ON THE ESTIMATION OF REGRESSION COEFFICIENTS OF A VECTOR-VALUED TIME SERIES WITH A STATIONARY RESIDUAL[1]

By Murray Rosenblatt[2]

University of Chicago

1. Summary. Time series which are realizations of a vector-valued stochastic process of dimension two with a stationary disturbance are considered. Linear estimates of the regression coefficients of the time series are discussed, in particular the least-squares or classical estimate and the Markov estimate. The least-squares estimate is the estimate computed under the assumption that the components of the disturbance are orthogonal processes and orthogonal to each other. It is known that the Markov estimate is in general better than the least-squares estimate. The asymptotic behavior of the covariance matrices of the least-squares estimate and of the Markov estimate is described. Conditions under which the least-squares estimate is as good asymptotically as the Markov estimate are obtained, that is, conditions under which the least-squares estimate is efficient asymptotically in the class of linear unbiased estimates. The analogues of the results described for vector-valued time series of dimension greater than two can be seen to hold.

2. Introduction. The presentation of the results of this paper is carried out for the case of a two-dimensional process because of the greater simplicity and clarity in exposition. The general n-dimensional case is briefly discussed in Section 9. Let us consider *a two-dimensional complex-valued discrete parameter process*, that is, *a sequence of stochastic vectors*

$$y_t = \begin{pmatrix} {}_1y_t \\ {}_2y_t \end{pmatrix} = x_t + m_t = \begin{pmatrix} {}_1x_t \\ {}_2x_t \end{pmatrix} + \begin{pmatrix} {}_1m_t \\ {}_2m_t \end{pmatrix}, \quad t = \cdots, -1, 0, 1, \cdots, \tag{2.1}$$

where $m_t = Ey_t$ *is the mean value sequence and* $x_t = y_t - m_t$ *is the residual process.* We *introduce the covariance sequence* (x_t' *denotes the conjugated transpose of* x_t)

$$E(y_s - m_s)(y_t - m_t)' = Ex_sx_t' = \begin{pmatrix} E\,{}_1x_s\,{}_1\bar{x}_t & E\,{}_1x_s\,{}_2\bar{x}_t \\ E\,{}_2x_s\,{}_1\bar{x}_t & E\,{}_2x_s\,{}_2\bar{x}_t \end{pmatrix} = \begin{pmatrix} {}_{11}r_{s,t} & {}_{12}r_{s,t} \\ {}_{21}r_{s,t} & {}_{22}r_{s,t} \end{pmatrix} = r_{s,t}. \tag{2.2}$$

The assumption that the random variables are complex-valued is made for mathematical convenience. The real-valued case is, of course, the one of greatest

Received October 4, 1954.

[1] Research carried out at the Statistical Research Center, University of Chicago, under sponsorship of the Statistics Branch, Office of Naval Research.

[2] Now at Indiana University.

statistical interest and is discussed in Section 7 in some detail. Sections 2 and 3 are an extended discussion of the assumptions made in the paper and their motivation. *All the assumptions made in Sections 2 and 3* (except possibly for that of a real-valued time series) *will be held to in all sections except Section 8. The residual process x_t is said to be stationary in the wide sense if $r_{s,t} = r_{s-t}$, and I shall assume that this is the case.* Then the covariance sequence has the representation

$$r_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dF(\lambda), \tag{2.3}$$

where $F(\lambda)$ is a matrix-valued function

$$F(\lambda) = \begin{pmatrix} F_{11}(\lambda) & F_{12}(\lambda) \\ F_{21}(\lambda) & F_{22}(\lambda) \end{pmatrix} \tag{2.4}$$

that is nondecreasing; that is, $\Delta F(\lambda) \geqq 0$ (cf. [2]). *The functions $F_{11}(\lambda)$, $F_{22}(\lambda)$ are the spectral distribution functions of ${}_1x_t$, ${}_2x_t$, respectively, while $F_{12}(\lambda)$, $F_{21}(\lambda)$ are the cross-spectral distribution functions of the two coordinates of x_t. We assume that the spectrum is absolutely continuous; that is, that*

$$F_{ij}(\lambda) = \int_{-\pi}^{\lambda} f_{ij}(u)\, du, \qquad i, j = 1, 2, \tag{2.5}$$

and that the spectral densities $f_{ij}(\lambda)$ are continuous. The spectral densities $f_{ii}(\lambda)$, $i = 1, 2$, are assumed to be positive. Note that $f_{12}(\lambda) = \overline{f_{21}(\lambda)}$. The inequality

$$|f_{12}(\lambda)|^2 \leqq f_{11}(\lambda) f_{22}(\lambda) \tag{2.6}$$

obviously holds. *We shall assume that*

$$|f_{12}(\lambda)|^2 < f_{11}(\lambda) f_{22}(\lambda) \tag{2.7}$$

for all λ.

We shall refer to the set of spectra satisfying this set of conditions as the admissible set of spectra. The equality $|f_{12}(\lambda)|^2 \equiv f_{11}(\lambda)f_{22}(\lambda)$ for all λ amounts to a linear relationship between the two coordinates ${}_1x_t$, ${}_2x_t$ of the form ${}_1x_t = \sum_j c_j\, {}_2x_{t-j}$. If the processes ${}_ix_t$ are orthogonal processes, the spectral densities $f_{ii}(\lambda) \equiv \sigma_i^2/2\pi$, $i = 1, 2$. Such processes are sometimes referred to as "white noise." If the processes ${}_1x_t$, ${}_2x_t$ are orthogonal to each other, the cross-spectral density $f_{12}(\lambda) \equiv 0$.

In Section 7 we shall assume that the process x_t is a real process. This condition imposes additional restraints on the spectrum, specifically that

$$f_{ii}(\lambda) = f_{ii}(-\lambda), \qquad i = 1, 2, \tag{2.8}$$

and $f_{12}(\lambda) = f_{21}(-\lambda)$. If the process is real, the admissible class of spectra must satisfy these additional restraints.

Let the regression ${}_im_t$, $i = 1, 2$, be of the form

$${}_im_t = \sum_{\nu=1}^{p_i} {}_ic_\nu\, {}_i\varphi_t^{(\nu)}. \tag{2.9}$$

The problem posed is that of estimating the regression coefficients ${}_ic_\nu$ from a time series $y_1, \cdots, y_N$. The regression vectors

$${}_i\varphi^{(\nu)} = \begin{pmatrix} {}_i\varphi_1^{(\nu)} \\ \vdots \\ {}_i\varphi_N^{(\nu)} \end{pmatrix}$$

are assumed known. We are interested in unbiased estimates that are linear in the observations ${}_iy_t$, $i = 1, 2$; $t = 1, \cdots, N$. The two linear estimates that we are specifically interested in are the least-squares estimate and the Markov estimate. Let

$${}_im = \begin{pmatrix} {}_im_1 \\ \vdots \\ {}_im_N \end{pmatrix}, \qquad {}_iy = \begin{pmatrix} {}_iy_1 \\ \vdots \\ {}_iy_N \end{pmatrix}, \qquad i = 1, 2,$$

and

$$m = \begin{pmatrix} {}_1m \\ {}_2m \end{pmatrix}, \qquad y = \begin{pmatrix} {}_1y \\ {}_2y \end{pmatrix}.$$

Define the vectors ${}_ic$ and c by

$${}_ic = \begin{pmatrix} {}_ic_1 \\ \vdots \\ {}_ic_{p_i} \end{pmatrix}, \qquad i = 1, 2;$$

$$c = \begin{pmatrix} {}_1c \\ \\ {}_2c \end{pmatrix}.$$

Also define the matrices

$${}_i\Phi = ({}_i\varphi^{(1)}, \cdots, {}_i\varphi^{(p_i)}),$$

$$\Phi = \begin{pmatrix} {}_1\Phi & 0 \\ 0 & {}_2\Phi \end{pmatrix}.$$

The fact that m_t is the mean value of y_t can be written in vector form as

$$m = Ey = \Phi c. \tag{2.10}$$

The least-squares estimate c_L^* is the estimate that minimizes the quadratic form

$$(y - m)'(y - m) = (y - \Phi c)'(y - \Phi c);$$

that is, $c_L^* = (\Phi'\Phi)^{-1}\Phi' y$. Note that we are assuming that $\Phi'\Phi$ is nonsingular. The estimate c_L^* is unbiased

$$Ec_L^* = (\Phi'\Phi)^{-1}\Phi' Ey = (\Phi'\Phi)^{-1}\Phi'\Phi c = c \tag{2.11}$$

and the covariance matrix of c_L^* is

$$E(c_L^* - c)(c_L^* - c)' = (\Phi'\Phi)^{-1}\Phi' R\Phi(\Phi'\Phi)^{-1}.$$

The matrix R is the covariance matrix of the vector y. Our assumptions concerning the spectrum of the process x_t imply that the matrix R is nonsingular. *The Markov estimate*

$$c_M^* = (\Phi' R^{-1} \Phi)^{-1} \Phi' R^{-1} y.$$

It is also unbiased and its covariance matrix

$$E(c_M^* - c)(c_M^* - c)' = (\Phi' R^{-1} \Phi)^{-1}. \tag{2.12}$$

The Markov estimate is minimum variance among all linear unbiased estimates in the following sense. Consider any unbiased linear estimate $c^* = My$, $Ec^* = M\Phi c = c$; that is, $M\Phi = I$. Its covariance matrix

$$E(c^* - c)(c^* - c)' = MRM'.$$

One can then show that

$$MRM' \geqq (\Phi' R^{-1} \Phi)^{-1}.$$

These remarks about the least-squares and Markov estimates are well known.

We shall investigate the asymptotic behavior of the covariance matrices of the least-squares and the Markov estimate as $N \to \infty$. Note that the least-squares estimate is identical with the Markov estimate when the processes ${}_ix_t$ are orthogonal processes and are orthogonal to each other. It is of considerable interest to find out when the least-squares estimate is asymptotically as good as the Markov estimate, that is, when it is asymptotically efficient in the set of linear unbiased estimates. Whenever we use the phrase asymptotic efficiency we mean asymptotic efficiency in the class of linear unbiased estimates. The least-squares estimate is much easier to compute than the Markov estimate, since it does not require knowledge of the structure of the process x_t. Even if the structure of the process x_t is known, the computation of the inverse R^{-1} may be very tedious. We will discuss the question of asymptotic efficiency. These problems are discussed in [3], [4], and [5] for one-dimensional time series. New aspects of these problems arise in the multidimensional case that we discuss in this paper. The principal results of the paper are given in Sections 4, 5, 6, and 7.

The discussion is based on what might be called a generalized harmonic analysis of the regression vectors. In carrying out this analysis we will have to impose some conditions on the asymptotic behavior of these vectors. However, these conditions will be sufficiently broad to allow most of the usual types of regression sequences. The techniques used are similar to those employed in [5].

3. The regression spectrum. *Let* ${}_i\Phi_N^{(\nu)} = \sum_{t=1}^N | {}_i\varphi_t^{(\nu)} |^2$, $i = 1, 2$. *We first assume that* ${}_i\Phi_N^{(\nu)} \to \infty$ *as* $N \to \infty$. Some condition of this type is required if we are to be able to estimate c consistently. *We also require that*

$$\lim_{N \to \infty} {}_i\Phi_{N+h}^{(\nu)} \,/\, {}_i\Phi_N^{(\nu)} = 1 \tag{3.1}$$

for every fixed h. Let the limits

$$ {}_{ij}M_h^{(\nu,\mu)} = \lim_{n\to\infty} \sum_{t=1}^{N} \frac{{}_i\varphi_{t+h}^{(\nu)} \overline{{}_j\varphi_t^{(\mu)}}}{\sqrt{{}_i\Phi_N^{(\nu)}\, {}_j\Phi_N^{(\mu)}}}, \tag{3.2} $$

$i, j = 1, 2$; $\nu = 1, \cdots, p_i$; $\mu = 1, \cdots, p_j$, *exist for all* $h \geqq 0$. If we set ${}_i\varphi_t^{(\nu)} = 0$ for $t < 0$, it can be seen that the limits ${}_{ij}M_{-h}^{(\nu,\mu)}$, $h > 0$, exist and that

$$ {}_{ij}M_h^{(\nu,\mu)} = \overline{{}_{ji}M_h^{(\mu,\nu)}}. \tag{3.3} $$

Let the matrices

$$ {}_{ij}M_h = \{{}_{ij}M_h^{(\nu,\mu)};\ \nu = 1, \cdots, p_i,\ \mu = 1, \cdots, p_j\}, \quad i, j = 1, 2, \tag{3.4} $$

and

$$ M_h = \begin{pmatrix} {}_{11}M_h & {}_{12}M_h \\ {}_{21}M_h & {}_{22}M_h \end{pmatrix}. $$

The matrices M_h, $h = \cdots, -1, 0, 1, \cdots$, form a positive definite sequence; that is, given any $p_1 + p_2$ vector z and any finite vector a

$$ \sum_{\nu,\mu} \overline{a_\nu}\, z' M_{\nu-\mu} z a_\mu \geqq 0. $$

The matrix sequence M_h *then has the representation*

$$ M_h = \int_{-\pi}^{\pi} e^{ih\lambda}\, dM(\lambda), \qquad h = \cdots, -1, 0, 1, \cdots, \tag{3.5} $$

where $M(\lambda)$ *is a matrix-valued function that is nondecreasing so that* $\Delta M(\lambda) \geqq 0$ *for all* λ. *Note that if all the regression vectors are real, we have* $dM(\lambda) = \overline{dM(-\lambda)}$. It will be convenient at times to write $M(\lambda)$ in the form

$$ M(\lambda) = \begin{pmatrix} {}_{11}M(\lambda) & {}_{12}M(\lambda) \\ {}_{21}M(\lambda) & {}_{22}M(\lambda) \end{pmatrix}, \tag{3.6} $$

where ${}_{ij}M(\lambda)$ is a $p_i \times p_j$ matrix. It is clear that

$$ {}_{ij}M_h = \int_{-\pi}^{\pi} e^{ih\lambda}\, d\,{}_{ij}M(\lambda), \qquad i, j = 1, 2. \tag{3.7} $$

The matrix-valued functions ${}_{11}M(\lambda)$, ${}_{22}M(\lambda)$ *are nondecreasing and* ${}_{12}M(\lambda) = {}_{21}M(\lambda)'$. *Let*

$$ M_0 = M(\pi) - M(-\pi) = M \tag{3.8} $$

and

$$ {}_{ii}M_0 = {}_{ii}M(\pi) - {}_{ii}M(-\pi) = {}_iM, \qquad i = 1, 2. \tag{3.9} $$

We shall assume that $_1M$ and $_2M$ are nonsingular. This means that there is no vector

$$_ia = \begin{pmatrix} _ia_1 \\ \vdots \\ _ia_{p_1} \end{pmatrix} \neq 0$$

such that

$$\sum_{\nu=1}^{p_i} {_ia_\nu} \frac{_i\varphi^{(\nu)}}{_i\Phi_N^{(\nu)}} \to 0$$

as $N \to \infty$, $i = 1, 2$. Thus, vectors $_i\varphi^{(\nu)}$ are asymptotically linearly independent in the sense described above. Such a condition is required if we are to be able to estimate the regression coefficients consistently as $N \to \infty$. The conditions imposed on the regression vectors are sufficiently broad to include the case of polynomial or trigonometric regression or mixed polynomial and trigonometric regression.

The singular case in which the regression of one component $_2m_t \equiv 0$ and one wishes to estimate the regression coefficients $_1c_\nu$ of the regression of the other component does not satisfy the conditions imposed on the regression vectors above. It is, however, of some interest and we shall discuss it in Section 8.

Let the diagonal matrix

$$D_N = \begin{bmatrix} \sqrt{_1\Phi_N^{(1)}} & & & & & & & 0 \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \sqrt{_1\Phi_N^{(p_1)}} & & & & \\ & & & & \sqrt{_2\Phi_N^{(1)}} & & & \\ & & & & & \cdot & & \\ 0 & & & & & & \cdot & \\ & & & & & & & \sqrt{_2\Phi_N^{(p_2)}} \end{bmatrix} = \begin{pmatrix} _1D_N & 0 \\ 0 & _2D_N \end{pmatrix}.$$

We shall show that the limits of

$$D_N E(c_L^* - c)(c_L^* - c)' D_N = D_N(\Phi'\Phi)^{-1} D_N D_N^{-1} \Phi' R \Phi D_N^{-1} D_N (\Phi'\Phi)^{-1} D_N$$

and

$$D_N E(c_M^* - c)(c_M^* - c)' D_N = D_N(\Phi' R^{-1} \Phi)^{-1} D_N$$

exist as $N \to \infty$ and shall obtain expressions for these limits in terms of the spectrum of x_t and the regression spectrum (see Sections 4 and 5). It will be convenient for us to write

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{3.10}$$

where R_{11}, R_{22} *are the covariance matrices of* ${}_1y$, ${}_2y$ *respectively, while* $R_{12} = R'_{21}$ *is the cross-covariance matrix of* ${}_1y$ *with* ${}_2y$.

Note that the assumptions concerning the regression vectors imply that

$$\lim_{N\to\infty} D_N^{-1}(\Phi'\Phi)D_N^{-1} = \begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M \end{pmatrix}, \tag{3.11}$$

which is nonsingular.

4. The least-squares estimate.

THEOREM 1. *Under the assumptions made in Section* 2 *on the spectrum* $f(\lambda)$ *of the process* x_t *and the assumptions made in Section* 3 *on the regression of the process* y_t,

$$\lim_{N\to\infty} D_N E(c_L^* - c)(c_L^* - c)' D_N$$

$$= 2\pi \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix} \begin{bmatrix} \int_{-\pi}^{\pi} f_{11}(-\lambda)\, d\,{}_{11}M(\lambda) & \int_{-\pi}^{\pi} f_{12}(-\lambda)\, d\,{}_{12}M(\lambda) \\ \int_{-\pi}^{\pi} f_{21}(-\lambda)\, d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} f_{22}(-\lambda)\, d\,{}_{22}M(\lambda) \end{bmatrix} \times \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix}. \tag{4.1}$$

In discussing the asymptotic behavior of the covariance matrix of the least-squares estimate, it will clearly be enough to consider $D_N^{-1}\Phi' R\Phi D_N^{-1}$. We will approximate R above and below by positive definite matrices of a simpler form.

Consider the quadratic form

$$z'Rz = {}_1z'R_{11}z_1 + {}_2z'R_{21}z_1 + {}_1z'R_{12}z_2 + z_2'R_{22}z,$$

where $z = \begin{pmatrix} {}_1z \\ {}_2z \end{pmatrix}$ and ${}_1z$, ${}_2z$ are N-vectors, so as to see how to approximate R conveniently. Clearly

$$z'Rz = \int_{-\pi}^{\pi} |{}_1z(-\lambda)|^2 f_{11}(\lambda)\, d\lambda + \int_{-\pi}^{\pi} \overline{{}_2z(-\lambda)}\, f_{21}(\lambda)\, {}_1z(-\lambda)\, d\lambda$$

$$+ \int_{-\pi}^{\pi} \overline{{}_1z(-\lambda)}\, f_{12}(\lambda)\, {}_2z(-\lambda)\, d\lambda + \int_{-\pi}^{\pi} |{}_2z(-\lambda)|^2 f_{22}(\lambda)\, d\lambda,$$

where ${}_iz(\lambda) = \sum_{k=1}^{N} {}_iz_k e^{ik\lambda}$, $i = 1, 2$. Note that $z'Rz$ can be written in the more convenient form

$$z'Rz = \int_{-\pi}^{\pi} z(-\lambda)' f(\lambda)\, z(-\lambda)\, d\lambda,$$

where

$$z(\lambda) = \begin{pmatrix} {}_1z(\lambda) \\ {}_2z(\lambda) \end{pmatrix}, \qquad f(\lambda) = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}.$$

Now $|f(\lambda)|$ is nonsingular for all λ, since $f_{11}(\lambda)f_{22}(\lambda) > |f_{12}(\lambda)|^2$ for all λ

Let

$$\begin{pmatrix} a_1 & c_1 \\ \bar{c}_1 & b_1 \end{pmatrix}, \qquad \begin{pmatrix} a_2 & c_2 \\ \bar{c}_2 & b_2 \end{pmatrix}$$

be two positive definite 2×2 matrices. They are positive definite if and only if $a_i, b_i \geqq 0$ and $a_i b_i \geqq |c_i|^2$, $i = 1, 2$. Moreover,

$$\begin{pmatrix} a_1 & c_1 \\ \bar{c}_1 & b_1 \end{pmatrix} \geqq \begin{pmatrix} a_2 & c_2 \\ \bar{c}_2 & b_2 \end{pmatrix}$$

if and only if $a_1 \geqq a_2$, $b_1 \geqq b_2$, and $(a_1 - a_2)(b_1 - b_2) - |c_1 - c_2|^2 \geqq 0$. Now $f_{11}(\lambda), f_{22}(\lambda)$ are positive continuous functions and inequality (2.7) holds. Given any $\epsilon > 0$, we can find finite trigonometric polynomials

$$g_{ij}(\lambda) = \sum_{k=-q}^{q} {}_{ij}g_k \, e^{ik\lambda}, \tag{4.2}$$

$$h_{ij}(\lambda) = \sum_{k=-q}^{q} {}_{ij}h_k \, e^{ik\lambda}, \tag{4.3}$$

$i, j = 1, 2$, satisfying the inequalities

$$\begin{gathered} g_{11}(\lambda) \geqq f_{11}(\lambda) \geqq h_{11}(\lambda) > 0, \\ g_{22}(\lambda) \geqq f_{22}(\lambda) \geqq h_{22}(\lambda) > 0, \\ g_{11}(\lambda)g_{22}(\lambda) > |g_{12}(\lambda)|^2, \\ h_{11}(\lambda)h_{22}(\lambda) > |h_{12}(\lambda)|^2, \\ \epsilon > (g_{11}(\lambda) - f_{11}(\lambda))(g_{22}(\lambda) - f_{22}(\lambda)) > |g_{12}(\lambda) - f_{12}(\lambda)|^2, \\ \epsilon > (f_{11}(\lambda) - h_{11}(\lambda))(f_{22}(\lambda) - h_{22}(\lambda)) > |f_{12}(\lambda) - h_{12}(\lambda)|^2, \\ |g_{ii}(\lambda) - h_{ii}(\lambda)| < \epsilon, \end{gathered}$$

$i = 1, 2$, for all λ. Let

$$\begin{aligned} G_{ij} &= \{ {}_{ij}g_{k-l} \, ; k, l = 1, \cdots, N \}, \\ H_{ij} &= \{ {}_{ij}h_{k-l} \, ; k, l = 1, \cdots, N \}, \end{aligned} \qquad i, j = 1, 2,$$

and

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \qquad H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

Let

$$g(\lambda) = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \qquad h(\lambda) = \begin{pmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{pmatrix}.$$

Now

$$\begin{aligned} z'\,Gz &= \int_{-\pi}^{\pi} z(-\lambda)'\,g(\lambda)\,z(-\lambda)\,d\lambda \geqq z'\,Rz \\ &= \int_{-\pi}^{\pi} z(-\lambda)'\,f(\lambda)\,z(-\lambda)\,d\lambda \geqq z'\,Hz \\ &= \int_{-\pi}^{\pi} z(-\lambda)'\,h(\lambda)\,z(-\lambda)\,d\lambda, \end{aligned}$$

so that

$$G \geqq R \geqq H. \tag{4.4}$$

Clearly, $D_N^{-1}\Phi' G\Phi D_N^{-1} \geqq D_N^{-1}\Phi' R\Phi D_N^{-1} \geqq D_N^{-1}\Phi' H\Phi D_N^{-1}$. We shall obtain the limit of $D_N^{-1}\Phi' G\Phi D_N^{-1}$ as $N \to \infty$. This matrix is easier to deal with, since ${}_{ij}g_{k-l} = 0$ if $|k - l| > q$. A typical element of the matrix in question is

$$\sum_{t,\tau=1}^{N} \frac{\overline{{}_i\varphi_t^{(\nu)}}\;{}_{ij}g_{t-\tau}\;{}_j\varphi_\tau^{(\mu)}}{\sqrt{{}_i\Phi_N^{(\nu)}\;{}_j\Phi_N^{(\mu)}}} = \sum_{k=0}^{q} {}_{ij}g_k \sum_{\tau=1}^{N-k} \frac{\overline{{}_i\varphi_{t+k}^{(\nu)}}\;{}_j\varphi_\tau^{(\mu)}}{\sqrt{{}_i\Phi_N^{(\nu)}\;{}_j\Phi_N^{(\mu)}}} + \sum_{k=-q}^{-1} {}_{ij}g_k \sum_{\tau=1-k}^{N} \frac{\overline{{}_i\varphi_{\tau+k}^{(\nu)}}\;{}_j\varphi_\tau^{(\mu)}}{\sqrt{{}_i\Phi_N^{(\nu)}\;{}_j\Phi_N^{(\mu)}}},$$

which approaches

$$\sum_k {}_{ij}g_k\;{}_{ij}M_k^{(\nu,\mu)} = 2\pi \int_{-\pi}^{\pi} g_{ij}(-\lambda)\,d\,{}_{ij}M^{(\nu,\mu)}(\lambda)$$

as $N \to \infty$, so that

$$\lim_{N\to\infty} D_N^{-1}\Phi'\,G\Phi\,D_N^{-1} = 2\pi \begin{bmatrix} \int_{-\pi}^{\pi} g_{11}(-\lambda)\,d\,{}_{11}M(\lambda) & \int_{-\pi}^{\pi} g_{12}(-\lambda)\,d\,{}_{12}M(\lambda) \\ \int_{-\pi}^{\pi} g_{21}(-\lambda)\,d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} g_{22}(-\lambda)\,d\,{}_{22}M(\lambda) \end{bmatrix}.$$

In like manner, one can show that

$$\lim_{N\to\infty} D_N^{-1}\Phi'\,H\Phi\,D_N^{-1} = 2\pi \begin{bmatrix} \int_{-\pi}^{\pi} h_{11}(-\lambda)\,d\,{}_{11}M(\lambda) & \int_{-\pi}^{\pi} h_{12}(-\lambda)\,d\,{}_{12}M(\lambda) \\ \int_{-\pi}^{\pi} h_{21}(-\lambda)\,d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} h_{22}(-\lambda)\,d\,{}_{22}M(\lambda) \end{bmatrix}.$$

Making use of the inequality (4.4), on letting $\epsilon \to 0$ we see that

$$\lim_{N\to\infty} D_N^{-1}\Phi'\,R\Phi D_N^{-1} = 2\pi \begin{bmatrix} \int_{-\pi}^{\pi} f_{11}(-\lambda)\,d\,{}_{11}M(\lambda) & \int_{-\pi}^{\pi} f_{12}(-\lambda)\,d\,{}_{12}M(\lambda) \\ \int_{-\pi}^{\pi} f_{21}(-\lambda)\,d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} f_{22}(-\lambda)\,d\,{}_{22}M(\lambda) \end{bmatrix}. \tag{4.5}$$

This limiting matrix will be shown to be nonsingular in Section 5. *Thus* (4.1) is valid.

5. The Markov estimate.

THEOREM 2. Under the assumptions made in Section 2 on the spectrum $f(\lambda)$ of the process x_t and the assumptions made in Section 3 on the regression of y_t,

$$\lim_{N\to\infty} D_N E(c_M^* - c)(c_M^* - c)' D_N \tag{5.1}$$

$$= 2\pi \begin{bmatrix} \int_{-\pi}^{\pi} \frac{f_{22}(-\lambda)\, d\, {}_{11}M(\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2} & -\int_{-\pi}^{\pi} \frac{f_{12}(-\lambda)\, d\, {}_{12}M(\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2} \\ -\int_{-\pi}^{\pi} \frac{f_{21}(-\lambda)\, d\, {}_{21}M(\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2} & \int_{-\pi}^{\pi} \frac{f_{11}(-\lambda)\, d\, {}_{22}M(\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2} \end{bmatrix}^{-1}$$

In discussing the Markov estimate it will be enough to consider $D_N^{-1}\Phi' R^{-1}\Phi D_N^{-1}$. We shall again approximate R above and below by positive definite matrices of a simpler form. Here we will approximate $f_{11}(\lambda), f_{22}(\lambda)$ by the absolute square of reciprocals of finite trigonometric polynomials, while $f_{12}(\lambda)$ will be approximated as before by a finite trigonometric polynomial. Given any $\epsilon > 0$, we can find finite trigonometric polynomials

$$\begin{aligned} \alpha_i(\lambda) &= \sum_{k=-q}^{q} {}_i\alpha_k\, e^{-ik\lambda}, \\ \beta_i(\lambda) &= \sum_{k=-q}^{q} {}_i\beta_k\, e^{-ik\lambda}, \\ \gamma_i(\lambda) &= \sum_{k=-q}^{q} {}_i\gamma_k\, e^{-ik\lambda}, \end{aligned} \tag{5.2}$$

$i = 1, 2,$ satisfying the inequalities

$$\begin{aligned} &|\alpha_1(\lambda)|^{-2} \geqq f_{11}(\lambda) \geqq |\alpha_2(\lambda)|^{-2}, \\ &|\beta_1(\lambda)|^{-2} \geqq f_{22}(\lambda) \geqq |\beta_2(\lambda)|^{-2}, \\ &|\alpha_1(\lambda)\beta_1(\lambda)|^{-2} > |\gamma_1(\lambda)|^2, \\ &|\alpha_2(\lambda)\beta_2(\lambda)|^{-2} > |\gamma_2(\lambda)|^2, \\ &\epsilon > (|\alpha_1(\lambda)|^{-2} - f_{11}(\lambda))(|\beta_1(\lambda)|^{-2} - f_{22}(\lambda)) > |\gamma_1(\lambda) - f_{12}(\lambda)|^2, \\ &\epsilon > (f_{11}(\lambda) - |\alpha_2(\lambda)|^{-2})(f_{22}(\lambda) - |\beta_2(\lambda)|^{-2}) > |\gamma_2(\lambda) - f_{12}(\lambda)|^2, \\ &|\alpha_1(\lambda)|^{-2} - |\alpha_2(\lambda)|^{-2} < \epsilon, \\ &|\beta_1(\lambda)|^{-2} - |\beta_2(\lambda)|^{-2} < \epsilon \end{aligned} \tag{5.3}$$

for all λ. Let

$${}_iR = \begin{pmatrix} {}_iR_{11} & {}_iR_{12} \\ {}_iR_{21} & {}_iR_{22} \end{pmatrix}$$

be the covariance matrix of y when the process x_t is such that $|\alpha_i(\lambda)|^{-2}$, $|\beta_i(\lambda)|^{-2}$ are the spectral densities of the components ${}_1x_t$, ${}_2x_t$, respectively while $\gamma_i(\lambda)$ is the cross-spectral density of ${}_1x_t$ and ${}_2x_t$, $i = 1, 2$. It is clear that ${}_1R$ and ${}_2R$ are nonsingular and that

$$(5.4) \qquad {}_1R \geqq R \geqq {}_2R.$$

For the moment let us assume that R is of the same form as one of the matrices R. Then

$$R_{11} = \Delta^{-1}\Delta'^{-1}, \qquad R_{22} = w^{-1}w'^{-1},$$

where $\Delta_{ij} = \alpha_{i-j}$ and $w_{ij} = \beta_{i-j}$ unless $i, j \leqq q$ or

$$N - i, N - j \leqq q \; (\alpha_k = 0, \beta_k = 0, \gamma_k = 0 \text{ if } |k| > q).$$

It is also clear that the (i, j)th element of R_{12} is γ_{i-j}. Let $P = \Delta R_{12} w'$. Then

$$R = \begin{pmatrix} \Delta^{-1}\Delta'^{-1} & \Delta^{-1}Pw'^{-1} \\ w^{-1}P'\Delta'^{-1} & w^{-1}w'^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & w^{-1} \end{pmatrix}\begin{pmatrix} I & P \\ P' & I \end{pmatrix}\begin{pmatrix} \Delta'^{-1} & 0 \\ 0 & w'^{-1} \end{pmatrix}.$$

It is clear that both

$$\begin{pmatrix} I & P \\ P' & I \end{pmatrix} \text{ and } \begin{pmatrix} I & -P \\ -P' & I \end{pmatrix}$$

are nonnegative definite. Moreover, they commute. Since the matrices commute, it is clear that their product is nonnegative definite and that $I - PP'$, $I - P'P \geqq 0$.

We would like to show that the maximal eigenvalue λ_m of either PP' or $P'P$ is less than one and bounded away from one as $N \to \infty$; that is, $\lambda_m < 1 - \epsilon$ as $N \to \infty$ for some $\epsilon > 0$. This can be seen by noting that the minimal eigenvalues of both

$$\begin{pmatrix} I & P \\ P' & I \end{pmatrix} \text{ and } \begin{pmatrix} I & -P \\ -P' & I \end{pmatrix}$$

are bounded away from zero as $N \to \infty$. It will be enough to show this for

$$\begin{pmatrix} I & P \\ P' & I \end{pmatrix}$$

Let u be any $2N$ vector. Then

$$u'\begin{pmatrix} I & P \\ P' & I \end{pmatrix}u = v'\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}v \geqq \epsilon v'v$$
$$= \epsilon u'\begin{pmatrix} R_{11}^{-1} & 0 \\ 0 & R_{22}^{-1} \end{pmatrix}u = \epsilon\epsilon' u'u$$

as $N \to \infty$, where $\epsilon, \epsilon' > 0$ and $v' = u' \begin{pmatrix} \Delta & 0 \\ 0 & w \end{pmatrix}$. Now

$$R^{-1} = \begin{pmatrix} \Delta' & 0 \\ 0 & w' \end{pmatrix} \begin{pmatrix} I & P \\ P' & I \end{pmatrix}^{-1} \begin{pmatrix} \Delta & 0 \\ 0 & w \end{pmatrix}.$$

But

$$\begin{pmatrix} I & P \\ P' & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - PP')^{-1} & -P(I - P'P)^{-1} \\ -P'(I - PP')^{-1} & (I - P'P)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} I & -P \\ -P' & I \end{pmatrix} \begin{pmatrix} (I - PP')^{-1} & 0 \\ 0 & (I - P'P)^{-1} \end{pmatrix}.$$

We can write

$$\begin{pmatrix} (I - PP')^{-1} & 0 \\ 0 & (I - P'P)^{-1} \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} (PP')^k & 0 \\ 0 & (P'P)^k \end{pmatrix}, \tag{5.5}$$

(5.6) since $0 \leqq PP', P'P \leqq (1 - \epsilon)I$. But then

$D_N^{-1}\Phi' R^{-1} \Phi D_N^{-1}$

$$= \sum_{k=0}^{\infty} \begin{pmatrix} {}_1D_N^{-1}\,{}_1\Phi'\Delta'(PP')^k\Delta\,{}_1\Phi\,{}_1D_N^{-1} & -{}_1D_N^{-1}\,{}_1\Phi'\Delta'P(P'P)^k w\,{}_2\Phi\,{}_2D_N^{-1} \\ -{}_2D_N^{-1}\,{}_2\Phi'w'P'(PP')^k\Delta\,{}_1\Phi\,{}_1D_N^{-1} & {}_2D_N^{-1}\,{}_2\Phi'w'(P'P)^k w\,{}_2\Phi\,{}_2D_N^{-1} \end{pmatrix}$$

$$= \sum_{k=0}^{\infty} Q_k.$$

Now the (ν, μ)th element of ${}_1D_N^{-1}\,{}_1\Phi'\Delta'P(P'P)^k w\,{}_2\Phi\,{}_2D_N^{-1}$ is

$$\sum_{t,\tau=1}^{N} \frac{\overline{{}_1\varphi_t^{(\nu)}}\, h_{t,\tau}\, {}_2\varphi_\tau^{(\mu)}}{\sqrt{{}_1\Phi_N^{(\nu)}\, {}_2\Phi_N^{(\mu)}}},$$

where

$$h_{t,\tau} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t-\tau)\lambda} \gamma(\lambda)\, |\alpha(\lambda)|^{2k+2}\, |\beta(\lambda)|^{2k+2}\, |\gamma(\lambda)|^{2k}\, d\lambda$$

$$= s_{t-\tau}$$

unless $t, \tau \leqq 12(k + 1)q$ or $N - t, N - \tau \leqq 12(k + 1)q$. Note that $s_j = 0$ if $|j| > 12(k + 1)q$. It is clear that each of the terms

$$\left| \frac{\overline{{}_1\varphi_t^{(\nu)}}\, h_{t,\tau}\, {}_2\varphi_\tau^{(\mu)}}{\sqrt{{}_1\Phi_N^{(\nu)}\, {}_2\Phi_N^{(\mu)}}} \right|$$

approaches zero as $N \to \infty$, since $|h_{t,\tau}|$ is uniformly bounded in t, τ, N and

$$|{}_1\varphi_t^{(\nu)}|^2/{}_1\Phi_N^{(\nu)} \to 0$$

as $N \to \infty$ for fixed t. But then

$$\lim_{N\to\infty} \sum_{t,\tau=1}^{N} \frac{\overline{{}_1\varphi_t^{(\nu)}}\, h_{t,\tau}\, {}_2\varphi_\tau^{(\mu)}}{\sqrt{{}_1\Phi_N^{(\nu)}\, {}_2\Phi_N^{(\mu)}}}$$

$$= \sum_{|k| \leqq 12(k+1)q} s_k \lim_{N\to\infty} \sum_{\tau=1}^{N} \frac{\overline{{}_1\varphi_{\tau+k}^{(\nu)}}\, {}_2\varphi_\tau^{(\mu)}}{\sqrt{{}_1\Phi_N^{(\nu)}\, {}_2\Phi_N^{(\mu)}}}$$

$$= \sum_k s_k\, {}_{12}M_k^{(\nu,\mu)} = 2\pi \int_{-\pi}^{\pi} \gamma(-\lambda)\, |\alpha(-\lambda)|^{2k+2}\, |\beta(-\lambda)|^{2k+2}\, |\gamma(-\lambda)|^{2k} \times d\, {}_{12}M^{(\nu,\mu)}(\lambda).$$

But then

$$\lim_{N\to\infty} {}_1D_N^{-1}\, {}_1\Phi' \Delta P (P'P)^k w\, {}_2\Phi\, {}_2D_N^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma(-\lambda)\, |\alpha(-\lambda)|^{2k+2}\, |\beta(-\lambda)|^{2k+2}\, |\gamma(-\lambda)|^{2k}\, d\, {}_{12}M(\lambda).$$

Similarly, one can show that

$$\lim_{N\to\infty} {}_1D_N^{-1}\, {}_1\Phi' \Delta' (PP')^k \Delta\, {}_1\Phi\, {}_1D_N^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\alpha(-\lambda)|^{2k+2}\, |\beta(-\lambda)|^{2k}\, |\gamma(-\lambda)|^{2k}\, d\, {}_{11}M(\lambda)$$

and

$$\lim_{N\to\infty} {}_2D_N^{-1}\, {}_2\Phi' w' (P'P)^k w\, {}_2\Phi\, {}_2D_N^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\alpha(-\lambda)|^{2k}\, |\beta(-\lambda)|^{2k+2}\, |\gamma(-\lambda)|^{2k}\, d\, {}_{22}M(\lambda).$$

Making use of (5.6), it then follows that

$$\lim_{N\to\infty} D_N^{-1} \Phi' R^{-1} \Phi D_N^{-1} = \lim_{N\to\infty} \sum_{k=0}^{\infty} Q_k = \sum_{k=0}^{\infty} \lim_{N\to\infty} Q_k \tag{5.7}$$

$$= \frac{1}{2\pi} \begin{bmatrix} \displaystyle\int_{-\pi}^{\pi} \frac{|\alpha(-\lambda)|^2}{1 - |\alpha(-\lambda)|^2 |\beta(-\lambda)|^2 |\gamma(-\lambda)|^2}\, d\, {}_{11}M(\lambda) & \displaystyle\int_{-\pi}^{\pi} \frac{\gamma(-\lambda)\, |\alpha(-\lambda)|^2\, |\beta(-\lambda)|^2}{1 - |\alpha(-\lambda)|^2 |\beta(-\lambda)|^2 |\gamma(-\lambda)|^2}\, d\, {}_{12}M(\lambda) \\ \displaystyle\int_{-\pi}^{\pi} \frac{\overline{\gamma(-\lambda)}\, |\alpha(-\lambda)|^2\, |\beta(-\lambda)|^2}{1 - |\alpha(-\lambda)|^2 |\beta(-\lambda)|^2 |\gamma(-\lambda)|^2}\, d\, {}_{21}M(\lambda) & \displaystyle\int_{-\pi}^{\pi} \frac{|\beta(-\lambda)|^2}{1 - |\alpha(-\lambda)|^2 |\beta(-\lambda)|^2 |\gamma(-\lambda)|^2}\, d\, {}_{22}M(\lambda) \end{bmatrix}.$$

We have thus found $\lim_{N\to\infty} D_N^{-1}\Phi' R^{-1}\Phi D_N^{-1}$ when R is of the same form as one of the matrices ${}_iR$. Let R now be of the general form, and approximate above and below by ${}_1R$ and ${}_2R$, respectively. On letting $\epsilon \to 0$, we see that

$$\lim_{N\to\infty} D_N^{-1}\Phi' R^{-1}\Phi D_N^{-1}$$

$$(5.8) \qquad = \frac{1}{2\pi}\begin{bmatrix} \int_{-\pi}^{\pi} \frac{f_{22}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{11}M(\lambda) & -\int_{-\pi}^{\pi} \frac{f_{12}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{12}M(\lambda) \\ -\int_{-\pi}^{\pi} \frac{f_{21}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} \frac{f_{11}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{22}M(\lambda) \end{bmatrix}.$$

We can show that the matrix (5.8) is nonsingular. Clearly

$$\begin{pmatrix} \Delta\,{}_{11}M(\lambda) & \Delta M_{12}(\lambda) \\ \Delta\,{}_{21}M(\lambda) & \Delta\,{}_{22}M(\lambda) \end{pmatrix} \geqq 0$$

and $f_{11}(\lambda)\, f_{22}(\lambda) > |f_{12}(\lambda)|^2$ for all λ. Now

$$f_{22}(-\lambda)z_1'\Delta\,{}_{11}M(\lambda)z_1 - f_{12}(-\lambda)z_1'\Delta\,{}_{12}M(\lambda)z_2 - f_{21}(-\lambda)z_2'\Delta\,{}_{21}M(\lambda)z_1 + f_{11}(-\lambda)z_2'\Delta\,{}_{22}M(\lambda)z_2$$

$$\geqq \left(f_{22}(-\lambda) - \frac{|f_{12}(-\lambda)|^2}{f_{11}(-\lambda)}\right) z_1'\Delta\,{}_{11}M(\lambda)z_1, \left(f_{11}(-\lambda) - \frac{|f_{12}(-\lambda)|^2}{f_{22}(-\lambda)}\right) z_2'\Delta\,{}_{22}M(\lambda)z_2,$$

so that

$$\frac{1}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\begin{pmatrix} f_{22}(-\lambda)\Delta\,{}_{11}M(\lambda) & -f_{12}(-\lambda)\Delta\,{}_{12}M(\lambda) \\ -f_{21}(-\lambda)\Delta\,{}_{21}M(\lambda) & f_{11}(-\lambda)\Delta\,{}_{22}M(\lambda) \end{pmatrix} \geqq \begin{pmatrix} \Delta\,{}_{11}M(\lambda) & 0 \\ 0 & \Delta\,{}_{22}M(\lambda) \end{pmatrix},$$

where $\epsilon > 0$. But then matrix (5.8) is greater than or equal to

$$\frac{1}{2\pi}\epsilon\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

and hence is nonsingular. Relation (5.1) is valid. It follows that the limiting matrix (4.5) is also nonsingular.

6. Asymptotic efficiency of least-squares estimate among linear unbiased estimates when observed process is complex-valued. We want to find out for what types of regression the least-squares estimate is asymptotically efficient among the linear unbiased estimates for any admissible spectral density matrix $f(\lambda)$. This amounts to asking for the conditions on $M(\lambda)$ such that matrix (4.1) is equal to matrix (5.1); that is,

$$(6.1)\quad \begin{bmatrix} \int_{-\pi}^{\pi} f_{11}(-\lambda)\, d\,{}_{11}M(\lambda) & \int_{-\pi}^{\pi} f_{12}(-\lambda)\, d\,{}_{12}M(\lambda) \\ \int_{-\pi}^{\pi} f_{21}(-\lambda)\, d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} f_{22}(-\lambda)\, d\,{}_{22}M(\lambda) \end{bmatrix} \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix} \times$$

$$\begin{bmatrix} \int_{-\pi}^{\pi} \frac{f_{22}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{11}M(\lambda) & -\int_{-\pi}^{\pi} \frac{f_{12}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{12}M(\lambda) \\ -\int_{-\pi}^{\pi} \frac{f_{21}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{21}M(\lambda) & \int_{-\pi}^{\pi} \frac{f_{11}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\,{}_{22}M(\lambda) \end{bmatrix} = \begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M \end{pmatrix}.$$

Let us first see what restraints are imposed on the regression spectrum if we require asymptotic efficiency of the least-squares estimate in the smaller class of spectra $f(\lambda)$ where there is no cross-correlation, that is, where $f_{12}(\lambda) \equiv 0$. Since ${}_1y_t$ and ${}_2y_t$ are uncorrelated, they can be treated separately. We make use of the results of [5] where the problem of estimating the regression coefficients of a 1-dimensional process with stationary residuals is discussed. The following restraints on the regression spectrum follow immediately from these results. The nondecreasing function ${}_{ii}M(\lambda)$ increases only on a finite set of points ${}_i\lambda_j,\ j = 1, \cdots, q_i$, where $q_i \leqq p_i,\ i = 1, 2$. The jump of ${}_{ii}M(\lambda)$ at ${}_i\lambda_j$ is $\Delta\,{}_{ii}M({}_i\lambda_j) = {}_{ii}M({}_i\lambda_j+) - {}_{ii}M({}_i\lambda_j-) > 0$ and

$$(6.2)\qquad \Delta\,{}_{ii}M({}_i\lambda_j)\,{}_iM^{-1}\Delta\,{}_{ii}M({}_i\lambda_k) = \delta_{jk}\Delta\,{}_{ii}M({}_i\lambda_j), \qquad i = 1, 2.$$

The sum of the ranks of the matrices $\Delta\,{}_{ii}M({}_i\lambda_j),\ j = 1, \cdots, q_i$, is p_i. It is then clear that the set of points of increase of the nondecreasing function $M(\lambda)$ is the set of points $\{\lambda_j\}$ consisting of the points ${}_1\lambda_k,\ k = 1, \cdots, q_1$, and ${}_2\lambda_k$, $k = 1, \cdots, q_2$. For convenience let ${}_{ij}M_k = {}_{ij}M(\lambda_k+) - {}_{ij}M(\lambda_k-)$. Relations (6.2) can then be rewritten ${}_{ii}M_j\,{}_iM^{-1}{}_{ii}M_k = \delta_{jk}\,{}_{ii}M_j,\ i = 1, 2$. Here either ${}_{11}M_j > 0$ or ${}_{22}M_j > 0$. We shall obtain additional restraints on the regression spectrum and thereby show that the sets of points $\{\lambda_j\}$, $\{{}_1\lambda_j\}$ and $\{{}_2\lambda_j\}$ are the same.

Let us now see what additional restraints on the regression spectrum are implied by asymptotic efficiency of the least-squares estimate when the spectrum is such that $f_{11}(\lambda) \equiv f_{22}(\lambda) \equiv 1$ and $f_{12}(-\lambda_j) = \alpha_j$, $|\alpha_j|^2 < 1$. The condition for asymptotic efficiency is then

$$\sum_{j,k} \begin{pmatrix} {}_{11}M_j & \alpha_j\, {}_{12}M_j \\ \bar{\alpha}_j\, {}_{21}M_j & {}_{22}M_j \end{pmatrix} \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix} \frac{1}{1 - |\alpha_k|^2} \begin{pmatrix} {}_{11}M_k & -\alpha_k\, {}_{12}M_k \\ -\bar{\alpha}_k\, {}_{21}M_k & {}_{22}M_k \end{pmatrix} = \begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M \end{pmatrix}. \tag{6.3}$$

If all the α's are zero except for one, say α_j, equation (6.3) reduces to

$$\begin{pmatrix} {}_{11}M_j & -\alpha_j\, {}_{12}M_j \\ -\bar{\alpha}_j\, {}_{21}M_j & {}_{22}M_j \end{pmatrix} - \begin{pmatrix} {}_{12}M_j\, {}_2M^{-1}\, {}_{21}M_j & -\alpha_j\, {}_{12}M_j\, {}_2M^{-1}\, {}_{22}M_j \\ -\bar{\alpha}_j\, {}_{21}M_j\, {}_1M^{-1}\, {}_{11}M_j & {}_{21}M_j\, {}_1M^{-1}\, {}_{22}M_j \end{pmatrix} = 0,$$

so that ${}_{ik}M_j\, {}_kM^{-1}\, {}_{kl}M_j = {}_{il}M_j$, $i, k, l = 1, 2$, $i \neq k$. If all the α's are zero except for two, say α_j and α_k, equation (6.3) reduces to

$$\begin{aligned} &\frac{1}{1 - |\alpha_k|^2} \begin{pmatrix} 0 & \alpha_j\, {}_{12}M_j \\ \bar{\alpha}_j\, {}_{21}M_j & 0 \end{pmatrix} \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix} \begin{pmatrix} |\alpha_k|^2\, {}_{11}M_k & -\alpha_k\, {}_{12}M_k \\ -\bar{\alpha}_k\, {}_{21}M_k & |\alpha_k|^2\, {}_{22}M_k \end{pmatrix} \\ &+ \frac{1}{1 - |\alpha_j|^2} \begin{pmatrix} 0 & \alpha_k\, {}_{12}M_k \\ \bar{\alpha}_k\, {}_{21}M_k & 0 \end{pmatrix} \begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1} \end{pmatrix} \begin{pmatrix} |\alpha_j|^2\, {}_{11}M_j & -\alpha_j\, {}_{12}M_j \\ -\bar{\alpha}_j\, {}_{21}M_j & |\alpha_j|^2\, {}_{22}M_j \end{pmatrix} = 0. \end{aligned} \tag{6.4}$$

But equation (6.4) cannot hold for all values of α_j, α_k less than one in absolute value unless

$${}_{il}M_j\, {}_lM^{-1}\, {}_{ls}M_k = 0, \quad j \neq k; \quad i, l, s = 1, 2; \quad i \neq l.$$

All other relations of this type can be obtained analogously from the matrix equation resulting from the interchange of the first and third matrix on the left side of equation (6.3). This equation obviously also holds if the least-squares estimate is asymptotically efficient. All the restraints on the matrices ${}_{ij}M_k$ can be written briefly

$${}_{ij}M_k\, {}_jM^{-1}\, {}_{jl}M_s = \delta_{ks}\, {}_{il}M_s, \tag{6.5}$$

where $i, j, l = 1, 2$ and $k, s = 1, \cdots, q$ where $q = q_1 = q_2$. It is clear that equations (6.5) cannot hold unless both ${}_{11}M_k$, ${}_{22}M_k > 0$ and have the same rank. Thus asymptotic efficiency of the least-squares estimate implies that $p_1 = p_2$.

THEOREM 3. *The following conditions are necessary if the least-squares estimate is to be asymptotically efficient for all admissible* $f(\lambda)$. *The function* $M(\lambda)$ *is a jump unction with a finite number of jumps* $\lambda_1, \cdots, \lambda_q$, *where* $q \leqq p = p_1 = p_2$. *Let the jumps be*

$${}_{ij}M_k = {}_{ij}M(\lambda_k +) - {}_{ij}M(\lambda_k -), \qquad i, j = 1, 2.$$

Then ($A > 0$ if A is a nonnegative definite matrix but not the null matrix) ${}_{ii}M_k > 0$,[2] $i = 1, 2$ and $k = 1, \cdots, q$ *and*

$${}_{ij}M_k\, {}_jM^{-1}\, {}_{jl}M_s = \delta_{ks}\, {}_{il}M_s, \tag{6.6}$$

where $i, j, l = 1, 2$ and $k, s = 1, \cdots, q$. ${}_{11}M_k$ and ${}_{22}M_k$ have the same rank. The sum of the ranks of ${}_{ii}M_k$, $k = 1, \cdots, q$, is p, $i = 1, 2$. These conditions can easily be seen to be sufficient for the least-squares estimate to be asymptotically efficient for all admissible $f(\lambda)$.

It is of especial interest to consider the case in which both components ${}_1y_t$, ${}_2y_t$ of the observed process have a mixed trigonometric and polynomial regression and the regression vectors of both components are the same; that is,

$$
\begin{aligned}
{}_1\varphi_t^{(\nu)} &= {}_2\varphi_t^{(\nu)} = t^\nu e^{-it\lambda_1}, && \nu = 0, 1, \cdots, s_1,\\
{}_1\varphi_t^{(s_1+1+\nu)} &= {}_2\varphi_t^{(s_1+1+\nu)} = t^\nu e^{-it\lambda_2}, && \nu = 0, 1, \cdots, s_2,\\
{}_1\varphi_t^{(s_1+\cdots+s_{u-1}+u-1+\nu)} &= {}_2\varphi_t^{(s_1+\cdots+s_{u-1}+u-1+\nu)} \\
&= t^\nu e^{-it\lambda_q}, && \nu = 0, 1, \cdots, s_q;
\end{aligned}
$$

where $\lambda_1, \cdots, \lambda_q$ are distinct. *The least-squares estimate can be seen to be asymptotically efficient in the case of such a regression.* The jumps of $M(\lambda)$ are at $\lambda_1, \cdots, \lambda_q$ and

$$
{}_{ij}M_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_k & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

where

$$
M_k = \left\{ \frac{\sqrt{(2\mu + 1)(2\nu + 1)}}{\mu + \nu + 1};\ \mu, \nu = 0, 1, \cdots, s_k \right\}
$$

and the null submatrix in the upper left-hand corner of ${}_{ij}M_k$ is of order $\sum_{i=1}^{k-1} (s_i + 1)$. It is clear that equations (5.6) are satisfied in this case. *One should note that if the regression vectors of the two components are unequal in number, the least-squares estimate is not asymptotically efficient.* This, for example, would be the case if one component had a linear regression and the other a quadratic regression.

7. Asymptotic efficiency of the least squares estimate when the observed process is real-valued. The case of greatest interest is that in which the process y_t and the regression vectors are real. This condition imposes additional restraints on the spectrum of the process and the regression spectrum. Then

$$
\begin{aligned}
f_{ii}(\lambda) &= f_{ii}(-\lambda), \qquad i = 1, 2,\\
f_{12}(\lambda) &= f_{21}(-\lambda),
\end{aligned} \tag{7.1}
$$

and

$$
dM(\lambda) = \overline{dM(-\lambda)}. \tag{7.2}
$$

We shall obtain necessary and sufficient conditions on the regression spectrum for the least-squares estimate to be asymptotically efficient for such a process. The derivation of these conditions is analogous to that followed in Section 6.

Just as in Section 6 one can see that asymptotic efficiency of the least-squares estimate implies that there are only a finite number of points of increase of $M(\lambda)$. Because of (7.2) we need only consider the nonnegative points of increase of $M(\lambda)$. Let the nonnegative points of increase be $\lambda_1 = 0 < \lambda_2 < \cdots < \lambda_q$. Of course zero needn't be one of the points of increase but we include it because if it is, the condition on the jump at zero is different from that on jumps at other points. Let ${}_{ij}M_k$ denote the jump of ${}_{ij}M(\lambda)$ at λ_k. Given the matrix A, let $\text{Re}(A)$ and $\text{Im}(A)$ be the matrices whose elements are the real and imaginary parts, respectively, of the corresponding elements of A. Equation (6.1) can then be rewritten as

$$\begin{aligned}(7.3)\quad &\sum_j{}' 2\begin{pmatrix} {}_{11}f_j \text{ Re } ({}_{11}M_j) & \text{Re } ({}_{12}f_j\, {}_{12}M_j) \\ \text{Re } ({}_{21}f_j\, {}_{21}M_j) & {}_{22}f_j \text{ Re } {}_{22}M_j \end{pmatrix}\begin{pmatrix} {}_1M^{-1} & 0 \\ 0 & {}_2M^{-1}\end{pmatrix} \times \\ &\sum_k{}' \frac{2}{{}_{11}f_k\, {}_{22}f_k - |{}_{12}f_k|^2}\begin{pmatrix} {}_{22}f_k \text{ Re } ({}_{11}M_k) & -\text{Re } ({}_{12}f_k\, {}_{12}M_k) \\ -\text{Re } ({}_{21}f_k\, {}_{21}M_k) & {}_{11}f_k \text{ Re } ({}_{22}M_k)\end{pmatrix} = \begin{pmatrix} {}_1M & 0 \\ 0 & {}_2M\end{pmatrix},\end{aligned}$$

making use of (7.1) and (7.2). Here, ${}_{ij}f_k = {}_{ij}f(\lambda_k)$. The primed summations indicate that the coefficient 2 in the summation is to be replaced by coefficient one when either j or $k = 1$, since $\lambda_1 = 0$. Because of (7.2) we can see that the matrices ${}_{ij}M_1$ have real elements. The fact that $f_{12}(\lambda) = \overline{f_{21}(\lambda)}$ indicates that ${}_{ij}f_1$, $i, j = 1, 2$, is real. Now equation (7.3) is assumed to hold for all ${}_{11}f_j$, ${}_{22}f_j > 0$ and all ${}_{12}f_j$ such that $|{}_{12}f_j|^2 < {}_{11}f_j\, {}_{22}f_j$. A discussion of equation (7.3) analogous to that carried out in Section 6 indicates that the *equation cannot be valid under these conditions unless the following restraints on the matrices* ${}_{ij}M_k$ *are satisfied:*

$$2 \text{ Re } ({}_{ij}M_l)\, {}_jM^{-1} \text{ Re } ({}_{jk}M_s) = \delta_{ls} \text{ Re } ({}_{ik}M_l)$$

if $l \neq 0$;

$$(7.4)\qquad \begin{aligned} {}_{ij}M_1\, {}_jM^{-1}\, {}_{jk}M_1 &= {}_{ik}M_1, \\ 2\text{Im}({}_{ij}M_l)\, {}_jM^{-1} \text{ Im}({}_{ji}M_k) &= -\delta_{lk} \text{ Re}({}_{ii}M_l), \end{aligned}$$

if $i \neq j$ *and* $l, k \neq 1$, *since* $\text{Im}({}_{ij}M_1) = 0$;

$$\text{Re}({}_{ij}M_k)\, {}_jM^{-1} \text{ Im}({}_{jl}M_k) = \text{Im}({}_{jl}M_k)\, {}_lM^{-1} \text{ Re}({}_{ls}M_k),$$

where $j \neq l$ *and* $k \neq 1$; *and finally,*

$$\text{Re}({}_{ij}M_k)\, {}_jM^{-1} \text{ Im}({}_{jl}M_s) = \text{Im}({}_{jl}M_k)\, {}_lM^{-1} \text{ Re}({}_{lu}M_s) = 0$$

if $k \neq s$. *It can also be readily seen that equation* (7.3) *will be satisfied for all admissible spectra of the process if the conditions* (7.4) *just derived are satisfied.*

THEOREM 4. *The least-squares estimate is asymptotically efficient for all admissible spectra of the process if and only if the regression spectrum is a jump spectrum with a finite number of jumps and the matrices* ${}_{ij}M_k$ *satisfy the conditions* (7.4).

It is again of special interest to consider the case in which both components ${}_1y_t$, ${}_2y_t$ of the observed process have a mixed trigonometric and polynomial regression and the regression vectors of both component are the same, so that

$$
\begin{aligned}
{}_1\varphi_t^{(\nu)} &= {}_2\varphi_t^{(\nu)} = t^\nu, & \nu &= 0, 1, \cdots, s_1, \\
{}_1\varphi_t^{(s_1+1+\nu)} &= {}_2\varphi_t^{(s_1+1+\nu)} = t^\nu \cos t\lambda_2, & \nu &= 0, 1, \cdots, s_2, \\
{}_1\varphi_t^{(s_1+s_2+2+\nu)} &= {}_2\varphi_k^{(s_1+s_2+2+\nu)} = t^\nu \sin t\lambda_2, & \nu &= 0, 1, \cdots, s_2.
\end{aligned}
$$

....

$$
{}_1\varphi_t^{(s_1+2s_2+\cdots+2s_{u-1}+2u-1+\nu)} = {}_2\varphi_t^{(s_1+2s_2+\cdots+2s_{u-1}2u-1+\nu)} = t^\nu \cos t\lambda_u, \qquad \nu = 0, 1, \cdots, s_u,
$$

$$
{}_1\varphi_t^{(s_1+2s_2+\cdots+2s_{u-1}+s_u+2u+\nu)} = {}_2\varphi_t^{(s_1+2s_2+\cdots+2s_{u-1}+s_u+2u+\nu)} = t^\nu \sin t\lambda_u, \qquad \nu = 0, 1, \cdots, s_u,
$$

where $\lambda_1 = 0 < \lambda_2 < \cdots < \lambda_u$. The jumps of $M(\lambda)$ are at $0, \ \pm\lambda_2, \cdots, \pm\lambda_u$. We need only discuss the jumps at nonnegative λ, since $dM(\lambda) = \overline{dM(-\lambda)}$. Now

$$
{}_{jk}M_1 = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix},
$$

where

$$
M_1 = \left\{ \frac{\sqrt{(2\mu+1)(2\nu+1)}}{\mu+\nu+1}; \mu, \nu = 0, 1, \cdots, s_1 \right\},
$$

$$
{}_{jk}M_l = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_l & -iM_l & 0 \\ 0 & iM_l & M_l & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

where

$$
M_l = \left\{ \frac{\sqrt{(2\mu+1)(2\nu+1)}}{\mu+\nu+1}; \mu, \nu = 0, 1, \cdots, s_l \right\}, \qquad l \neq 1,
$$

and the null submatrix in the upper left-hand corner of ${}_{jk}M_l$ is of order $s_1 + 1 + 2\sum_{h=2}^{l-1}(s_h + 1)$. It is clear that *the least-squares estimate is asymptotically efficient in the case of such a regression, since the conditions* (7.4) *are satisfied.* As in the case of a complex-valued process, one does not have asymptotic efficiency of the least-squares estimate if the regression vectors of the two components are unequal in number. There is, however, an additional restriction that enters into this context and did not arise in Section 6. *If one of the terms in* *the regression is* $t^\nu \cos t\lambda$, $\lambda \neq 0$, *one must also have the term* $t^\nu \sin t\lambda$ *for asymptotic efficiency of the least-squares estimate.* Thus, one does not have asymptotic efficiency of the least-squares estimate in the case of the regression

$$ {}_1m_t = {}_1c \cos t\lambda, \qquad {}_2m_t = {}_2c \cos t\lambda, \qquad \lambda \neq 0, $$

and one does in the case of the regression

$$ \begin{aligned} {}_1m_t &= {}_1c_1 \cos t\lambda + {}_1c_2 \sin t\lambda, \\ {}_2m_t &= {}_2c_1 \cos t\lambda + {}_2c_2 \sin t\lambda, \qquad \lambda \neq 0. \end{aligned} $$

8. The Markov and least-squares estimate when the regression of one component vanishes. The special case in which the regression of one component vanishes, say ${}_2m_t \equiv 0$, has not yet been discussed but is of some interest. It is clear that the least-squares estimate can not be asymptotically efficient for all admissible spectra of the observed process in this case. Let c^* now denote a linear unbiased estimate of ${}_1c$. The least-squares estimate of ${}_1c$ in this case is

$$ c_L^* = ({}_1\Phi'{}_1\Phi)^{-1}{}_1\Phi'{}_1y. \tag{8.1} $$

The covariance matrix of the least-squares estimate is

$$ E(c_L^* - {}_1c)(c_L^* - {}_1c)' = ({}_1\Phi'{}_1\Phi)^{-1}{}_1\Phi' R_{11}\, {}_1\Phi({}_1\Phi'{}_1\Phi)^{-1}. \tag{8.2} $$

The Markov estimate c_M^* of ${}_1c$ is

$$ c_M^* = \left(({}_1\Phi' \; 0) R^{-1} \begin{pmatrix} {}_1\Phi \\ 0 \end{pmatrix} \right)^{-1} ({}_1\Phi' \; 0) R^{-1} y. \tag{8.3} $$

The covariance matrix of the Markov estimate is

$$ \begin{aligned} E(c_M^* - {}_1c)(c_M^* - {}_1c)' &= \left(({}_1\Phi' \; 0) R^{-1} \begin{pmatrix} {}_1\Phi \\ 0 \end{pmatrix} \right)^{-1} \\ &= ({}_1\Phi'(R_{11} - R_{12}R_{22}^{-1}R_{21})^{-1}\, {}_1\Phi^{-1}). \end{aligned} \tag{8.4} $$

By using techniques analogous to those of Sections 4 and 5, one can show that

$$ \lim_{N\to\infty} {}_1D_N E(c_L^* - {}_1c)(c_L^* - {}_1c)'{}_1D_N = 2\pi\, {}_1M^{-1} \int_{-\pi}^{\pi} f_{11}(-\lambda)\, d\, {}_{11}M(\lambda)\, {}_1M^{-1}, \tag{8.5} $$

while

$$ \begin{aligned} &\lim_{N\to\infty} {}_1D_N E(c_M^* - {}_1c)(c_M^* - c_1)'{}_1D_N \\ &\qquad = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \frac{f_{22}(-\lambda)}{f_{11}(-\lambda)f_{22}(-\lambda) - |f_{12}(-\lambda)|^2}\, d\, {}_{11}M(\lambda) \right)^{-1}. \end{aligned} \tag{8.6} $$

Of course c_M^* is the best of all linear unbiased estimates of ${}_1c$ in that it has the smallest covariance matrix of them all. It is interesting to compare the two estimates when ${}_1m_t \equiv c$, a constant. Then

$$ {}_1M(\pi) - {}_1M(-\pi) = {}_1M(0+) - {}_1M(0-) = 1, $$

so that

$$ \lim_{N\to\infty} \frac{E(c_M^* - c)^2}{E(c^* - c)^2} = 1 - \frac{|f_{12}(0)|^2}{f_{11}(0)f_{22}(0)}. $$

Some aspects of this example have been discussed in [1] from the point of view of discriminant analysis.

9. Processes of dimension higher than two. We shall discuss briefly the case of an n-dimensional process y_t with stationary residuals, $n \geqq 3$, and indicate that most of the results obtained in the two-dimensional case are still valid. Assumptions analogous to those of Sections 2 and 3 are made. Let

$$ {}_im_t = \sum_{\nu=1}^{p_i} {}_ic_\nu \varphi_t^{(\nu)}, \qquad i = 1, \cdots, n, $$

be the regression of the ith component ${}_iy_t$ of y_t. The spectrum of the residual process $x_t = y_t - m_t$ is again assumed absolutely continuous with continuous spectral densities $f_{ij}(\lambda)$, $i, j = 1, \cdots, n$. Of course $f_{ij}(\lambda)$ is the cross-spectral density of ${}_ix_t$ and ${}_jx_t$. The determinant $|f(\lambda)|$ of the matrix

$$ f(\lambda) = \{f_{ij}(\lambda); i, j = 1, \cdots, n\} \tag{9.1} $$

is assumed to be positive for all λ.

Let

$$ {}_ic = \begin{pmatrix} {}_ic_1 \\ \vdots \\ {}_ic_{pi} \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} {}_ic \\ \vdots \\ {}_nc \end{pmatrix}. $$

Let c_L^* and c_M^* be the least-squares estimate and the Markov estimate of c, respectively, in terms of the observed process $y_1, \cdots, y_N$. The matrices ${}_i\Phi$, $i = 1, \cdots, n$, are defined just as in Section 2, and we set

$$ \Phi = \begin{bmatrix} {}_1\Phi & & 0 \\ & \ddots & \\ 0 & & {}_n\Phi \end{bmatrix}. $$

R is the covariance matrix of the vector $\begin{pmatrix} {}_1y \\ \vdots \\ {}_ny \end{pmatrix}$. The expressions given for c_L^* and c_M^* and their respective covariance matrices in Section 1 are still valid.

The cross-spectral distribution function of the regression vectors of ${}_iy_t$ and ${}_jy_t$ are computed just as in the two-dimensional case. The matrices

${}_{ii}M(\pi) - {}_{ii}M(-\pi) = {}_iM, \quad i = 1, \cdots, n$, are assumed to be nonsingular. Let $M(\lambda) = \{{}_{ij}M(\lambda); i, j = 1, \cdots, n\}$. We set

$$D_N = \begin{pmatrix} {}_1D_N & & 0 \\ & \ddots & \\ 0 & & {}_nD_N \end{pmatrix}.$$

Let $\theta_{ij}(\lambda)$ be the cofactor of the element $f_{ij}(\lambda)$ in the matrix $f(\lambda)$. By using an elaboration of the methods of Sections 4 and 5, one can show that

$$\begin{aligned} &\lim_{N\to\infty} D_N E(c_L^* - c)(c_L^* - c)' D_N \\ (9.2) \qquad &= 2\pi M^{-1}\left\{ \int_{-\pi}^{\pi} f_{ij}(-\lambda)\, d_{ij}M(\lambda); i, j = 1, \cdots, n \right\} M^{-1} \end{aligned}$$

and

$$\begin{aligned} &\lim_{N\to\infty} D_N E(c_M^* - c)(c_M^* - c)' \\ (9.3) \qquad &= \frac{1}{2\pi}\left\{ \int_{-\pi}^{\pi} \theta_{ij}(-\lambda)\, d_{ij}M(\lambda); i, j = 1, \cdots, n \right\}^{-1}. \end{aligned}$$

The analogues of the results obtained in Sections 6 and 7 on asymptotic efficiency of the least-squares estimate c_L^* are valid in the n-dimensional case and can be proved in the same way.

10. Concluding remarks. It would be very interesting to find out for what sample size N the various results obtained on asymptotic efficiency of the least-squares estimate are practically valid. An effective way to find out would be to set up a computational program for the calculation of the covariance matrices of the least-squares and Markov estimates for a variety of interesting regressions and spectra $f(\lambda)$. The approximations derived in this paper for these covariance matrices should also be computed and compared with the true covariance matrices.

Consider the case of a real two-dimensional process $y_t = \begin{pmatrix} {}_1y_t \\ {}_2y_t \end{pmatrix}$ where both components have constant mean values $E\,{}_iy_t = {}_ic, \quad i = 1, 2$, which we want to estimate. The simplest case of cross-correlation, and a rather uninteresting one, is that in which

$$\operatorname{cov}({}_iy_t, {}_iy_\tau) = \delta_{t\tau},$$

$$\operatorname{cov}({}_1y_t, {}_2y_\tau) = \rho\delta_{t\tau},$$

where $|\rho| < 1$. Nonetheless, it is amusing to note that in this case the least-squares estimate is efficient for all finite N and that

$$\begin{aligned} E(c_L^* - c)(c_L^* - c)' &= E(c_M^* - c)(c_M^* - c)' \\ &= 2\pi D_N^{-1} M^{-1} \left\{ \int_{-\pi}^{\pi} f_{ij}(-\lambda)\, d\, {}_{ij}M(\lambda); i, j = 1, 2 \right\} M^{-1} D_N^{-1} \\ &= 2\pi D_N^{-1} \left\{ \int_{-\pi}^{\pi} \theta_{ij}(-\lambda)\, d\, {}_{ij}M(\lambda); i, j = 1, 2 \right\}^{-1} D_N^{-1} \\ &= \frac{1}{N} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \end{aligned}$$

REFERENCES

[1] W. G. Cochran and C. I. Bliss, "Discriminant functions with covariance," *Ann. Math. Stat.*, Vol. 19 (1948), pp. 151–176.

[2] H. Cramér, "On the theory of stationary random processes," *Ann. Math. Stat.*, Vol. 41 (1940), pp. 215–230.

[3] U. Grenander, "On the estimation of regression coefficients in the case of an autocorrelated disturbance," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 252–272.

[4] U. Grenander and M. Rosenblatt, "An extension of a theorem of G. Szegö and its application to the study of stochsatic processes," *Trans. Amer. Math. Soc.*, Vol. 76 (1954), pp. 112–126.

[5] U. Grenander and M. Rosenblatt, "Regression analysis of time series with a stationary disturbance," *Proc. Nat. Acad. Sci. U.S.* (to be published), Vol. 40 (1954), pp. 812–816.