the continuous case). Define $f_T^*(\omega) = (\frac{1}{2}\pi) \times \Sigma_{|v| \leqq T}\, e^{-iv\omega}\,k(B_T v)\,R_T(v)$, where B_T is a sequence of constants such that $B_T \to 0$ and $TB_T \to \infty$ as $T \to \infty$, and the kernel $k(u)$ is defined for all real u as an even, bounded, square integrable function, with Fourier transform $K(\omega)$. It is assumed to satisfy $k(0) = 1$, and $\int_{-T}^{T} |\,k(u)\,|\,du \leqq MT^{\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$ and constant M. Various estimates that have been proposed for the spectral density (Bartlett, Daniell, Grenander, Tukey) may be regarded as special instances of $f_T^*(\omega)$. To study the properties of $f_T^*(\omega)$, and to form a theory of the optimum estimate of this form, we need to know the mean square error $E \mid f_T^*(\omega) - f(\omega) \mid^2$. An asymptotic expression for it can be obtained from the following two theorems.

Theorem I: $TB_T\sigma^2[f_T^*(\omega)] \to \mid f(\omega) \mid^2 \int k^2(u)\, du\ \{1 + \delta(0, \omega)\}$ *where* $\delta(0, \omega) = 1$ or 0 according as $\omega = 0$ or $\neq 0$.

Theorem II: *Let* $r > 0$ *be such that* $\Sigma \mid v \mid^r \mid R(v) \mid < \infty$, *and* $k^{(r)} = \lim_{u \to 0} (1 - k(u))/ \mid u \mid^r$ *is finite. Then* $B_T^{-2r} \mid Ef_T^*(\omega) - f(\omega) \mid^2 \to \mid k^{(r)} f^{(r)}(\omega) \mid^2$ *where* $f^{(r)}(\omega) = (\frac{1}{2}\pi)\Sigma e^{-iv\omega} \mid v \mid^r R(v)$.

6. A Central Limit Theorem for Multilinear Stochastic Processes, Emanuel Parzen. Columbia University, (By Title).

A definition of a multilinear process is given. Intuitively, a stochastic process $x(t)$, defined for all real t, is said to be multilinear if it arises from a process with independent increments by means of passage through a finite bank of "linear filters" and "polynomial law instantaneous devices". Many physically observed stochastic processes may be assumed to arise in this way. Let $S_T = \int_0^T x(t)\, dK(t)$ and $S_N' = \Sigma_n^N x(t_{n-1})\, a_n$, for some sequence of points $t_n \to \infty$, constants a_n, and weighting function $K(t)$. Conditions are given in terms of moments, in order that the normalized random variables $(S_T - ES_T)/\sigma S_T$ and $(S_N' - ES_N')/\sigma S_N'$ tend in distribution to a normal law with zero mean and unit variance.

7. An Extension of Cramér's Theorem 20.6 to Random Functions with Values in a Metric Space, Emanuel Parzen, Columbia University, (By Title).

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $(R, \rho; \mathcal{B})$ be a metric measurable space, by which we mean that R is metrized by ρ, and $\mathcal{B}$ is the minimal σ-field over the open sets in $\mathcal{B}$. A random function X on Ω to R is $\mathcal{A}$-measurable if $\mathcal{A}$ contains the inverse image under X of every set in $\mathcal{B}$; then X generates a probability P_X on $\mathcal{B}$. For a sequence of $\mathcal{A}$-measurable functions X_n, define X_n to converge in distribution to X (denoted $X_n \xrightarrow{D} X$) if, for every bounded real-valued $\mathcal{B}$-measurable function $f(x)$ on R whose set of discontinuities is of P_x-measure 0, (*) $\int f(X_n)\, dP \to \int f(X)\, dP$. By a result of P. P. Billingsley (Ph.D. thesis, Princeton, 1955; Th. 1.1), $X_n \xrightarrow{D} X$ if, and only if, (*) holds for every bounded uniformly continuous function on R. Note also that, if X is a constant, then $X_n \xrightarrow{D} X$ if, and only if, $\rho\,(X_n, X) \to 0$ in probability. Extension of Theorem 20.6 in Cramér (*Mathematical Methods of Statistics*, Princeton, 1946): Let X_n, X be $\mathcal{A}$-measurable functions to $(R_1, \rho_1; \mathcal{B}_1)$ and let Y_n, Y be $\mathcal{A}$-measurable function to $(R_2, \rho_2; \mathcal{B}_2)$. Let R be the Cartesian product of R_1 and R_2, and let ρ be a metric on R which agrees with the metrics ρ_1 and p_2, and such that (X_n, Y_n), (X, Y) are $\mathcal{A}$-measurable to $(R, \rho; \mathcal{B})$. Suppose $X_n \xrightarrow{D} X$, $Y_n \xrightarrow{D} Y$, and Y is a constant. Then $(X_n, Y_n) \xrightarrow{D} (X, Y)$. *Proof.* It suffices to show $\int f(X_n, Y_n)\, dP \to \int f(X, Y)\, dP$ for any bounded uniformly continuous function $f(x, y)$ on R. Clearly, $\int f(X_n, Y)\, dP \to \int f(X, Y)\, dP$. Let $\delta(\epsilon)$ be such that $\mid f(X_n, Y_n) - f(X_n, Y) \mid < \epsilon$ for $\rho_2(Y_n, Y) < \delta(\epsilon)$. Since $\int \mid f(X_n, Y_n) - f(X_n, Y) \mid dP \leqq \epsilon + P\{\rho_2(Y_n, Y) > \delta(\epsilon)\}$, the desired conclusion may now be inferred.

8. Orthogonality and Fractional Replication of Factorial Experiments, ALLAN BIRNBAUM, Columbia University.

A simple characterization of orthogonal factorial designs is derived from the condition for orthogonality of appropriate vector subspaces of the sample space. This leads naturally to: (a) the definitions of various classes of orthogonal designs, some of them standard (e.g., Latin squares) and some less familiar (e.g., "Latin rectangles"); (b) some lower bounds on the fraction of replication which is consistent with orthogonality; (c) some elementary methods of construction of orthogonal fractional replicates, which in some cases can be shown to consist of a smallest possible fractional. Examples of such fractional replicates, including cases of factors at unequal numbers of levels, are given.

9. On the Second Sample Size Function of a Bayes Two-Stage Test for the Mean, MORRIS SKIBINSKY, Purdue University.

This paper investigates in detail the second sample size function of a Bayes two-stage rule that decides between two possible values for the mean of a normal distribution which has unit variance. The second sample size is the greatest integer less than or equal to a number, $\hat{y}(\text{const.} \times \log (r_m/Wg), W, M)$, where $\hat{y}(t, \gamma, \mu)$ is, for fixed values of its arguments, a value of y for which a certain function, $U(y, t, \gamma, \mu)$, is absolutely minimum; m is the size of the first sample, r_m the value of the probability ratio from the first sample; W and g the ratios, respectively, of the simple wrong decision losses and the a priori probabilities associated with the two possible means; and M is the minimum wrong decision loss. Certain monotonicity, symmetry, and continuity properties of $\hat{y}$, and functions related to it, are proved, and an asymptotic expression for the function is found when the minimum wrong decision loss is large. A subsequent paper, continuing this investigation, will consider Bayes two-stage rules having optimum properties with respect to expected overall sample size among rules of the same power.

10. A New Estimation Procedure for a Linear Combination of Exponentials, (Preliminary Report), RICHARD G. CORNELL, Oak Ridge National Laboratory and Virginia Polytechnic Institute.

A new estimation procedure is developed for the parameters of the model $y_{ij} = \alpha_1 e^{-\lambda_1 t_i} + \alpha_2 e^{-\lambda_2 t_i} + \cdots + \alpha_p e^{-\lambda_p t_i} + e_{ij}$. The errors e_{ij} are independently and normally distributed about mean zero with equal variances. The parameters λ_k are restricted to be positive and the observation points t_i are equally spaced. The number of points at which observations are taken is specified to be an integral multiple of the number of parameters. Also, equal numbers of observations are required at each observation point. Estimates are obtained by forming as many independent sums from the observations y_{ij} as there are parameters, equating these sums to their expectations, and solving for estimates of the parameters. A computationally simple, non-iterative solution is found. The resultant estimators are not only asymptotically normally distributed, but are also consistent, sufficient and asymptotically efficient. The limiting properties are demonstrated as either the number of observation points or the number of observations per observation point grows infinitely large.

11. A Note on Weighted Randomization, D. R. COX, University of North Carolina.

The standard methods of randomization used in experimental design consist of selecting an arrangement at random from a set S of similar arrangements, giving each arrangement in the set equal chance of selection. As is well known this device makes the standard designs unbiased in the sense that, under weak assumptions, the randomization expectations of linear and quadratic functions of the observations agree with their values as calculated from an appropriate linear model with random residuals. It is also known that these results do not hold when an adjustment for concomitant variation is made by analysis of covariance. In the present paper it is pointed out that a randomization justification for the covariance procedure can be provided if weighted randomization is used, i.e. if the arrangement for use is selected giving different arrangements in the set S appropriate unequal chances of selection. Possible applications are considered briefly.

12. **On the Analysis of Incomplete Block Designs,** MARVIN ZELEN, National Bureau of Standards.

Let there be 2v normal populations which can be divided into two sets of v populations each, such that the unknown parameters of each set are (μ_i, σ_1^2) and (μ_i, σ_2^2) $i = 1, 2, \cdots, v$. Consider the null hypothesis $H_0: (\mu_i = 0 \text{ for } i = 1, 2, \cdots, v)$ against the alternative hypothesis $H_1: (\mu_i \neq 0 \text{ for } i = 1, 2, \cdots, v)$. If a sample of size r_j is made for each of the v populations of the jth set $(j = 1, 2)$, then one can test H_0 using two independent F-ratios. The main problem is to combine the two independent tests of significance into one single test having (perhaps) greater power than either of the individual tests. This problem arises in the analysis of incomplete block designs where one set corresponds to the intra-block analysis and the other to the inter-block analysis. The object of this paper is to show that exact statistical tests do exist for combining intra- and inter-block information. Methods are discussed for combining the two tests and a comparison of the power function is made for particular numerical values of the alternative hypothesis.

13. **A Remark to Wald's Paper: "On a Statistical Problem Arising in the Classification of an Individual into One of Two Groups,"** JUNJIRO OGAWA, University of North Carolina.

In the paper above mentioned, the late Professor A. Wald proposed a statistic U for the use in classification procedure and considered its exact sampling distribution. His result is too complicated to describe here. His proof was divided into nine lemmas, and each lemma was proved by an ingenious method. But, at least in the author's opinion, the proofs of his sixth and seventh lemmas can be improved with the help of the invariant measure defined on the Grassmann manifold, which consists of p-planes in $(n + 2)$-dimensional Euclidean space. The author presents new proofs of these lemmas as an example of applications of the theory of orthogonal group manifolds developed by A. T. James in 1954 (*Ann. Math. Stat.*, Vol. 25).

14. **Consistency and Optimum Properties of Some Two-Sample Tests,** JULIUS R. BLUM, Indiana University, and LIONEL WEISS, University of Virginia.

Let $X_1, \cdots, X_m$ be a sample from the uniform distribution on the unit interval and let $Y_1, \cdots, Y_n$ be a sample with density $g(y)$ on the unit interval. Let $Z_0 = 0$, Z_{n+1}, and $Z_1 < \cdots < Z_n$ be the order statistics corresponding to $Y_1, \cdots, Y_n$. For each $i = 1, \cdots, n + 1$ let S_i be the number of X's in the interval $[Z_{i-1}, Z_i]$, and for each non-negative integer r, let $Q_n(r)$ be the proportion among $S_1, \cdots, S_{n+1}$ which are equal to r. Let $\alpha = m/n$, and for each r, let $Q(r) = \alpha^r \int_0^1 \{g^2(y)/[\alpha + g(y)]^{r+1}\}\, dy$. Then it is shown that under mild restrictions on $g(y)$ we have $P\{\lim_{n\to\infty} \sup_{r\geq 0} \mid Q_n(r) - Q(r) \mid = 0\} = 1$. This is applied to prove consistency of certain two-sample tests such as the Wald-Wolfowitz run test (*Ann. Math. Stat.*, Vol. 11 (1940), pp. 147–162). One of these tests is shown to have a further desirable property.

15. Remarks on Characteristic Functions, Eugene Lukacs, The Catholic University of America and The Office of Naval Research.

Let $F(x)$ be a distribution function and denote by $\phi(t)$ its characteristic function (Fourier transform). Functions of characteristic functions are studied which are themselves characteristic functions. The following theorem is established: Let $\phi(t)$ be a characteristic function and let $G(z)$ be a function of the complex variable z which is analytic in $|z| < R$, where $R > 1$. The function $G[\phi(t)]$ is also a characteristic function if, and only if, $G(z)$ has a power series expansion about the origin with non-negative coefficients and if $G(1) = 1$. The class of functions $G(z)$ which have the property that $G[\phi(t)]$ is a characteristic function whenever $\phi(t)$ is a characteristic function includes also functions which are not analytic, for example the function $|z|^2$. By means of the theorem, one obtains also the following result: Let $\phi(t)$ be an arbitrary characteristic function and p be a real number such that $p > 1$; then, $(p - 1)/[p - \phi(t)]$ is the characteristic function of an infinitely divisible distribution.

16. The Limiting Distribution of the Serial Correlation Coefficient in the Explosive Case, John S. White, University of Manitoba.

An auto-regressive process satisfying the stochastic difference equation $x_t = \alpha x_{t-1} + u_t$, $(t = 1, 2, \cdots)$, where the u's are independent identically distributed random variables, x_0 is a constant, and α is an unknown parameter, is said to be explosive if $|\alpha| \geqq 1$. If the u's are normally distributed with mean zero, it is shown that the maximum likelihood estimator for α has an asymptotic Cauchy distribution when $|\alpha| > 1$. For $|\alpha| = 1$, a characteristic function is obtained for the limiting distribution. For $\alpha = 1$, it is also shown that the limiting distribution of the maximum likelihood estimator for α is the distribution of a certain functional of a Wiener process.

17. The Distribution of the Ratio of Two Measures of Normal Dispersion, H. O. Hartley, Iowa State College.

Let us denote by x_i $(i = 1, 2, \cdots, n)$ a random sample of n items from $N(0, 1)$ and by $\bar{x}$ and s^2 the sample mean and variance, i.e., $\bar{x} = n^{-1}\Sigma x_i$; $s^2 = (n-1)^{-1}\Sigma(x_i - \bar{x})^2$. Consider now, the measure of dispersion $\phi = \phi(x_1 - \bar{x}, \cdots, x_n - \bar{x})$, where ϕ is a 1st order homogeneous function of its arguments $x_i - \bar{x}$, and finally $u = \phi/s$. Special cases of such a ratio which have been considered in the literature are: (a) ϕ = range $= x_{\max} - x_{\min}$ (David, Hartley and Pearson) *Biometrika*, 41, 482; (b) $\phi = x_{\max} - \bar{x}$ (Pearson and Chandrasekar) *Biometrika* 28, 308; (c) $\phi = 1/n\Sigma\,|x_i - \bar{x}|$ (Geary) *Biometrika* 27, 310, 353. Here we develop a general distribution theory for the ratio u based on a fundamental integral equation: Introducing the probability integrals $F(U) = \Pr\{1/u \leqq U\}$; $G(U) = \Pr\{1/\phi \leqq U\}$ it follows from the independence of u and s that $G(U) = \int_0^\infty F(Us)\, f_\nu(s)\, ds$ where $f_\nu(s)$ is the ordinate distribution of s based on $\nu = n - 1$ degrees of freedom. Given the known integral $G(U)$ equation (1) is an integral equation Fredholm 1st kind for the unknown integral $F(U)$. Various methods of solving this equation are discussed and applied to cases (a) and (b), supplying answers unobtainable by the methods hitherto employed. Also, (d) $\phi = (\Sigma_{i=1}^{n-1}(x_i - x_{i+1})^2/(n-1))^{\frac{1}{2}}$ (von Neumann; *Ann. Math Stat.* 12, 367); (e) $\phi = \Sigma_{i=1}^{n-1}|x_i - x_{i+1}|/(n-1)$ (Kamat, *Biometrika*, 40, 116).

18. Estimating a Linear Functional Relation, H. Fairfield Smith, North Carolina State College.

The problem considered is to estimate a theoretical line $\xi_1 \cos \beta - \xi_2 \sin \beta - P = 0$ from paired observations $x_{1i}x_{2i}$ which are assumed to be random vectors from bivariate normal distributions around arbitrary centers $\xi_{1i}\ \xi_{2i}$. Most attempts to fit such a functional relation to observations with errors in both variates introduce the ξ_{pi} as what Neyman and Scott (1947) called "incidental parameters." These bring troubles to both least squares and maximum likelihood formulations. Attention is focused on the condition that the only ascertainable quantities from which a solution must be deduced are deviations of observations from the line in some prescribed direction. These have univariate distributions whose expectations in general deviate from the line by amounts proportional to their distances from the respective ξ_{pi}. But when, and only when, the deviations are measured in one particular direction their expectations are zero *independently of* ξ_{pi}. By utilizing this condition, and only thus, the incidental variables can be eliminated from the problem. The probability of a sample can then be expressed in terms of univariate normal distributions *about the line*, and maximum likelihood may be applied free of incidental variables. Kummell's solution is then seen to be unique and efficient. The estimator of the angle β is unbiased and its asymptotic variance may be evaluated. With certain supplementary conditions the exact sampling distribution has been obtained. (Supported by the Office of Ordnance Research).

19. Asymptotic Distributions of Roots of Certain Determinantal Equations, R. Gnanadesikan, University of North Carolina.

The tests obtained by Roy for the hypotheses: (i) $\xi_1 = \xi_2 = \cdots = \xi_k$, i.e., $\Sigma^* = 0$ where Σ^* is the "between" covariance matrix, and (ii) $\Sigma_{12}(p \times q) = 0$ where Σ_{12} is the covariance matrix between a p-set and a q-set ($p \leqq q$), on multivariate normal populations depend on the largest characteristic roots of (i) S^*S^{-1} where S^* and S are the sample "between" and "within" dispersion matrices respectively; and (ii) $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where $S_{11}(p \times p)$ $S_{22}(q \times q)$ are sample covariance matrices of the p-set and the q-set respectively, and $S_{12}(p \times q)$ is the sample covariance matrix between the p-set and the q-set. The exact c.d.f. of this largest root has been obtained by Roy. For large sample sizes the problem becomes identical with that of finding the c.d.f. of the largest characteristic root of the sample dispersion matrix for a sample from one multivariate normal population. This limiting distribution has been obtained by Nanda for two particular cases, but there exists no explicit and general method of obtaining it. This has been done now. Also considering the test of, $\Sigma = \Sigma_0 = I(p)$ in particular, on one p-variate normal population we get Roy's test depending on the largest and the smallest characteristic roots of $S(p \times p)$, the sample covariance matrix. The joint c.d.f. of the largest and smallest roots has been obtained. Explicit expressions for some particular cases have also been obtained.

20. Investigation of the Possibility of Using Likelihood Ratio Tests of Certain Multivariate Hypotheses, for Obtaining Confidence Bounds, R. Gnanadesikan, University of North Carolina, (By Title).

The likelihood ratio tests considered are of the following composite hypotheses on one or more multivariate normal populations $N[\xi(p \times 1), \Sigma(p \times p)]$: (i) $H_0: \Sigma = I(p)$ (one population), (ii) $H_0: \Sigma_1 = \Sigma_2$ (two populations), (iii) $H_0: \xi_1 = \cdots = \xi_k$ (analysis of variance of mean vectors for k populations) and (iv) $H_0: \Sigma_{12}(p \times q)(p \leqq q) = 0$ (where Σ_{12} is the covariance matrix between a p-set and a q-set), the alternative in each case being $H \neq H_0$. One wants in each case confidence bounds (in terms of the observations) on meaningful parametric functions which, as it were, would measure departures from the null hypothesis, such functions being (for the different cases) the respective characteristic roots of (i) Σ, (ii) $\Sigma_1\Sigma_2^{-1}$, (iii) $\Sigma^*\Sigma^{-1}$ (where Σ^* is the "between" and Σ the "within" dispersion matrix of the k populations) and (iv) $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$(where Σ_{11} and Σ_{22} are the dispersion matrices of the p-set and the q-set and Σ_{12} has been already defined). While the confidence bounds on these parametric functions are already available if one starts from other tests of these hypotheses and then inverts, it is found that if one starts from the likelihood ratio tests and then tries to invert, the problem of separation of the parametric functions from the observations becomes quite difficult.

21. Asymptotic Efficiencies of a Nonparametric Life Test for Investigating Smaller Percentiles of a Gamma Distribution, JOHN E. WALSH, Lockheed Aircraft Corporation, (By Title).

In many life testing situations the quantity of interest is a specified smaller percentage point of the statistical population investigated. For example, a substantial loss may be incurred if more than a specified small percentage of the items of the population have the property of failing too soon. This paper considers some well known nonparametric tests of the sign test type and investigates their properties when applied to smaller percentage points for the case of a sample from a gamma probability distribution. Asymptotically, the nonparametric results are found to be highly efficient compared to the "best" parametric results based on the same fraction of items failed for the case of gamma distributions. Intuitive reasoning indicates that this high efficiency property holds for any reasonable type of statistical population and any sample size. Appropriate use of these nonparametric tests and estimates sometimes can yield a saving in cost and/or time without loss of statistical efficiency since the experiment can be stopped when only a fraction of the items being life tested are failed.

22. A Test of Judge Concordance for Paired Comparison Designs, (Preliminary Report), J. W. WILKINSON, University of North Carolina.

In a recent paper, (to appear in *Biometrika*), R. C. Bose has given several highly symmetrical designs where each of v judges compares r pairs of n objects, $(1 < r \leqq n(n-1)/2)$, and each pair is compared by k judges, $(1 < k \leqq v)$. To obtain a test of judge concordance for these designs, a pseudo-preference matrix $P_i = (p_{st}^i)$, $(s, t = 1, 2, \cdots, n)$, is constructed for each judge i, $(i = 1, 2, \cdots, v)$, where $p_{st}^i = 1$ or 0 according as a_s is, or is not, preferred to a_t. The diagonal cells, and cells p_{st}^i and p_{ts}^i when a_s and a_t are not compared by judge i, are left blank. A statistic Σ' is defined as $\Sigma' = \Sigma\gamma_{ij}(\gamma_{ij} - 1)/2$, where summation is extended over the non-diagonal cells of $P = \Sigma_{i=1}^v P_i$, and where γ_{ij} is the entry in cell (i, j) of P. The distribution of Σ' under the hypothesis that the preferences are allotted at random has been obtained, and has been tabulated for most of the known designs. Calculation of the first few moments of Σ' would indicate that a linear function of it is a χ^2 for large n and k. For $r = n(n-1)/2$, Σ' and its distribution are identical with those obtained by Kendall and B. Smith (*Biometrika*, Vol. 31, 1940).

23. On the Efficiency of Certain Classes of Tests Based on U-Statistics, JOAN RAUP ROSENBLATT, National Bureau of Standards.

A class of non-parametric decision problems is characterized by partition of a set of possible probability distributions into sets defined by the value of a functional. Particular attention is given to a class of functionals of the form $E\phi(X; F)$, where X is a vector random variable with distribution $F(x)$, and especially to the subclass in which the function $\phi(x)$ takes only the values zero and one. Certain families of tests are considered, which are based on functions of observed values of X which depend on these values only through the function $\phi(x)$. One such family is that based on the U-statistic corresponding to the functional $E\phi(X)$.

Methods are developed for computing an asymptotic expression for an index of efficiency for these families of tests relative to decision problems stated in terms of values of $E\phi(X; F)$. These methods are applied in particular to comparison of a family of two-sample tests based on the Wilcoxon-Mann-Whitney statistic with a family of tests based on a related statistic which has binomial distribution. Additional examples are given. (Work done at the Univ. of No. Carolina, with the support of the U. S. Air Force.)

24. The Dynamic Statistical Decision Problem when the Component Problem Involves a Finite Number, *m*, of Distributions, JAMES F. HANNAN, Michigan State University.

The dynamic problem consists in a sequence of N statistical decision problems with identical formal structure. Decisions are made successively within components and the risk of a sequence decision function is taken to be the average of the risks incurred in the components. An earlier paper by Hannan and Gaddum (submitted for an Annals of Mathematics Study) considered a sequence of formally identical two person S-games under the assumptions: (i) II's risk points in the component game form a closed and convex subset of the unit m cube, (ii) II's choice of strategy in each component can depend on the e.d. (empirical distribution) of I's moves in prior components. The principal theorem of that paper exhibited a usable sequence strategy (not depending on N) whose average risk across the N games exceeds by less than $(6m/N)^{\frac{1}{2}}$ the single-game minorant risk against the e.d. of I's N moves. The present paper is concerned with the substitution of estimates for the successive e.d. in (ii). If mixtures of the m distribution have unique representation, there exists a bounded vector kernel for unbiased estimates of the e.d. and the sequence decision function obtained by the substitution of unbiased estimates of this type satisfies the analogous theorem with the bound on the excess increased by multiplication by a bound of the kernel.

25. On Certain Systems of Experiments as Interdependent Stochastic Processes, (Preliminary Report), DAVID ROSENBLATT, American University, (By Title).

In certain systems of experiments, one may regard behavioral interaction between a "responsive" generalized subject and an experimenter in terms of interdependent discrete time-parameter stochastic processes. The subject (2) engages in actions or decisions A_{2k}; the experimenter (1), on the basis of an estimated conditional distribution of subject's actions, produces stimuli or treatments A_{1k}, where A_{ik} takes one of the values $a_{i1}, \cdots, a_{il}$, $i = 1, 2$, $k = 1, 2, \cdots$; subject and experimenter act alternately. The ith entity ($i = 1, 2$) possesses (a) a probability distribution R_{i0} over a finite set of m_i configuration states (threshold or preference configurations), i.e., a point in the m_i-dimensional simplex; (b) parameter stochastic operators ($m_i \times m_i$ matrices, row sum unity) Π_{ij}, $j = 1, \cdots, l$; (c) parameter mapping Γ_i or behavior function (matrix r.s.u.) which takes R_{ik} into the conditional distribution G_{ik} of A_{ik}, G_{ik} assuming values in the l-dimensional simplex. Let $f_j(A_{ik}) = 1$ if $A_{ik} = a_{ij}$, $= 0$ otherwise. Consider the system: (i) $R_{11} = R_{10}$; (ii) $R_{1,k+1} = R_{1k}(\sum_{j=1}^{l} f_j(A_{2k})\Pi_{1j})$; (iii) $R_{2k} = R_{2,k-1}(\sum_{j=1}^{l} f_j(A_{1k})\Pi_{2j})$; (iv) $G_{ik} = R_{ik}\Gamma_i$; $i = 1, 2$, $k = 1, 2, \cdots$, where each relation holds with probability one. The "conjoint process"

$E_k = (A_{2k}, R_{2k}, A_{1k}, R_{1k})$, $k = 2, 3, \cdots$, is a Markov process with stationary transition function. Conditions are adduced under which $\lim (G_{1k} - G_{2k}) = 0$, the null vector, with probability one, where G_{1k} is the experimenter's "estimate" of the "labile" conditional distribution of the subject's actions, G_{2k}. This model is constructed within the formal system OFK presented at the August, 1955, meeting of IMS (forthcoming abstract in *Econometrica*).

26. A Spherically-Symmetric Order Statistic r, (Preliminary Report), Brian Gluss and Fred L. Strodtbeck, University of Chicago.

n properties $A, B, \cdots, E$ are ordered $1, 2, \cdots, n$ by m people and scores $a, b, \cdots, e$ are calculated such that $q = \Sigma$ (ranks of A) $- [(n + 1)m/2]$, and so on. A statistic r is defined in the following manner: Consider n lines in an $(n - 1)$-dimensional space with orthogonal axes $(x_1, \cdots, x_{n-1})$, in which each line corresponds to one of the characteristics $A, B, \cdots, E$. The lines all pass through the origin and are such that the angles between all pairs are equal. Points $A, B, \cdots, E$ are defined on the lines such that the distances $OA, \cdots, OE$ (where O is the origin) are $a, b, \cdots, e$, respectively. A point $P(X_1 \cdots, X_{n-1})$ is then obtained such that OX_i is the sum of the projections of $OA, \cdots, OE$ on Ox_i. Then $OP = r$. The paper shows that $r = \sqrt{Sn/(n = 1)}$, where S is Kendall's statistic [M. G. Kendall, *Rank Correlation Methods*, 1948, Chap. 6.]. By this approach the paper shows it is possible by considerations of volumes of the space to test various hypotheses including (i) $A = B = \cdots = E$; (ii) $A > B > \cdots > E$ versus all other contingencies.

27. Generalized Normalization Polynomials, D. Teichroew, University of California at Los Angeles and National Cash Register Company, Dayton, Ohio.

Normalization polynomials have been studied by Campbell in 1923, Cornish and Fisher in 1937 and Hotelling and Frankel in 1938. Expansions based on these polynomials enable one to use the normal integral table for computing probability points of other distributions which approach the normal. These expansions have been expansions about the mean and the further the variate is from the mean the higher is the error of approximation. This paper shows that it is possible to get polynomials for expansions about an arbitrary point. These expansions are useful for obtaining probability points for probabilities very close to zero or one. Some expansions are obtained for the t and Gamma distributions. (Part of this work was sponsored by the Office of Naval Research).

28. "No Interaction" in a Three-way Table, Marvin A. Kastenbaum, University of North Carolina.

Let n_{ijk} denote the observed frequency, and p_{ijk} the probability of having an observation in the (ijk)th cell of a three-way table, $(i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; k = 1, 2, \cdots, t)$. Also let the marginal probabilities $\Sigma_i\, p_{ijk} = p_{0jk}$, etc., $\Sigma_{i,k}\, p_{ijk} = \Sigma_k\, p_{0jk} = p_{0j0}$, etc., and $\Sigma_{i,j,k}\, p_{ijk} = 1$, and $\Sigma_{i,j,k}\, n_{ijk} = n$. Then, if the marginal frequencies are stochastic variates, the condition of independence between "(ij)" and "k" is expressed as: (1) $p_{ijk} = p_{ij0}p_{00k}$; and the conditions of independence between "i" and "k" and between "j" and "k" are given as (2) $p_{i0k} = p_{i00}p_{00k}$, and (3) $p_{0jk} = p_{0j0}p_{00k}$, respectively. A condition of "no interaction" is defined as one which, when taken together with (2) and (3), will yield (1). This condition is (4) $p_{ijk} = (q_{ij0}q_{i0k}q_{0jk})/(q_{i00}q_{0j0}q_{00k})$, where it is not assumed that $q_{ij0} = p_{ij0}$, etc., nor even that $q_{i00} = \Sigma_j\, q_{ij0}$, etc. The q's in (4) may be eliminated in such a fashion as to yield distinct relationships among the p_{ijk}'s, namely: (5) $(p_{rst}p_{ijt})/(p_{ist}p_{rjt}) =$

$(p_{rsk}p_{ijk})/(p_{i0k}p_{rjk})$, $i = 1, 2, \cdots, (r-1); j = 1, 2, \cdots, (s-1); k = 1, 2, \cdots, (t-1)$. If the hypothesis (4) of "no interaction" is to be tested, then the p_{ijk}'s may be estimated by maximizing the multinomial likelihood function, subject to (5) and to the further constraint $\Sigma_{i,j,k}\, p_{ijk} = 1$. The resulting $\hat{p}_{ijk}$'s, expressed in terms of their respective n_{ijk}'s and the deviations from expectation, are then substituted into the expression $\Sigma_{i,j,k}\, (n_{ijk} - n\hat{p}_{ijk})^2/n\hat{p}_{ijk}$, which for large n, is distributed as χ^2 with $(r-1)(s-1)(t-1)$ degrees of freedom.

29. On Bartlett's Test of Complex Contingency Table Interaction, Sujit Kumar Mitra, University of North Carolina.

Contrary to some of the current beliefs it is shown that the stochastic cubic equation suggested to Professor Bartlett (*J. Roy. Statist. Soc.*, suppl. 1935) by Professor R. A. Fisher in developing a test of his hypothesis of no interaction in a $2 \times 2 \times 2$ table might, with probability approaching one, have all the three roots real, under the null hypothesis of no-interaction. In such a case only the real root with the numerically smallest value will validate the use of χ^2 test. This could be immediately extended to a general $r \times s \times t$ table. A little thought would reveal that the numerical example considered by Bartlett actually represents 4 samples from 4 binomial populations. An attempt has been made to interpret interaction and main effects in this case and to furnish suitable tests for the suggested hypotheses.

30. A Theorem in Minimum Chi Square, Sujit Kumar Mitra, University of North Carolina, (By Title).

Let the possible results of a certain random experiment E be divided into r mutually exclusive groups and suppose that the probability of obtaining a result belonging to the ith group is $p_i^0 = p_i(\alpha_1^0, \cdots, \alpha_s^0)$ where $\alpha^0 = (\alpha_1^0, \cdots, \alpha_s^0)$ is an inner point of some non-generate interval A. We assume that $p_i(\alpha_1, \cdots, \alpha_s)$ considered as functions of $\alpha_1, \cdots, \alpha_s$ over A satisfy Cramér's conditions (a), (b), (c) and (d). (See Cramér: *Mathematical Methods of Statistics*, Section 30.3.) Let $f_k(\alpha_1, \cdots, \alpha_s) k = 1, 2, \cdots, t \leqq s$ be t functions of $\alpha_1, \cdots, \alpha_s$ such that for all points in A, the f_k satisfy the following conditions: (e) Every f_k has continuous derivatives $(\partial f_k)/(\partial\alpha_j)$ and $(\partial^2 f_k/(\partial\alpha_j\partial\alpha_j)$; (f) the matrix $\{(\partial f_k)/(\partial\alpha_j)\}$ where $k = 1, 2, \cdots, t$, $j = 1, 2, \cdots, s$ is of rank t. Let $f_k^0 (k = 1, 2, \cdots, t)$ be certain numbers in the range of the respective f_k's over A. We denote by H the hypothesis $f_k(\alpha_1^0, \cdots, \alpha_s^0) = f_k^0 (k = 1, 2, \cdots, t)$. Let ν_i denote the number of observations belonging to the ith group in n actual repetitions of E. Cramér has shown that the modified minimum chi-square equations have exactly one system of solutions $\hat{\alpha} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_s)$ such that $\hat{\alpha} \underset{p}{\rightarrow} \alpha^0$ as $n \rightarrow \infty$ and such that $\chi_0^2 = \Sigma_{i=1}^r (\nu_i - np_i(\hat{\alpha}))^2/np_i(\hat{\alpha})$ is asymptotically distributed as a χ^2 with $r - s - 1$ d.f. The following results are proved: If H_0 is true (1) the modified minimum χ^2 equations subject to restriction H have exactly one system of solutions $\hat{\alpha}_H$ such that $\hat{\alpha}_H \underset{p}{\rightarrow} \alpha^0$ as $n \rightarrow \infty$, and such that $\chi_H^2 = \Sigma_{i=1}^r (\nu_i - np_i(\hat{\alpha}_H))^2/(np_i\hat{\alpha}_H)$ is asymptotically distributed as χ^2 with $r - s + t - 1$ d.f.; (2) $\chi_H^2 - \chi_0^2$ is asymptotically independently distributed of χ_0^2 as a χ^2 with t d.f. This result is analogous to a result in least squares proved by C. R. Rao in his book. This also extends a result proved by Neyman (Proc. First Berk. Symp.).

31. Sequential Estimation from a Finite Population, Herbert T. David and Ingram Olkin, University of Chicago.

This paper is concerned with sequential estimation of the fraction defective p, in a finite (hypergeometric) population. The development is similar to that of Girshick, Mosteller, Savage (*Ann. Math. Stat.*, v. 17, 1946, 13–23), who treat the case of infinite (binomial) populations. Path count ratios play essentially the same role in both cases, and are shown to provide unique unbiased estimates of certain functions of p when the regions are simple. Expressions for the variance of the estimate of p are given for both cases, and it is shown that for symmetric boundaries the variances in the finite and infinite situations are formally similar polynomials in pq of the same degree. A generalized finite population correction is discussed and, in particular, boundaries for which the variance is equal to cpq are considered.

32. Tables for Computing Bivariate Normal Probabilities, Donald B. Owen, Sandia Corporation.

A table of $T(h, a) = 1/(2\pi) \int_0^a \exp[-\frac{1}{2}h^2(1 + x^2)]/(1 + x^2)\, dx$, which may be used to obtain bivariate normal probabilities, has been computed to be used with a special two-dimensional linear interpolation scheme. The function is tabulated in two tables, one table having a coarse interval in one of the parameters and an interval fine enough for ordinary linear interpolation in the second parameter. The second table has the coarse interval on the second parameter and the fine interval on the first. By choosing the four points at the coarse intervals of the two tables that are nearest to a value to be interpolated and four other points on the fine intervals, the interpolation scheme gives accuracies comparable to ordinary linear interpolation with only ten per cent as many entries as that required for ordinary linear interpolation.

33. Bounds and Approximations for Constants Used in Quality Control, J. T. Chu, University of North Carolina and Case Institute of Technology.

Very close, yet very simple in form, upper and lower bounds are obtained for constants a, c_2, b, A, E_1, and B_i, $i = 1, \cdots, 4$, often used in quality control to set up control charts for individual observations and the means and standard deviations of groups of observations. For example, let a random sample: $x_1, \cdots, x_n$, be drawn from a normal distribution with variance σ^2. If $s^2 = \Sigma_{i=1}^n (x_i - \bar{x})^2/(n - 1)$ where $\bar{x} = \Sigma_{i=1}^n x_i/n$ and $a = E(s)/\sigma$, then $[(2n - 3)/(2n - 2)]^{\frac{1}{2}} \leqq a \leqq [(2n - 2)/(2n - 1)]^{\frac{1}{2}}$ for all integers $n \geqq 2$. In using these bounds and their arithmetic mean as approximations to a, the proportional errors are shown to be respectively less than $E = \frac{1}{2}(2n - 1)(2n - 3)$ and $E/2(E/2 = .004$ if $n = 5)$. Similar results are obtained for the constants mentioned above. Tables are given for illustration (Research partially supported by the Office of Naval Research).

34. Four Streams of Traffic Converging on a Cross-Road, Brian Gluss, University of Chicago, (By Title).

Four streams of traffic arrive at a cross-road in independent Poisson process. The lights operate such that if they have just turned red against two of the streams they will turn green again when either (i) n cars are waiting, or (ii) a time α has passed, whichever is the sooner. n and α are prearranged. The frequency function and expectation of the waiting time τ of a car wishing to go straight or to turn right are obtained: $E(\tau) = [n(n + 1)/2 - e^{-m\alpha}\Sigma_{R=0}^n (n - R + 1)(n + R)(m\alpha)^R/2(R)!]/(t_1 + t_2)m^2$, where m = sum of mean flows of the two streams for which the lights are red, and t_1 and t_2 are the expected red time-periods for the two sets of two streams. The frequency functions of the waiting times u, v of a car arriving in a green or red period respectively and wishing to turn left, and therefore having to wait for a sufficient time-gap in the opposing stream or until the lights turn red again, are found. $E(u)$ and $E(v)$ are then calculated for some sets of parameter-values by approximate integration.

35. Markov Processes Arising in Learning Models, John G. Kemeny and J. L. Snell, Dartmouth College.

The paper studies two learning models, one due to W. K. Estes, and the other due to R. R. Bush and F. Mosteller. In the cases studied both models lead to one-parameter families of Markov processes; the Estes model having a finite number of states, the Bush-Mosteller model an infinite number. For each value of the learning-parameter there is a single Bush-Mosteller process, but an infinite number of Estes models—one for each possible number of states. It is shown that for a given value of the learning-parameter, as the number of states tends to infinity, the stationary distribution of the Estes processes tends to the stationary distribution of the corresponding Bush-Mosteller process. Moments of the stationary distribution are found in the Bush-Mosteller processes, and the distributions themselves are also found in several special classes of processes. It is shown that as the learning-parameter tends to zero the stationary distributions in both models approach very simple distributions. Since some psychologists are interested primarily in low values of the learning-parameter, this result provides simple approximate answers.

36. On a Decision Rule for Selecting a Group Containing the Population with the Largest Mean, (Preliminary Report), R. C. Bose and S. S. Gupta, University of North Carolina, (By Title).

Suppose there are $(n + 1)$ normal populations $N(\mu_i, \sigma^{*2})$, $i = 0, 1, 2, \cdots, n$, and that $x_0, x_1, \cdots, x_n$ are the $(n + 1)$ means based on samples of equal size k, one from each population. One would like to select as small a group as possible subject to the restriction that the least upper bound of the probability of not including in the group the population with the largest mean is $\alpha(0 < \alpha < 1)$. K. C. Seal (*Ann. Math. Stat.*, Sept., 1955) has given an infinite class of decision rules for this problem and has obtained an optimum rule for the situation when all means but one are equal. Another rule has been studied here in detail. This is based on the auxiliary statistic $u = (Y_{(n)} - Y_0)/s$, where $Y_0, Y_1, \cdots, Y_n$ are independently and identically distributed $N(0, \sigma^2)$, $\sigma^2 = \sigma^{*2}/k$, corresponding random observations $y_1, y_2, \cdots, y_n$ being ordered as $y_{(1)} < y_{(2)} < \cdots < y_{(n)}$ and where s^2 is an independent estimate of σ^2. The rule is, "Reject any observation x_0 from the given $x_i(i = 0, 1, 2, \cdots, n)$ if $x_{(n)} - x_0 > su_\alpha$ and retain otherwise; where $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ are n ordered observations among $(x_1, x_2, \cdots, x_n)$ and u_α is the upper α % point of $(Y_{(n)} - Y_0)/s$." The upper 5 % points of U, in the case when σ is known, have been tabulated.

37. Recurrent Values of Sums of Independent Random Variables, (Preliminary Report), Louis J. Cote and Henry Teicher, Purdue University.

Let $\{X_i\}$ $i = 1, 2, \cdots$ be a sequence of independent random variables defined on a probability space with $S_{m,n} = \sum_{i=m}^{n} X_i$, $S_{1,n} = S_n$ and $P_\epsilon = P\{| S_{m,n} - b | < \epsilon \text{ i.o.}\}$ where i.o. signifies "infinitely often" (i.e., for infinitely many values of $n \geqq m$). The real number b is called recurrent or quasi recurrent for the sequence $\{S_{m,n}\}$ according as $P_\epsilon = 1$ or $P_\epsilon > 0$ for all $\epsilon > 0$. The classes of such values are examined and conditions for the existence of recurrent or quasi-recurrent values are considered.

38. A Problem Involving the Distribution of Shadows, (Preliminary Report), Herman Chernoff, Stanford University, and Joseph F. Daly, National Bureau of the Census.

A source of light is at a point P and a worm is crawling in a given direction along a line L which does not go through P. Circular disks are distributed randomly throughout the plane containing P and L. Suppose that the worm can travel only in the shadow. The distribution of the distance the worm can travel from a given starting point is characterized. One such characterization involves the wave equation. The results generalize to the cases where the disks are replaced by line segments parallel to L and the source of light is at infinity and these results have applications to geiger counter and traffic problems. The corresponding problems of the worm who travels only in light are rather easy to treat.

39. Note on Two-stage Test Procedures, S. G. Ghurye, Lucknow University, (By Title).

This note concerns tests of hypotheses regarding a parameter which are designed to have power independent of another parameter. The conditions satisfied in the problem of the mean of a normal distribution solved by Stein (*Ann. Math. Stat.*, 1945) are stated more generally, and the corresponding general solution is given. It is shown that these conditions are also satisfied in the problem of testing for the location parameter of an exponential distribution by a number of two-stage tests, and the performances of some of these tests are compared in some particular cases.

40. Some Properties of Generalized Sequential Probability Ratio Tests, Jack C. Kiefer, Cornell University, and Lionel Weiss, University of Virginia.

Generalized sequential probability ratio tests (GSPRT) are known to form a complete class with respect to the probabilities of making errors and the distribution of the sample size, when one simple hypothesis is being tested against another. In this paper it is shown that (1) under certain conditions, a GSPRT is uniquely determined by the distributions of the sample size under the two hypotheses; (2) for a GSPRT to be admissible with respect to the probabilities of error and the distribution of the sample size, the decision bounds characterizing the test must obey certain inequalities; (3) under certain monotonicity conditions on the probability ratio, a GSPRT forms a complete class with respect to the probabilities of error and the "average" distribution of the sample size (averaged over a set of alternatives to the two hypotheses being tested); and (4) a class of tests complete with respect to the probabilities of error and the expected sample size under a third distribution consists of truncated GSPRT whose decision bounds satisfy certain inequalities.

41. Sequential Decision Problems for a Class of Stochastic Processes. Testing Hypotheses, (Preliminary Report), A. T. Bharucha-Reid, University of California, Berkeley.

Let $\{X_i(t),\ t \geqq 0\}$, $i = 1, 2$, be two different stochastic processes with continuous time parameter. Beginning at $t = 0$, a process $\{X(t),\ t \geqq 0\}$, which is either $\{X_1(t)\}$ or $\{X_2(t)\}$, is observed continuously, and on the basis of the observed realization of the process, $x(t)$, the statistician wishes to decide whether $\{X(t)\}$ is $\{X_1(t)\}$ or $\{X_2(t)\}$. This problem has been considered by Dvoretzky, Kiefer, and Wolfowitz (*Ann. Math. Stat.*, Vol. 24 (1953), pp. 254–264) when $\{X(t)\}$ is a stochastic process with stationary independent increments.

In this paper we consider the case where $\{X(t)\}$ is a Markov process, with $x(t)$ a sufficient statistic for the process. We consider in particular the application of these results to some branching stochastic processes, e.g., the birth, death, birth-and-death, and Pólya processes. Let $p(x, t; \omega) = \Pr(X(t) = x \mid \omega)$, $x = 0, 1, 2, \cdots; \omega \varepsilon \Omega$, and denote by $D(t)$ the decision function $\log \{p(x, t; \omega_2)/p(x, t; \omega_1)\}$. For decision boundaries A and B, $B < 0 < A$, the Wald sequential procedure is used to test the hypothesis $H_i (i = 1, 2)$ that $\omega = \omega_i$, where ω_1 and ω_2 are any two positive numbers, $\omega_1 \neq \omega_2$. Let $f(d; \omega)$ denote the probability that the decision procedure will terminate with the acceptance of H_2 when the parameter is really ω and $D(0) = d$; and let $m(s, \tau) = E\{\exp(s\tau)\}$ be the moment generating function of the observation time τ necessary to reach a decision when $D(0) = d$ and the parameter is really ω. The usual probabilistic reasoning leads to functional equations for $f(d; \omega)$ and $m(s, \tau)$, the analytic properties of which will be discussed in a subsequent publication. (This work was supported by the USAF School of Aviation Medicine.)

42. **Note on a Markov Chain with Matrix States and Some Applications,** A. T. Bharucha-Reid and Rodabé P. Bharucha-Reid, University of California, Berkeley, (By Title).

In connection with a probability problem in learning theory concerned with latent and reinforced types, it was necessary to consider a Markov chain with matrix states. Various ways of defining the transition probabilities are considered, and the asymptotic properties of the chain investigated. The results obtained are applicable to the study of changes in systems whose structure has a matrix representation, e.g., communication nets, social groups, etc.

43. **On the Comparison of Two Stochastic Epidemics,** A. T. Bharucha-Reid, University of California, Berkeley, (By Title).

In this paper the Girshick procedure for comparing or ranking two populations with respect to an unknown parameter is applied to the problem of comparing the effect of two types of housing on hospital admission rates for acute respiratory disease. The procedure is applied when different stochastic models are used to describe the development of the epidemic. Data used are from an epidemic situation studied at Sampson Air Force Base, Geneva, New York. (This work was supported by the USAF School of Aviation Medicine.)

44. **A Sequential Multiple Decision Procedure for Selecting the Population with the Largest Mean from k Normal Populations with a Common Unknown Variance, (Preliminary Report),** R. E. Bechhofer, Cornell University, and M. Sobel, Bell Telephone Laboratories, (By Title).

Let $x_{ij} (i = 1, \cdots, k; j = 1, 2, \cdots)$ be independent observations from normal populations Π_i with unknowns means μ_i and a common unknown variance, and let $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ranked means. A sequential procedure is proposed which guarantees probability P^* of selecting the population with the largest mean $\mu_{[k]}$ whenever $\mu_{[k]} - \mu_{[k-1]} \geqq \delta^*$; the constants $P^* < 1$ and $\delta^* > 0$ are preassigned. Let

$$\bar{x}_{im} = \Sigma_{j=1}^{m} x_{ij}/m, \qquad s_m^2 = \Sigma_{i=1}^{k} \Sigma_{j=1}^{m} (x_{ij} - \bar{x}_{im})^2/k(m - 1),$$

and $t_{ijm} = (\bar{x}_{im} - \bar{x}_{jm} - \delta^*)/s_m\sqrt{2}/m$. For $i = 1, \cdots, k$ let

$$L_{im} = \left[1 + \sum_{\substack{\alpha=1 \\ \alpha,\beta \neq i}}^{k} \sum_{\beta=1}^{k} A_{\alpha\beta}\, t_{i\alpha m}\, t_{i\beta m}/k(m - 1) \right]^{-(km-1)/2},$$

where $A_{\alpha\beta} = 2(k - 1)/k$ for $\alpha = \beta$ and $-2/k$ for $\alpha \neq \beta$, let $L_{[1]m} \leqq \cdots \leqq L_{[k]m}$ denote the ranked L_{im} , and let $P_m = L_{[k]m}/\sum_{i=1}^{k} L_{im}$. At every stage, the k values, L_{im} differ with probability one and are in one-to-one correspondence with the k populations Π_i ; let $\Pi_{[k]m}$ denote the population associated with $L_{[k]m}$ at the mth stage. *Procedure*: At the mth stage $(m = 1, 2, \cdots)$ take the vector observation $(x_{1m}, \cdots, x_{km})$ and compute P_m . If $P_m \geqq P^*$, stop and select $\Pi_{[k]m}$; if $P_m < P^*$, take the $(m + 1)$st vector observation and compute P_{m+1} . This procedure meets the requirement, is scale and location invariant, and the probability of termination is unity. The procedure can be generalized to handle problems such as obtaining a complete ranking of the k means. (Research supported in part by the U. S. Air Force through the Office of Scientific Research of the ARDC.)

45. A Scale Invariant Sequential Multiple Decision Procedure for Selecting the Population with the Smallest Variance from *k* Normal Populations, (Preliminary Report), R. E. Bechhofer, Cornell University, and M. Sobel, Bell Telephone Laboratories, (By Title).

Let $x_{ij}(i = 1, \cdots, k; j = 1, 2, \cdots)$ be independent observations from normal populations Π_i with unknown means μ_i and unknown variances σ_i^2 , and let $\sigma_{[1]}^2 \leqq \cdots \leqq \sigma_{[k]}^2$ denote the ranked variances. A sequential procedure is proposed which guarantees probability P^* of selecting the populations with the smallest variance $\sigma_{[1]}^2$ whenever $\sigma_{[2]}^2/\sigma_{[1]}^2 \geqq \theta^*$; the constants $P^* < 1$ and $\theta^* > 1$ are preassigned. Let $\bar{x}_{im} = \sum_{j=1}^{m} x_{ij}/m$, $s_{im}^2 = \sum_{j=1}^{m} (x_{ij} - \bar{x}_{im})^2$, and $R_{jim} = s_{jm}^2/s_{im}^2$. For $i = 1, \cdots, k$ let

$$L_{im} = [\Pi_{j=1}^{k} R_{jim}]^{(m-3)/2}[1 + \Sigma_{j=1, j\neq i}^{k} R_{jim}/\theta^*]^{-k(m-1)/2},$$

let $L_{[1]m} \leqq \cdots \leqq L_{[k]m}$ denote the ranked L_{im} , and let $P_m = L_{[k]m}/\sum_{i=1}^{k} L_{im}$. At every stage, the k values, L_{im} , differ with probability one and are in one-to-one correspondence with the k populations Π_i ; let $\Pi_{[k]m}$ denote the population associated with $L_{[k]m}$ at the mth stage. *Procedure*: At the mth stage $(m = 2, 3, \cdots)$, take the vector observation $(x_{1m}, \cdots, x_{km})$ and compute P_m . If $P_m \geqq P^*$, stop and select $\Pi_{[k]m}$; if $P_m < P^*$ take the $(m + 1)$st vector observation and compute P_{m+1} . This procedure meets the requirement, is scale and location invariant, and the probability of termination is unity. The procedure can be generalized to handle problems such as obtaining a complete ranking of the k variances. A similar procedure can be used for the case of known means and for ranking the scale parameters of exponential populations. (Research supported in part by the U.S. Air Force through the Office of Scientific Research of the ARDC.)

46. Exact Probabilities in a Test for Markoff Dependency, Reed B. Dawson, Jr., Department of Defense.

This paper is concerned with Markoff dependency (of the first order) in a digital stream, where the object is to test the hypothesis of independence against any alternative which alters the probabilities of the pairs. Let N digits be distributed about an oriented circle so that each of the $(N - 1)!$ arrays are equally likely, and form a matrix $[f_{ij}]$, where f_{ij} is the number of digits i which are followed by a digit j. The exact probability of this matrix of pairs is found, generalizing a result of Stevens (*Ann. Eugenics*, Vol. 9 (1939), pp. 10–17). This probability, asymptotically the same as the probability that a matrix with the same entries will arise from the usual contingency table assumptions, illuminates a special case of the asymptotic test of Hoel (*Biometrika*, Vol. 41 (1954), pp. 430–433) for Markoff dependency of general order. A formula for the expectancy of a product of factorial powers of the f_{ij} is derived.

47. A Combinatorial Problem and Its Application to Probability Theory, T. V. Narayana, McGill University.

A quasi order called k-domination is defined on the r-partitions of two integers m and n. An explicit expression for the number of k-dominations of the r-partitions of n by those of m is derived. This result is extended and shown to be a generalization of the "problème du scrutin" of D. André. Two classes of coin-tossing problems are solved as an application of this result. A number of combinatorial identities and the solution of a class of difference equations are obtained by probability methods. The relation of this problem to the recurrent events of Feller in the case of coin tossing is briefly discussed.

48. The Bayesian Inference Problem in Stochastic Systems, Max A. Woodbury, George Washington University.

In an experimental or environmental stochastic system, the possible inputs to the controlled stochastic process are represented by stochastic mappings of the internal states of the system into each other. The observable outputs are assumed to be the result of a stochastic mapping from the internal states of the system to the set of possible outputs. In the case where the inputs only are known, the general formula for the a posteriori distribution at a given time is the result of applying the product of the input stochastic matrices to the a priori distribution vector. If, however, account is taken of the information provided by the output the result is expressible in linear terms only if the requirement for a normalized probability vector is dropped. The relationship of this result to the stochastic behavior models of Rosenblatt, Flood and Mosteller is discussed. (The research covered by this abstract was supported by the Office of Naval Research.)

49. Some Nonparametric Generalizations of Multivariate Analysis and Analysis of Variance, S. N. Roy, University of North Carolina.

With observed frequency data arranged in a multi-way table, assuming that the observations are independent in probability, there will be, under any hypothesis, (i) a single multinomial distribution or (ii) the product of a number of separate multinomial distributions according as (i) only the total number of observations is supposed to be fixed from sample to sample or (ii) marginal frequencies in certain directions of the table are supposed to be fixed. An attempt is made at a systematic elaboration of the historically prior ideas of Barnard and Pearson (*Biometrika*, 1947), to (i) multivariate analysis, starting from a single multinomial and framing hypothesis suitable to multivariate analysis situations and to (ii) analysis of variance, starting from the product of an appropriate number of multinomials and framing hypotheses suitable to analysis of variance situations. The theorems used are those of Cramér [*Mathematical Methods of Statistics*, Chapter 30] and some other theorems which can be proved on the same lines. The conditional probability approach is altogether abandoned.

50. Further Remarks on Measures of Association for Cross-Classifications, Leo A. Goodman, University of Chicago, and William H. Kruskal, Universities of California and Chicago.

Measures of association discussed by the authors previously (*J. Amer. Stat. Assn.*, 49 (1954), 732–64) are considered further, especially in regard to the sampling distributions of their sample analogues. Asymptotic distributions are obtained for a number of cases, and numerical investigations of the accuracy (qua approximations) of these asymptotic distributions are described.

51. Uniformly Consistent Sequences of Multiple-Decision Rules, William Jackson Hall, University of North Carolina.

Suppose x has an unknown distribution function F, belonging to one of m disjoint classes $\omega_1, \cdots, \omega_m$, and suppose $A_1, \cdots, A_m$ are corresponding alternative decisions, one of which is to be chosen by a multiple-decision rule (m-d.r.) D_n after taking a sample of size n. D_n is defined by $\phi^n(x) = [\phi_1^n(x), \cdots, \phi_m^n(x)]$, $\phi_i^n \geqq 0$, x denoting the sample, where the θ_i^n's sum over i to unity. $\phi_i^n(x)$ is the probability that D_n chooses A_i when x is observed. *Definition* 1: $\{\phi^n(x)\}$, $n = 1, 2, \cdots$, defines a "uniformly consistent sequence (u.c.s.) of m-d.r.'s $\{D_n\}$ for discriminating among $\omega_1, \cdots, \omega_m$" if $\lim_{n\to\infty} \inf_{F\varepsilon\omega_i} \mathcal{E}_F\phi_i^n(X) = 1$ $(i = 1, \cdots, m)$. *Definition* 2: $\phi^n(x)$ defines a "non-trivial m-d.r. D_n for discriminating among $\omega_1, \cdots, \omega_m$" if $\sum_{i=1}^m \inf_{F\varepsilon\omega_i} \mathcal{E}_F\phi_i^n(X) > 1$. *Theorem* 1: *A necessary and sufficient condition for the existence of a* u.c.s. *of* m-d.r.'s *for discriminating among* $\omega_1, \cdots, \omega_m$ *is that there exist non-trivial* 2-d.r.'s *for discriminating between* ω_i *and* ω_j *for some* n_{ij} *(sample size) for every* $i \neq j$. Results of Hoeffding (unpublished) and Berger and Wald (*Ann. Math. Stat.*, Vol. 20 (1949), pp. 104–9) are adapted to supply some necessary and sufficient conditions, respectively, for the existence of non-trivial 2-d.r.'s. *Theorem* 2: *A necessary and sufficient condition for the existence of a most economical* m-d.r. *relative to any* $(\alpha_1, \cdots, \alpha_m)$, *or relative to any* (β_{ij}), *for discriminating among* $\omega_1, \cdots, \omega_m$ (Hall, Abstract, *Ann. Math. Stat.*, Vol. 25 (1954), p. 814) *is that there exist a* u.c.s. *of* m-d.r.'s *for discriminating among* $\omega_1, \cdots, \omega_m$.

52. Some Hypergeometric Series Distributions Occurring in Birth-and-Death Processes at Equilibrium, (Preliminary Report), William Jackson Hall, University of North Carolina, (By Title).

Some time-homogeneous birth-and-death processes at equilibrium are considered in which the birth and death rates are "stimulated" by "overcrowding." Generally, under mild restrictions, p_n, the distribution of population size n is proportional to $\Lambda_n/(M_n n!)$, where $\Lambda_n = \lambda_0\lambda_1 \cdots \lambda_{n-1}$ and $M_n = \mu_1\mu_2 \cdots \mu_n$; $\lambda_n(\mu_n)$ is the birth (death) rate when the population size is n. If λ_n is quadratic in n (i.e., constant immigration rate and reproduction rate is linear in n) and μ_n linear in n, then p_n is shown to be proportional to the $(n + 1)$th term in a general hypergeometric series, a four parameter distribution. If λ_n and μ_n are both linear (constant reproduction rate), p_n is proportional to the $(n + 1)$th term in a confluent hypergeometric series, a three parameter distribution. In the same manner, using a constant death rate, p_n is proportional to the $(n + 1)$th term in a negative binomial series, as is well known; and, with no reproduction, an exponential series (Poisson distribution) is obtained. Each distribution is a limiting form of the preceding one. Generating functions, moments, and approximate estimates by the method of moments of the parameters of the hypergeometric series distributions are derived.

53. Some General Aspects of Stochastic Approximations, Tosio Kitagawa, Iowa State College.

As one continuation of random integration introduced by the author, some general aspects of stochastic approximations will be discussed specifically in reference to the risk function formulations. Our stochastic approximations are concerned with the various problems of (a) solutions of equations, (b) interpolation problems, (c) mapping problems, and (d) numerical differentiations.

54. The Analysis of Incidence Rates Under Multiple Classifications of the Population, (Preliminary Report), WYMAN RICHARDSON, University of North Carolina.

A population is classified two·ways into cells, n_{ij}. The number of cases, a_{ij} (of some disease, for instance), is assumed to have, in one model, a Poisson distribution with parameter $n_{ij}\rho_{ij}$, and in another, a binomial (Q_{ij}, n_{ij}) distribution. Q_{ij} is assumed to be equal to $f(\theta_i, \psi_j)$. The hypothesis $\psi_1 = \cdots = \psi_s$ can be tested in each model by χ^2, with expected frequencies in each cell of $N_{ij}A_{i\cdot}/N_{i\cdot}$, (where $A_{i\cdot} = \Sigma_j A_{ij}$, etc.). Maximum likelihood equations are derived for the case $f(\theta_i, \psi_j) = \theta_i\psi_j$. It is shown that, except for a multiplicative factor, there is a single solution of these equations, which can be obtained by efficient iterative procedures. This result holds when there are k classifications. In the Poisson model, these estimates are sufficient. A large sample test of the hypothesis $\psi_1 = \cdots = \psi_s$ against the alternative $Q_{ij} = \theta_i\psi_j$ is to compute $\chi^2 = 2[\Sigma A_{i\cdot} \log_e (\hat{\theta}_i N_{i\cdot}/A_{i\cdot}) + \Sigma A_j \log_e \hat{\psi}_j]$.

55. Estimation of Percentiles by Order Statistics, A. E. SARHAN, University of North Carolina.

In previous work of the author ("Estimation of the mean and standard deviation by order statistics," *Ann. Math. Stat.*, Vol. 25 (1954), and Part II, *Ann. Math. Stat.*, Vol. 26 (1955)) the means and standard deviations of certain distributions were estimated by the best linear combinations of the ordered sample values. In the present paper, the same methods are used to derive a general expression for estimation of the jth percentile and its variance. From this expression and by making use of previous results, the jth percentile is estimated for certain distributions. As special cases, the estimates of the 50th percentile (the population median) and of the semi-interquartile range are calculated.

56. On Renewal Theory, Counter Problems, and Quasi-Poisson Processes, WALTER L. SMITH, University of North Carolina.

Let $\{t_i\}$ be a *renewal process*, i.e., a sequence of non-negative, independent, identically distributed random variables which are not zero with probability one. Let $\mu_r = Et_i^r$, and define n_t by $\Sigma_1^{n_t} t_i \leqq t < \Sigma_1^{n_t+1} t_i$ (taking $n_t = 0$ if $t_1 > t$). If $H(t) = En_t$, then it is shown that (i) if $\mu_1 < \infty$, then a necessary and sufficient condition for $\mu_2 < \infty$ is that $\lim_{t\to\infty} \{H(t) - t\mu_1^{-1}\} = \beta$ exist and be finite, when $\mu_2 = 2\mu_1^2(1 + \beta)$; (ii) if $t_i = u_i + v_i$ where $\{u_i\}$ and $\{v_i\}$ are independent renewal processes, the v_i having a negative exponential distribution, then $\lim_{t\to\infty} H'(t)$ always exists. The results (i) and (ii) render the calculation of the asymptotic properties of a certain electronic counter process, previously studied by Hammersley, straightforward. If $H(t)$ is linear in t for all $t > \tau$, for some finite τ, the process is called quasi-Poisson, and the class of quasi-Poisson processes is not empty. Let $Y(x; t) = \Pr(\Sigma_1^{n_t+1} t_i \leqq t + x)$. Then it is shown that a necessary and sufficient condition for $Y(x; t)$ to be independent of $t > \tau$ is that $\{t_i\}$ be quasi-Poisson. When $\{t_i\}$ is quasi-Poisson, $\mu_r < \infty$ for all r, and the study of the effects of an automatic self-paralyzing mechanism on $\{t_i\}$, of a type in use for blood-cell counting, becomes trivial.

57. On the Construction of Significance Tests on the Circle and the Sphere, G. S. WATSON, The Australian National University, and E. J. WILLIAMS, Commonwealth Scientific and Ind. Res. Organization, S. Melbourne, (By Title).

The probability density proportional to exp $(k \cos \theta)$, where k is a precision constant and θ is the angle between an observed vector and a population mean vector or polar vector, has been considered in two and three dimensions by several authors. Significance tests are required to test (i) that $k = k0$, is a prescribed value, or that several populations have the same value of k, and (ii) that the polar direction of a population has prescribed direction cosines or that several populations have the same polar vectors. Tests of these hypotheses are given which are free of nuisance parameters. They are based on conditional distributions formed by holding constant sufficient statistics. Inequalities and approximations are suggested to make the tests easy to apply in practice. The arithmetic examples given suggest that, in three dimensions, the tests given by one of us (G.S.W.) elsewhere will be satisfactory.

58. Estimation of Individual Variations in an Unreplicated Two-Way Classification, (Preliminary Report), THOMAS S. RUSSELL and RALPH ALLAN BRADLEY, Virginia Polytechnic Institute, (By Title).

Consider a two-way classification, the usual model $x_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$, $i = 1, \cdots, n$, $j = 1, \cdots, r$ (e.g., r chemists and n batches or r judges and n items) and the usual assumptions, except let var $(\epsilon_{ij}) = \sigma_j^2$. It was assumed that an estimator Q_j of σ_j^2 should be a quadratic form in the $(r - 1)(n - 1)$ linear contrasts usually ascribed to error. Reasonable requirements on such a quadratic form led to the estimator in $(n - 1)(r - 1)(r - 2)Q_j = r(r - 1)E_j - E$, where $E_j = \sum_{i=1}^{n} (x_{ij} - x_{i\cdot} - x_{\cdot j} + x_{\cdot\cdot})^2$ and $E = \sum_{j=1}^{r} E_j$, the usual error sum of squares. Q_j is the estimator previously suggested by Ehrenberg (*Biometrika*, Vol. 37 (1950), pp. 347–357). Q_j has been shown to be the maximum likelihood estimator of σ_j^2 only when $r = 3$. When $\sigma_j^2 = \sigma^2$ for all j, the distribution of Q_j/σ^2 has been shown to be that of $r\chi_{n-1}^2/(n - 1)(r - 1) - \chi_{(n-1)(r-2)}^2/(n - 1)(r - 1)(r - 2)$, the two χ^2's being independent. Q_j/E has been shown to be a monotone function of an F with $(n - 1)(r - 2)$ and $(n - 1)$ degrees of freedom formed from those χ^2's. The joint distribution of the Q's has been considered and further research on various aspects of the problem is underway. (Work supported by A.R.S., U.S.D.A. and Q.M.R. and D., U.S. Army.)

59. Empirical Bayes Estimation, (Preliminary Report), M. V. JOHNS, JR., Columbia University.

Let $\mathbf{X} = (X_1, X_2, \cdots, X_r)$ where the X_i's are independent discrete valued random variables with a common c.d.f. $F(x \mid \lambda)$, and where there exists an a priori probability measure p over a σ-algebra of subsets of the values of the parameter λ, so that the parameter is also a random variable Λ. Suppose that it is desired to estimate $\theta(\lambda) = E(X_i \mid \Lambda = \lambda)$, using the risk function $E(\varphi(\mathbf{X}) - \theta(\Lambda))^2$, where $\varphi(\mathbf{x})$ is any estimator. Let the Bayes estimator (depending on p and $F(x \mid \lambda)$) be $\varphi^*(\mathbf{x}) = E(\theta(\Lambda) \mid \mathbf{X} = \mathbf{x})$. Suppose now that p and the form of F are unknown, but that n independent $(r + 1)$-component random vectors $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$, each having the same probability structure as $\mathbf{X}$, are available. Then a "nonparametric" estimator $\varphi_n(\mathbf{x}) = \varphi_n(\mathbf{x}; \mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n)$ is given having the property that $\lim_{n\to\infty} E(\varphi_n(\mathbf{X}) - \theta(\Lambda))^2 = E(\varphi^*(\mathbf{X}) - \theta(\Lambda))^2$ for any p and F subject to certain mild restrictions. The general case where the X_i's are not necessarily discrete valued is also considered. Similar results are obtained for several cases (considered by Robbins in the *Third Berkeley Symposium on Mathematical Statistics and Probability*) where p is unknown but F belongs to a specified one-parameter family of probability distributions and where the value of the parameter is to be estimated. The behavior of these empirical Bayes estimators is also investigated for finite n for certain special cases.