SOME APPLICATIONS OF BIPOLYKAYS TO THE ESTIMATION OF VARIANCE COMPONENTS AND THEIR MOMENTS[1]

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1. Summary. Bipolykays were introduced in [3]. They form a family of symmetric (row-wise and column-wise) polynomial functions of the elements of a two-way array, with the property of being inherited on the average, and such that any similarly symmetric polynomial function of the same numbers can be written linearly in terms of the bipolykays. This paper will describe some applications of bipolykays to problems in the analysis of variance of two-way classifications, using the formulas and tables derived in [3]. A linear model which includes contributions from interaction as well as independently sampled cell contributions is given in Section 3, and applications are made to certain cases of this model. These applications include (a) finding unbiased estimators for the variance components in the case of no interaction as well as unbiased estimators for the variances of these estimators (Section 6), (b) finding expressions for means and variances of some of the functions of degrees 1 and 2 that are of interest in the problem of sampling from a matrix (Section 7), and (c) finding unbiased estimators for variance components in the general case, including expressions for the variances of these estimators in the case of infinite populations (Section 8).

2. Introduction. The purpose of this paper is to describe some uses of bipolykays, which were defined in [3], in connection with problems arising in the analysis of variance. A linear model is given in Section 3 and an analysis of variance notation, in Section 4. Sections 6, 7, and 8 are given over to derivation of results related to the estimation of variance components in various special cases of the linear model.

It will be necessary to make frequent references to [3]. However, in order to enable the reader to get the gist of the present paper without referring to [3], the remainder of this section is devoted to a summary of definitions and frequently-used results.

The symbol $\sum^{\neq}$, for "distinct sum," means a sum taken over all subsequent subscripts, but with subscripts kept different when they are indicated by different letters. Thus

$$\sum_{1}^{2}{}^{\neq} x_i x_j = x_1 x_2 + x_2 x_1 .$$

Received July 28, 1954.

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When the x's are matrix elements, the distinctness relates to row subscripts and to column subscripts. Thus

$$\sum_1^{2}{}^{\neq} x_{ij}x_{ik} = x_{11}x_{12} + x_{12}x_{11} + x_{21}x_{22} + x_{22}x_{21}.$$

For a set of numbers $x_1, \cdots, x_n$, the *symmetric means* of degrees 1 and 2 are

$$\langle 1\rangle = \sum{}^{\neq} x_i/n,$$
$$\langle 11\rangle = \sum{}^{\neq} x_i x_j \,/\, n(n-1),$$
$$\langle 2\rangle = \sum{}^{\neq} x_i^2/n.$$

The *polykays* are linear combinations of symmetric means denoted by k's with subscripts. Those of degrees 1 and 2 are defined by

$$k_1 = \langle 1\rangle,$$
$$k_{11} = \langle 11\rangle,$$
$$k_2 = \langle 2\rangle - \langle 11\rangle.$$

Symmetric means and polykays are inherited on the average; this means that if $x_1, \cdots, x_n$ are a sample from a population $x_1, \cdots, x_N$, and if primes are used to denote values defined over the population, then

$$\text{ave } k_2 = k_2', \text{ etc.,}$$

where "ave" means average over all possible samples of size n.

If $x_1, \cdots, x_n$ is a sample from a population P' (with polykays k_1', k_{11}', etc.) and $y_1, \cdots, y_n$ is a sample from a population P'' (with polykays k_1'', k_{11}'', etc.), and if $z_1, \cdots, z_n$ is a sample formed by letting $z_i = x_i + y_i$ $(i = 1, 2, \cdots, n)$, then

(1) $$\text{ave aver } k_1 = k_1' + k_1''.$$

Here k_1 (with no prime) means a polykay over the z's, "aver" means average over all possible permutations (or randomizations) of the x's and y's before adding, and "ave" means average over samples as before. Equation (1) is known as a pairing formula. Pairing formulas for the polykays of degrees 2 are

$$\text{ave aver } k_{11} = k_{11}' + 2k_1'k_1'' + k_{11}'',$$
$$\text{ave aver } k_2 = k_2' + k_2''.$$

We now let $\|x_{ij}\|$ be a two-way array of numbers $(i = 1, 2, \cdots, r;$ $j = 1, 2, \cdots, c)$ which may be regarded as a bisample from an array $\|x_{IJ}\|$ $(I = 1, 2, \cdots, R;\ J = 1, 2, \cdots, C)$, a bisample being chosen from a population matrix by taking those elements which are at the intersections of a selected set of r of the R rows and c of the C columns.

Generalized symmetric means (g.s.m.'s) of degrees 1 and 2 over a bisample are[3]

$$\begin{bmatrix}1 & -\\ - & -\end{bmatrix} = \sum^{\neq} x_{ij} \,/\, rc,$$

$$\begin{bmatrix}1 & -\\ - & 1\end{bmatrix} = \sum^{\neq} x_{ij}x_{km} \,/\, rc(r-1)(c-1),$$

$$\begin{bmatrix}1 & -\\ 1 & -\end{bmatrix} = \sum^{\neq} x_{ij}x_{kj} \,/\, rc(r-1),$$

$$\begin{bmatrix}1 & 1\\ - & -\end{bmatrix} = \sum^{\neq} x_{ij}x_{ik} \,/\, rc(c-1),$$

$$\begin{bmatrix}2 & -\\ - & -\end{bmatrix} = \sum^{\neq} x_{ij}^2 \,/\, rc.$$

Those of degrees 3 and 4 can be expressed in similar notation, but to save space are indicated by $t_1, t_2, \cdots, t_{10}$ for the 10 g.s.m.'s of degree 3 and by $f_1, f_2, \cdots, f_{33}$ for the 33 g.s.m.'s of degree 4. An arbitrary symmetric mean may be denoted by $\langle\|\alpha\|\rangle$, the $\|\alpha\|$ representing the matrix of entries.

The bipolykays are linear combinations of the g.s.m.'s, represented in the same way with parentheses replacing $\langle\ \rangle$'s. Those of degrees 1 and 2 are defined by

$$\begin{pmatrix}1 & -\\ - & -\end{pmatrix} = \begin{bmatrix}1 & -\\ - & -\end{bmatrix},$$

$$\begin{pmatrix}1 & -\\ - & 1\end{pmatrix} = \begin{bmatrix}1 & -\\ - & 1\end{bmatrix},$$

$$\begin{pmatrix}1 & -\\ 1 & -\end{pmatrix} = \begin{bmatrix}1 & -\\ 1 & -\end{bmatrix} - \begin{bmatrix}1 & -\\ - & 1\end{bmatrix},$$

$$\begin{pmatrix}1 & 1\\ - & -\end{pmatrix} = \begin{bmatrix}1 & 1\\ - & -\end{bmatrix} - \begin{bmatrix}1 & -\\ - & 1\end{bmatrix},$$

$$\begin{pmatrix}2 & -\\ - & -\end{pmatrix} = \begin{bmatrix}2 & -\\ - & -\end{bmatrix} - \begin{bmatrix}1 & 1\\ - & -\end{bmatrix} - \begin{bmatrix}1 & -\\ 1 & -\end{bmatrix} + \begin{bmatrix}1 & -\\ - & 1\end{bmatrix}.$$

Those of degrees 3 and 4 are indicated by $T_1, T_2, \cdots, T_{10}$ and $F_1, F_2, \cdots, F_{33}$, respectively. (See Section 8 of [3].) A general bipolykay may be indicated by $(\|\alpha\|)$.

Bipolykays and g.s.m.'s are inherited on the average in the sense that

$$\text{ave}\begin{pmatrix}1 & -\\ - & -\end{pmatrix} = \begin{pmatrix}1 & -\\ - & -\end{pmatrix}', \text{etc.,}$$

[3] Square brackets are used, for convenience in printing, in place of $\langle\ \rangle$'s for g.s.m.'s having more than one row.

the prime and "ave" having the same meanings as for polykays above, with

$$\begin{pmatrix}1 & -\\ - & -\end{pmatrix}' = \begin{bmatrix}1 & -\\ - & -\end{bmatrix}' = \sum^{\neq} x_{IJ} \,/\, RC, \text{ etc.}$$

Pairing formulas have a meaning analogous to that defined above for polykays and are as follows for bipolykays of degrees 1 and 2, "aver" meaning average over permutations of rows and of columns:

$$\text{ave aver}\begin{pmatrix}1 & -\\ - & -\end{pmatrix} = \begin{pmatrix}1 & -\\ - & -\end{pmatrix}' + \begin{pmatrix}1 & -\\ - & -\end{pmatrix}'',$$

$$\text{ave aver}\begin{pmatrix}1 & -\\ - & 1\end{pmatrix} = \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}' + 2\begin{pmatrix}1 & -\\ - & -\end{pmatrix}'\begin{pmatrix}1 & -\\ - & -\end{pmatrix}'' + \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}'',$$

$$\text{ave aver}\begin{pmatrix}1 & -\\ 1 & -\end{pmatrix} = \begin{pmatrix}1 & -\\ 1 & -\end{pmatrix}' + \begin{pmatrix}1 & -\\ 1 & -\end{pmatrix}'',$$

$$\text{ave aver}\begin{pmatrix}1 & 1\\ - & -\end{pmatrix} = \begin{pmatrix}1 & 1\\ - & -\end{pmatrix}' + \begin{pmatrix}1 & 1\\ - & -\end{pmatrix}'',$$

$$\text{ave aver}\begin{pmatrix}2 & -\\ - & -\end{pmatrix} = \begin{pmatrix}2 & -\\ - & -\end{pmatrix}' + \begin{pmatrix}2 & -\\ - & -\end{pmatrix}''.$$

In general, it is shown in [3] that the pairing formula for a bipolykay $(\|\alpha\|)$ is

$$\text{ave aver } (\|\alpha\|) = (\|\alpha\|)' + (\|\alpha\|)''$$

unless the matrix $\|\alpha\|$ can be expressed as a direct sum

$$\|\alpha\| = \begin{Vmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_m\end{Vmatrix},$$

where the α_i are matrices. If the $\|\alpha_i\|$ cannot be further broken down, then, in this case,

$$\text{ave aver } (\|\alpha\|) = (\|\alpha\|)' + (\|\alpha\|)'' + \sum (\|\beta\|)'(\|\gamma\|)'',$$

where the sum extends over all $\|\beta\|$ and $\|\gamma\|$ such that $\|\beta\|$ is the direct sum of $1, 2, \cdots$, or $m - 1$ of the $\|\alpha_i\|$ and $\|\gamma\|$ is the direct sum of the remaining ones.

The reader is referred to [3] for the general definitions exemplified above, for multiplication formulas, for conversion formulas for degrees 3 and 4, etc.

3. The linear model. Each example discussed in this paper will be based on a linear model which is a special case of the following:

$$x_{ijk} = \theta + \eta_i + \xi_j + \lambda_{ij} + \omega_{ijk}\,. \tag{2}$$

Here i, j, k run from 1 to r, c, b, respectively. The θ's, η's, ξ's, and ω's are independently sampled contributions from populations described in Table 1.

The systematic interactions λ_{ij} are not independently sampled but are "tied" to the η's and ξ's; i.e., the λ's come from an $R \times C$ matrix having a row cor-

TABLE 1

Notation associated with model (2)

Contribution	Sample Size	Population Size	Population Polykays
θ, general	1	Arbitrary	k_1'''', k_{11}'''', etc.
η_i, row	r	R	k_1', k_{11}', etc. ($k_1' = 0$)
ξ_j, column	c	C	k_1'', k_{11}'', etc. ($k_1'' = 0$)
ω_{ijk}, cell	rcb	N	k_1''', k_{11}''', etc. ($k_1''' = 0$)

responding to each η and a column to each ξ, so that for each selection of an η_i and a ξ_j there is a unique λ_{ij} that accompanies them. Since the η's and ξ's represent row and column contributions, respectively, it is assumed that the row means and column means of the matrix of λ's all vanish; otherwise the population matrix of λ's is arbitrary and so must be described in terms of bipolykays rather than polykays.

4. Analysis of variance notation. The matrix

$$\|x_{ij}\|, \quad i = 1, 2, \cdots, r; j = 1, 2, \cdots, c$$

will represent a bisample from a population matrix

$$\|x_{IJ}\|, \quad I = 1, 2, \cdots, R; J = 1, 2, \cdots, C.$$

For any matrix $\|x_{ij}\|$, whether it be a population matrix, a bisample from such a matrix, or simply a two-way array of sampled numbers such as arises in connection with some linear model, our interests will center around certain families of symmetric quadratic functions of the x's. Two of these are

(a) the bipolykays $\begin{pmatrix}1 & -\\ - & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ - & -\end{pmatrix}, \begin{pmatrix}1 & -\\ 1 & -\end{pmatrix}, \begin{pmatrix}2 & -\\ - & -\end{pmatrix}$, and

(b) the various sums of squares and mean squares associated with conventional analysis of variance procedures.

The mean squares (denoted by MS) and sums of squares (denoted by SS) are defined as follows, where a dot represents an average over the subscript it replaces:

Designation	*Mean Square and Sum of Squares*
Rows	$MSR = SSR/(r-1) = c \sum_i (x_{i.} - x_{..})^2/(r-1)$
Columns	$MSC = SSC/(c-1) = r \sum_j (x_{.j} - x_{..})^2/(c-1)$
Residual or balance	$MSB = SSB/(r-1)(c-1)$
	$= \sum_i \sum_j (x_{ij} - x_{i.} - x_{.j} + x_{..})^2/(r-1)(c-1)$
Mean	$MSM = SSM = \sum_i \sum_j x_{..}^2 = rcx_{..}^2$
Total	$SST = \sum_i \sum_j (x_{ij} - x_{..})^2$

(As always, we have $SST = SSR + SSC + SSB$.)

When it is desired to emphasize the fact that a population matrix is being discussed, subscripts will be capital letters, dashes (instead of dots) will indicate averages over the population, and primes will indicate population values; e.g.,

$$SST' = \sum_{I=1}^{R} \sum_{J=1}^{C} (x_{IJ} - x_{--})^2.$$

In dealing with a population matrix we shall use the following quantities:

$$\begin{aligned} \theta &= x_{--} \qquad\qquad (\text{i.e., } \theta = \textstyle\sum \sum x_{IJ}/RC), \\ \eta_I &= x_{I-} - x_{--}, \\ \xi_J &= x_{-J} - x_{--}, \\ \lambda_{IJ} &= x_{IJ} - x_{I-} - x_{-J} + x_{--}. \end{aligned}$$

The elements of the matrix can then be thought of as built up from the η's, ξ's, λ's, and θ as in model (2), taking the ω's in that model to be 0. We are then interested in a third family of quadratic symmetric functions, namely

(c) the variance components,

$$\begin{aligned} \sigma_R^2 &= \text{variance component for rows} \\ &= \textstyle\sum_I (x_{I-} - x_{--})^2 / (R - 1) \\ &= k_2', \end{aligned}$$

where k_2' has the meaning assigned in Table 1;

$$\begin{aligned} \sigma_C^2 &= \text{variance component for columns} \\ &= \textstyle\sum_J (x_{-J} - x_{--})^2 / (C - 1) \\ &= k_2''; \\ \sigma_\lambda^2 &= \textstyle\sum_I \sum_J (x_{IJ} - x_{I-} - x_{-J} + x_{--})^2 / (R - 1)(C - 1). \end{aligned}$$

In this section we shall express the SS's in terms of the bipolykays (so that moments of the former can be easily obtained), the bipolykays in terms of the MS's (so that values of the former can be computed by standard techniques used in computing the latter), and finally the MS's for a population matrix in terms of the variance components (to help in expressing unbiased estimators for the latter). These various expressions are derived by elementary algebra, so only one example of the derivations will be given:

To express SSR in terms of bipolykays, for example, we have

$$\begin{aligned} SSR &= c \textstyle\sum_i (x_{i.} - x_{..})^2 \\ &= c \textstyle\sum_i x_{i.}^2 - rcx_{..}^2. \end{aligned}$$

Since

$$\begin{aligned} c \textstyle\sum_i x_{i.}^2 &= c\textstyle\sum_i (\sum_j x_{ij} / c)^2 \\ &= \frac{1}{c} \textstyle\sum_i \left(\sum_j x_{ij}^2 + \sum_{j,k}^{\neq} x_{ij}x_{ik}\right) \\ &= \frac{1}{c}\left\{rc\begin{bmatrix} 2 & - \\ - & - \end{bmatrix} + rc(c-1)\begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix}\right\}, \end{aligned}$$

and since

$$
\begin{aligned}
rc\bar{x}_{\cdot\cdot}^2 &= rc(\textstyle\sum_{i,j} x_{ij} / rc)^2 \\
&= \frac{1}{rc} (\textstyle\sum_{i,j} x_{ij}^2 + \sum^{\neq} x_{ij}x_{ik} + \sum^{\neq} x_{ij}x_{kj} + \sum^{\neq} x_{ij}x_{km}) \\
&= \frac{1}{rc}\left\{rc\begin{bmatrix} 2 & - \\ - & - \end{bmatrix} + rc(c-1)\begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} + rc(r-1)\begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix}\right. \\
&\qquad\qquad \left. + rc(r-1)(c-1)\begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}\right\},
\end{aligned}
$$

we have

$$
SSR = (r-1)\left\{\begin{bmatrix} 2 & - \\ - & - \end{bmatrix} - \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix} + (c-1)\left(\begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} - \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}\right)\right\}.
$$

The equations defining the bipolykays of degrees 1 and 2 (Section 2) enable us to express SSR at once as a linear combination of bipolykays. The result, together with similar ones for the other SS's, is contained in Table 2.

The equations represented by the first four rows of Table 2 can be solved to produce the following inverse relationships, where it is convenient to use MS's in place of SS's:

(3) $$\begin{pmatrix} 2 & - \\ - & - \end{pmatrix} = MSB,$$

(4) $$\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} = (MSR - MSB) / c,$$

(5) $$\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix} = (MSC - MSB) / r,$$

(6) $$\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix} = (MSM - MSR - MSC + MSB) / rc.$$

TABLE 2

Coefficients for SS's as linear combinations of bipolykays

	$\binom{2\,-}{-\,-}$	$\binom{1\,1}{-\,-}$	$\binom{1\,-}{1\,-}$	$\binom{1\,-}{-\,1}$
SSR	$r-1$	$c(r-1)$	0	0
SSC	$c-1$	0	$r(c-1)$	0
SSB	$(r-1)(c-1)$	0	0	0
SSM	1	c	r	rc
SST	$rc-1$	$c(r-1)$	$r(c-1)$	0

Finally, it follows immediately from the definitions that, for a population matrix,

$$\sigma_R^2 = MSR' \,/\, C$$

$$(7) \qquad = \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}' + \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}' \Big/ C,$$

$$\sigma_C^2 = MSC' \,/\, R$$

$$(8) \qquad = \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}' + \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}' \Big/ R,$$

$$\sigma_\lambda^2 = MSB'$$

$$(9) \qquad = \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}'.$$

5. Expectations in the case of no interactions. We suppose now that all the λ's in (2) are zero, and for the present that $b = 1$, so that the model is

$$(10) \qquad x_{ij} = \theta + \eta_i + \xi_j + \omega_{ij}, \qquad i = 1, 2, \cdots, r; j = 1, 2, \cdots, c.$$

Our first problem is to find the average values of bipolykays defined over the $r \times c$ array of x's (and distinguished here by having no prime or asterisk) in terms of the polykays (one to four primes) of the four populations described in Table 1. The procedure is to apply bipolykay pairing formulas to (10). The populations from which come the η's, ξ's, ω's, and θ are all special cases (numbered I to IV, respectively, below) and will now be considered one at a time.

CASE I. The population of η's is not a matrix population, but it can be thought of as a matrix whose Ith row is a vector of C components, each equal to η_I, ($I = 1, 2, \cdots, R$). Any sample of r η's can then be regarded as a bisample of r rows and c (arbitrary) columns from this matrix. Bipolykays of this bisample will be written $(\| \alpha \|)^*$, with a single asterisk. Referring to Case I of Section 7 of [3], we see that

$$(\| \alpha \|)^* = k^*_{mn\cdots p} \qquad \text{if the entries of } (\| \alpha \|)^* \text{ are all 1's in different columns}$$

$$= 0 \qquad \text{otherwise,}$$

$m, n, \cdots p$ being the row sums of the entries in $\| \alpha \|$, and k^* denoting a polykay for the sample $\eta_1, \cdots, \eta_r$ which defined the bisample in question. The same is obviously true for the population.

CASE II. The remarks just made for the η's apply to the population of ξ's if we change rows to columns, single primes and asterisks to double primes and asterisks, etc.

CASE III. The population of ω's enters as in Case IV of Section 7 of [3]. There it was shown that $(\| \alpha \|)^{***}$ becomes, on the average (over the kind of randomization that is pertinent here),

$k^{***}_{mn\cdots p}$ if $m, n, \cdots, p$ are the entries of $\| \alpha \|$ and if all are in different row and in different columns

0 otherwise.

CASE IV. In sampling θ we take a sample of size 1 and make it a bisample by putting this one number into every cell of the $r \times c$ matrix. Referring to Case III of Section 7 of [3], we see that

$$(\| \alpha \|)^{****} = \theta^m \quad \text{if all } m \text{ entries of } \| \alpha \| \text{ are 1's in different rows and different columns}$$

$$= 0 \quad \text{otherwise}$$

Hence ave $(\| \alpha \|)^{****} = \langle m \rangle''''$ or 0, respectively.

Keeping these facts in mind, together with the fact that $k_1' = k_1'' = k_1''' = 0$, we can apply the pairing formulas to the bipolykays of degree 4 or less to obtain some useful results that are collected in Table 3 below. We first derive a few of these results to show how the pairing formulas are used.

The only first-degree bipolykay is, of course, indecomposable, so its pairing formula (Section 2) gives us

$$\begin{aligned}\text{ave aver}\begin{pmatrix}1 & -\\ - & -\end{pmatrix} &= \begin{pmatrix}1 & -\\ - & -\end{pmatrix}' + \begin{pmatrix}1 & -\\ - & -\end{pmatrix}'' + \begin{pmatrix}1 & -\\ - & -\end{pmatrix}''' + \begin{pmatrix}1 & -\\ - & -\end{pmatrix}''''\\ &= k_1' + k_1'' + k_1''' + k_1''''\\ &= k_1'''',\end{aligned}$$

since $k_1' = k_1'' = k_1''' = 0$.

The indecomposable bipolykays of degree 2 can be treated in exactly the same way; *e.g.*

$$\begin{aligned}\text{ave aver}\begin{pmatrix}2 & -\\ - & -\end{pmatrix} &= \begin{pmatrix}2 & -\\ - & -\end{pmatrix}' + \begin{pmatrix}2 & -\\ - & -\end{pmatrix}'' + \begin{pmatrix}2 & -\\ - & -\end{pmatrix}''' + \begin{pmatrix}2 & -\\ - & -\end{pmatrix}''''\\ &= k_2''',\end{aligned}$$

since the term $\begin{pmatrix}2 & -\\ - & -\end{pmatrix}'''$ is really aver $\begin{pmatrix}2 & -\\ - & -\end{pmatrix}^{***}$, which is k_2''' by the remarks under Case III above. The other terms, i.e. those with one, two, and four primes, vanish in accordance with the remarks made in Cases I, II, and IV, above.

Decomposable bipolykays lead to more complex expressions. Averaging $\begin{pmatrix}1 & -\\ - & 1\end{pmatrix}$, for example, produces

$$\begin{aligned}\text{ave aver}\begin{pmatrix}1 & -\\ - & 1\end{pmatrix} &= \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}' + \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}'' + \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}''' + \begin{pmatrix}1 & -\\ - & 1\end{pmatrix}''''\\ &\qquad + 2\begin{pmatrix}1 & -\\ - & -\end{pmatrix}'\begin{pmatrix}1 & -\\ - & -\end{pmatrix}'' + \cdots.\\ &= k_{11}' + k_{11}'' + k_{11}''' + \langle 2 \rangle''''.\end{aligned}$$

As an example for degree 4, we consider the decomposable bipolykay

$$F_{11} = \begin{pmatrix} - & 1 & 1 \\ 1 & - & - \\ 1 & - & - \end{pmatrix}.$$

The pairing formula is

$$\text{ave aver } F_{11} = \text{ave aver } (F_{11}^{*} + F_{11}^{**} + F_{11}^{***} + F_{11}^{****})$$
$$+ \text{ ave aver } \sum \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^{u} \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}^{v},$$

where u and v represent different numbers (1, 2, 3, or 4) of asterisks. By the remarks made under Cases I through IV, each term of the form ave aver F_{11}^{*} vanishes, as does each term ave aver $\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^{u}$ ave aver $\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}^{v}$ except in the case where u and v represent a single and double asterisk, respectively. (The assumption of independence in sampling provides that the average of the sum of products is equal to the sum of products of averages.) Hence we have

$$\text{ave aver } F_{11} = \text{ave aver } \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^{*} \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}^{**}$$
$$= k_2' k_2''$$

Continuing in this way, we obtain the results shown in Table 3.

The following, omitted from Table 3, have expectation 0: T_6, T_8, T_9 ; F_{10}, F_{13}, F_{14}, F_{15}, F_{16}, F_{17}, F_{20}, F_{21}, F_{22}, F_{23}, F_{24}, F_{25}, F_{26}, F_{29}, F_{30}, F_{31}, and F_{32}.

The following have expectations which are complex expressions that will not be used in this paper: $\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$; T_1, T_2, T_3, T_7 ; F_1, F_2, F_3, F_6, F_7, F_{12}, and F_{28}.

The first formula in Table 3 says that $\begin{pmatrix} 1 & - \\ - & - \end{pmatrix}$, the mean of the x's, is an un-

TABLE 3

Bipolykays (column A) and their expected values (column B) in model (10)

A	*B*	*A*	*B*
$\begin{pmatrix} 1 & - \\ - & - \end{pmatrix}$	k_1''''	F_4	k_{22}'
		F_5	k_{22}''
$\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$	k_2'	F_8	k_4'
		F_9	k_4''
$\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}$	k_2''	F_{11}	$k_2' k_2''$
		F_{18}	$k_2' k_2'''$
$\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}$	k_2'''	F_{19}	$k_2'' k_2'''$
		F_{27}	k_{22}'''
T_4	k_3'	F_{33}	k_4'''
T_5	k_3''		
T_{10}	k_3'''		

biased estimator of the mean of the θ population, which is obvious, since the other populations have zero means. The second formula (reading down) says that $\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$ is an unbiased estimator of the component of variance for rows. In (4) it was pointed out that

$$\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} = (MSR - MSB) \,/\, c,$$

so this is the usual result. Similarly for the third formula. The fourth says that $\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}$ is an unbiased estimator of the "error variance." Since $\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}$ is MSB, this is well known. Interpretation of the formulas for the F's will be given in Section 6.

The two-way model with replications,

$$x_{ijk} = \theta + \eta_i + \xi_j + \omega_{ijk}, \qquad k = 1, 2, \cdots, b, \tag{11}$$

where everything is as in (10) except for the greater number of ω's sampled, can be treated in the same way if we suppose the population of ω's to be infinite. First we find $x_{ij} = x_{ij.}$, the average in each cell. Then

$$x_{ij} = \theta + \eta_i + \xi_j + \omega_{ij.},$$

and we have the same situation as before, with this exception: if P_1 represents the population of ω's, the $\omega_{ij.}$ come from the population P_2 of samples of size b from P_1. It remains only to find the polykays of P_2 in terms of those of P_1.

Any $\omega_{ij+} = \sum_k \omega_{ijk}$ can be thought of as a sum of b numbers from b populations all equal to P_1. Hence if $k_{22}^{***(+)}$ $k_{22}^{***(\cdot)}$ represent polykays for the ω_{ij+} and $\omega_{ij.}$, respectively, we have, by the pairing formula for k_{22}, for example,

$$\begin{aligned} \text{ave aver } k_{22}^{***(+)} &= bk_{22}''' + b(b-1)k_2'''k_2''' \\ &= b^2 k_{22}'''. \end{aligned}$$

Finally, since k_{22} is of degree 4, we divide by b^4 to obtain

$$\text{ave aver } k_{22}^{***(\cdot)} = k_2'''/b^2.$$

In similar fashion we find that

$$\begin{aligned} \text{ave aver } k_2^{***(\cdot)} &= k_2'''/b, \\ \text{ave aver } k_3^{***(\cdot)} &= k_3'''/b^2, \\ \text{ave aver } k_4^{***(\cdot)} &= k_4'''/b^3. \end{aligned}$$

It follows that for model (11), Table 3 remains as it is, except that k_2''', k_3''', k_{22}''', and k_4''', wherever they appear, must be divided, respectively, by b, b^2, b^2, and b^3. For example,

$$\text{aver aver } F_{18} = k_2'k_2'''/b,$$

F_{18} being defined over the matrix $\|x_{ij.}\|$.

6. Estimating variance components and their variances in the case of no interactions. In this section we consider applications to the case described by model (10) or (11). We begin with model (10), where components of variance for rows, columns, and "error" are k_2', k_2'', and k_2''', respectively, and for which respective unbiased estimators (Table 3) are $\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$, $\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}$, and $\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}$. In order to find the variances or covariances of these estimates, one may proceed as in this example:

$$\text{(12)} \quad \begin{aligned} \text{var}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} &= \text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^2 - \left\{\text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}\right\}^2 \\ &= \text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^2 - (k_2')^2. \end{aligned}$$

Referring to the relevant multiplication formulas, we find that

$$(k_2')^2 = \frac{R+1}{R-1} k_{22}' + \frac{1}{R} k_4'$$

([5], p. 516) and that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^2 = [&rc(r+1)(c-1)F_4 + c(r-1)(c-1)F_8 \\ &+ 4r^2(c-1)F_{13} + 2r^2F_{17} + 4r(c-1)F_{18} \\ &+ 4(r-1)(c-1)F_{22} + 2rF_{27} \\ &+ 2(r-1)F_{29}] \,/\, rc(r-1)(c-1) \end{aligned}$$

by Section 9 of [3]).

From Table 3 it follows that

$$\begin{aligned} \text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^2 = [rc(r+1)(c-1)k_{22}' + c(r-1)(c-1)k_4' + 4r(c-1)k_2'k_2''' \\ + 2rk_{22}'''] \,/\, rc(r-1)(c-1)\cdot \end{aligned}$$

Hence (12) becomes

$$\text{(13)} \quad \begin{aligned} \text{var}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} = \frac{2}{c(r-1)(c-1)} k_{22}''' + \frac{4}{c(r-1)} k_2'k_2''' \\ + \left(\frac{2}{r-1} - \frac{2}{R-1}\right) k_{22}' + \left(\frac{1}{r} - \frac{1}{R}\right) k_4'. \end{aligned}$$

In similar fashion one obtains

$$\text{(14)} \quad \begin{aligned} \text{var}\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix} = \frac{2}{r(r-1)(c-1)} k_{22}''' + \frac{4}{r(c-1)} k_2''k_2''', \\ + \left(\frac{2}{c-1} - \frac{2}{C-1}\right) k_{22}'' + \left(\frac{1}{c} - \frac{1}{C}\right) k_4'', \end{aligned}$$

$$\text{var}\begin{pmatrix} 2 & - \\ - & - \end{pmatrix} = \left(\frac{1}{rc} - \frac{1}{N}\right) k_4''' + \left\{\frac{2}{(r-1)(c-1)} - \frac{2}{(N-1)}\right\} k_{22}''', \tag{15}$$

$$\text{cov}\left\{\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}, \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}\right\} = 2k_{22}''' \,/\, rc(r-1)(c-1), \tag{16}$$

$$\text{cov}\left\{\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}, \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}\right\} = -2k_{22}''' \,/\, c(r-1)(c-1), \tag{17}$$

$$\text{cov}\left\{\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}, \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}\right\} = -2k_{22}''' \,/\, r(r-1)(c-1). \tag{18}$$

Formulas (13) through (18) have, of course, been derived without the use of bipolykays, and are now new. Our interest here is in using them to derive unbiased estimators of the variances and covariances of estimated variance components. In the above formulas each variance is given in terms of population polykays k_2', k_{22}'', etc. If one could observe the actual η's, ξ's, and ω's that are sampled, the sample polykays k_2^*, k_{22}^{**}, etc., would then provide unbiased estimators of the population polykays. In practice, however, one cannot do this, so that the formulas above do not provide unbiased estimates.

Inspection of formulas (13) through (18) shows that one would like to have unbiased estimators for the following:

$$k_{22}', \ k_{22}'', \ k_{22}'''; \ k_2'k_2''', \ k_2''k_2'''; \ k_4', \ k_4'', \ k_4'''.$$

Such estimators are provided at once by Table 3, and are, respectively, the following bipolykays computed over the matrix of x's:

$$F_4, \ F_5, \ F_{27}, \ F_{18}, \ F_{19}, \ F_8, \ F_9, \ F_{33}.$$

Substituting these into formula (13), we have

$$\text{Unbiased estimator for var}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} = \frac{2}{c(r-1)(c-1)} F_{27} + \frac{4}{c(r-1)} F_{18} + \left(\frac{2}{r-1} - \frac{2}{R-1}\right) F_4 + \left(\frac{1}{r} - \frac{1}{R}\right) F_8,$$

and similarly for the other formulas.

Turning now to model (11), with replications, and again supposing the population of ω's to be infinite, we note that estimators for k_2', k_2'', and k_2''' are, respectively, $\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$, $\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix}$, and $\begin{pmatrix} 2 & - \\ - & - \end{pmatrix} \Big/ b$, these bipolykays being applied to the matrix $\|x_{ij.}\|$. Hence (13) through (18) give us the formulas for the variances and covariances of these estimators if only we divide the entire right-hand sides of (15), (17), and (18) by b^2, b, and b, respectively, and change (in all formulas) all of the three-primed quantities in the manner indicated at the end of Section 5.

7. Moments in bisampling. The problem that led originally to the development of bipolykays (see [1]) came from this model of the educational testing process: Given a population of C questions (a "test") and a population of R examinees, suppose that the score of examinee I on question J will be x_{IJ}. A "test form," consisting of a random sample of c of the C questions, is given to each of a random sample of r of the R examinees. The "test score" of the ith examinee is $\sum_{j=1}^{c} x_{ij}$, the average test score of the group is $\sum_i \sum_j x_{ij}/r$, etc. One wants means and variances of quantities such as these.

Insofar as problems connected with bisampling can be expressed in terms of finding low moments of first- and second-degree symmetric polynomial functions, they can be easily solved by application of bipolykays. For degree 1, we have, for example,

$$\operatorname{var}\begin{pmatrix}1 & -\\ - & -\end{pmatrix} = \left(\frac{1}{rc} - \frac{1}{RC}\right)\begin{pmatrix}2 & -\\ - & -\end{pmatrix}' + \left(\frac{1}{r} - \frac{1}{R}\right)\begin{pmatrix}1 & 1\\ - & -\end{pmatrix}' + \left(\frac{1}{c} - \frac{1}{C}\right)\begin{pmatrix}1 & -\\ 1 & -\end{pmatrix}'$$

by Section 10 of [3]

$$= \frac{1}{c}\left(\frac{1}{r} - \frac{1}{R}\right) MSR' + \frac{1}{r}\left(\frac{1}{c} - \frac{1}{C}\right) MSC' - \left(\frac{1}{c} - \frac{1}{C}\right)\left(\frac{1}{r} - \frac{1}{R}\right) MSB' \qquad \text{by (3), (4), (5).}$$

As far as first moments of functions of degree 2 are concerned, we can derive the following formulas:

$$E(MSR) = MSB' + \frac{c}{C}(MSR' - MSB') = \left(1 - \frac{c}{C}\right)\sigma_\lambda^2 + c\sigma_R^2. \tag{19}$$

$$E(MSC) = MSB' + \frac{r}{R}(MSC' - MSB') = \left(1 - \frac{r}{R}\right)\sigma_\lambda^2 + r\sigma_C^2. \tag{20}$$

$$E(MSB) = \sigma_\lambda^2. \tag{21}$$

We use one of these to illustrate the derivations:

$$(ESMR) = \operatorname{ave}\left\{\begin{pmatrix}2 & -\\ - & -\end{pmatrix} + c\begin{pmatrix}1 & 1\\ - & -\end{pmatrix}\right\} \qquad \text{from Table 2}$$

$$= \begin{pmatrix}2 & -\\ - & -\end{pmatrix}' + c\begin{pmatrix}1 & 1\\ - & -\end{pmatrix}' \qquad \text{by inheritance on the average}$$

$$= MSB' + \frac{c}{C}(MSR' - MSB') \qquad \text{by (3) and (4)}$$

$$= \left(1 - \frac{c}{C}\right)\sigma_\lambda^2 + c\sigma_R^2 \qquad \text{by (7) and (9).}$$

Equations (19), (20), and (21) lead at once to the following unbiased estimators of the σ^2's:

$$\text{(22)} \qquad \hat{\sigma}_\lambda^2 = MSB = \begin{pmatrix} 2 & - \\ - & - \end{pmatrix},$$

$$\text{(23)} \qquad \hat{\sigma}_R^2 = \frac{1}{c} MSR - \left(\frac{1}{c} - \frac{1}{C}\right) MSB = \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} + \begin{pmatrix} 2 & - \\ - & - \end{pmatrix} \Big/ C,$$

$$\text{(24)} \qquad \hat{\sigma}_C^2 = \frac{1}{r} MSC - \left(\frac{1}{r} - \frac{1}{R}\right) MSB = \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix} + \begin{pmatrix} 2 & - \\ - & - \end{pmatrix} \Big/ R.$$

We turn next to the variances of the functions of degree 2. Variances and covariances of the bipolykays were tabulated in Section 10 of [3]. From these it is easy to find expressions, in terms of the bipolykays of degree 4, for the variances of the mean squares of the bisample, though these expressions are quite long. To find the variance of MSR, for example, we recall from Table 2 that

$$MSR = \begin{pmatrix} 2 & - \\ - & - \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}.$$

Hence

$$\operatorname{var} MSR = \operatorname{var}\begin{pmatrix} 2 & - \\ - & - \end{pmatrix} + 2c \operatorname{cov}\left\{\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}\right\} + c^2 \operatorname{var}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}.$$

The three terms on the right-hand side of this equation are given in Section 10 of [3], leading to an expression for var MSR in terms of bipolykays of degree 4. Variances of the estimated variance components can be found in the same way. These expressions for variances of quadratic expressions are long and clumsy. Formulas for unbiased estimators of these variances, however, are less complicated. Suppose, for example, that we want an unbiased estimator for the variance of $\hat{\sigma}_R^2$. We have

$$\begin{aligned}
\operatorname{var} \hat{\sigma}_R^2 &= \operatorname{ave}(\hat{\sigma}_R^2)^2 - (\operatorname{ave}\hat{\sigma}_R^2)^2 \\
&= \operatorname{ave}\hat{\sigma}_R^4 - \left\{\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}' + \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}' \Big/ C\right\}^2 \qquad \text{by (7)} \\
&= \operatorname{ave}\hat{\sigma}_R^4 - \left\{\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}'^2 + 2C\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}'\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}' + C^2\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}'^2\right\} \Big/ C^2 \\
&= \operatorname{ave}\hat{\sigma}_R^4 - \frac{1}{C^3}\left\{\frac{R+1}{R-1}[F'_{30} + C(2F'_{17} + 4F'_{25} + F'_{27}) + C^2(4F'_{13} + 2F'_{18}) \right. \\
&\qquad \left. + C^3F'_4] + \frac{1}{R}[F'_{33} + C(3F'_{29} + 4F'_{31}) + 6C^2F'_{22} + C^3F'_8]\right\},
\end{aligned}$$

the last step following from the multiplication table in Section 9 of [3]. It follows that we have this unbiased estimator of var $\hat{\sigma}_R^2$:

$$\hat{\sigma}_R^4 - \frac{1}{C^3}\left\{\frac{R+1}{R-1}[F_{30} + C(2F_{17} + 4F_{25} + F_{27}) + C^2(4F_{13} + 2F_{18}) + C^3F_4] + \frac{1}{R}[F_{33} + C(3F_{29} + 4F_{31}) + 6C^2F_{22} + C^3F_8]\right\}.$$

In a numerical case, an estimator for var $\hat{\sigma}_R^2$ is not likely to be wanted unless $\hat{\sigma}_R^2$ itself has already been computed, so there is no reason for expanding $\hat{\sigma}_R^4$ in bipolykays. It is of interest to note that the exponent of C (inside square brackets above) is one less than the number of columns appearing in the primary notation for each of the accompanying bipolykays; those bipolykays occurring in the first set of square brackets each have two rows, and those in the second set have one row.

8. Analysis of variance with interaction. We return now to the full model (2), supposing only for the present that $b = 1$, so that the model is

$$x_{ij} = \theta + \eta_i + \xi_j + \lambda_{ij} + \omega_{ij}\,. \tag{25}$$

That part of the sampling which pertains to the η's, ξ's, λ's, and θ can be described more simply (from the algebraic point of view, at least) by saying that

$$\rho_{ij} = \theta + \eta_i + \xi_j + \lambda_{ij}\,, \qquad i = 1, 2, \cdots, r; j = 1, 2, \cdots, c,$$

form a bisample from some $R \times C$ matrix. (See Section 4.) The model (25) then reduces to

$$x_{ij} = \rho_{ij} + \omega_{ij}\,, \tag{26}$$

where the ρ_{ij} are a bisample from a matrix $\| \rho_{IJ} \|$ for which σ_R^2 , σ_C^2 , and σ_λ^2 are defined as in Section 4, and the ω's are a sample of size rc from a population of variance σ^2. We shall use here one and two primes or asterisks to refer to functions over populations or bisamples related to ρ and ω, respectively.

To find $E(MSR)$, say, for this model, we recall that

$$MSR = \begin{pmatrix} 2 & - \\ - & - \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} \qquad \text{from Table 2.}$$

Taking averages and using the pairing formulas (Section 2), we have

$$E(MSR) = \text{ave aver}\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}^* + \text{ave aver}\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}^{**} + c\left\{\text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^* + \text{ave aver}\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^{**}\right\}.$$

For the population of ω's we have aver $\begin{pmatrix} 2 & - \\ - & - \end{pmatrix}^{**} = k_2''$ and aver $\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}^{**} = 0,$

as was pointed out in Section 5. Hence

$$E(MSR) = \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}' + k'' + c \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}'$$

$$= \left(1 - \frac{c}{C}\right) \sigma_\lambda^2 + c\sigma_R^2 + \sigma^2 \qquad \text{by (3) and (5).}$$

Similarly,

$$E(MSC) = \left(1 - \frac{r}{R}\right) \sigma_\lambda^2 + r\sigma_C^2 + \sigma^2$$

and

$$E(MSB) = \sigma_\lambda^2 + \sigma^2.$$

Such results have been obtained before, for example in [4], though perhaps not so simply.

In model (26) the lack of replication of course leaves the ω's and λ's confounded. We suppose now that there are b observations per cell, so that the model can be written

$$x_{ijk} = \rho_{ij} + \omega_{ijk}, \qquad k = 1, 2, \cdots, b. \tag{27}$$

As in section 5, we consider the matrix of

$$x_{ij.} = \rho_{ij} + \omega_{ij.},$$

supposing again that the population of ω's is infinite. We then obtain

$$E(\overline{MSR}) = \left(1 - \frac{c}{C}\right) \sigma_\lambda^2 + c\sigma_R^2 + \sigma^2/b, \tag{28}$$

$$E(\overline{MSC}) = \left(1 - \frac{r}{R}\right) \sigma_\lambda^2 + r\sigma_C^2 + \sigma^2/b, \tag{29}$$

$$E(\overline{MSB}) = \sigma_\lambda^2 + \sigma^2/b, \tag{30}$$

where $\overline{MSR}$ means MSR for the matrix $\| x_{ij.} \|$, etc. Since

$$MSW = \frac{1}{rc} \sum_{i,j} \frac{1}{b-1} \sum_k (x_{ijk} - x_{ij.})^2$$

has expectation σ^2, we find from (28), (29), and (30) the following unbiased estimators for the variance components:

$$\hat{\sigma}^2 = MSW,$$

$$\hat{\sigma}_\lambda^2 = \overline{MSB} - MSW/b,$$

$$\hat{\sigma}_R^2 = \overline{MSR}/c - \left(\frac{1}{c} - \frac{1}{C}\right) \overline{MSB} - MSW/bC,$$

$$\hat{\sigma}_C^2 = \overline{MSC}/r - \left(\frac{1}{r} - \frac{1}{R}\right) \overline{MSB} - MSW/bR.$$

One would like to know the variances of these estimators, but in general the third dimension introduced by the subscript k seems to preclude finding them by means of bipolykays. However, if we are willing to assume that R and C are infinite, we have

$$\hat{\sigma}_R^2 = (\overline{MSR} - \overline{MSB})/c = \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix},$$

$$\hat{\sigma}_C^2 = (\overline{MSC} - \overline{MSB})/r = \begin{pmatrix} \overline{1 \ -} \\ 1 \ - \end{pmatrix},$$

and the variances of these can be found as follows:

$$\begin{aligned} \operatorname{var} \hat{\sigma}_R^2 &= \text{ave aver} \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix}^2 - \left\{ \text{ave aver} \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix} \right\}^2 \\ &= \text{ave aver} \sum a_i \bar{F}_i - \left\{ \text{ave aver} \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix} \right\}^2, \end{aligned}$$

where $\sum a_1 \bar{F}_1$ is the expression for $\begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix}^2$ in terms of bipolykays of degree 4, the a_i being functions of r and c given in Section 8 of [3]. In this sum 8 distinct F's appear, and their pairing formulas in the present situation become as follows, by virtue of the remarks under Case III in Section 5:

$$\text{ave aver } \bar{F}_{27} = F'_{27} + 2 \begin{pmatrix} 2 \ - \\ - \ - \end{pmatrix}' \tilde{k}_2'' + \tilde{k}_{22}'',$$

$$\text{ave aver } \bar{F}_{18} = F'_{18} + \begin{pmatrix} 1 \ 1 \\ - \ - \end{pmatrix}' \tilde{k}_2'',$$

$$\text{ave aver } \bar{F}_i = F'_i, \qquad i = 4, 8, 13, 17, 22, 29,$$

where $\tilde{k}_2''$, etc., are polykays of the population of $\omega_{ij.}$ and can be expressed in terms of the polykays of the ω_{ijk} as at the end of Section 5. Finally we have, again from Case III in Section 5,

$$\text{ave aver} \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix} = \begin{pmatrix} 1 \ 1 \\ - \ - \end{pmatrix}',$$

and so

$$\left\{ \text{ave aver} \begin{pmatrix} \overline{1 \ 1} \\ - \ - \end{pmatrix} \right\}^2 = F'_4,$$

by Section 9 of [3], since R and C are infinite. Hence

$$\begin{aligned} rc(r-1)(c-1) \operatorname{var} \hat{\sigma}_R^2 = {} & 2rc(c-1)F'_4 + c(r-1)(c-1)F'_8 \\ & + 4r^2(c-1)F'_{13} + 2r^2F'_{17} + 4r(c-1)F'_{18} \\ & + 4(r-1)(c-1)F'_{22} + 2rF'_{27} + 2(r-1)F'_{29} \end{aligned}$$

$$+ 4r(c-1)\begin{pmatrix}1 & 1\\ - & -\end{pmatrix}' k_2''/b$$

$$+ 4r\begin{pmatrix}2 & -\\ - & -\end{pmatrix}' k_2''/b + 2rk_2''/b^2.$$

9. Computation. In order to make use of the formulas developed in this paper, it is of course necessary to be able to compute the bipolykays in particular numerical situations. Those of degree 2 can be easily found from equations (3) through (6) of Section 4, after *MSR*, *MSC*, etc., have been computed by standard procedures. A method of computation has been developed for the bipolykays of degree 4, but it will not be given here, as it is very lengthy, and it is hoped that better procedures can be found; this method was reported in [2], copies of which may be obtained on request by writing the secretary of the Statistical Research Group, Box 708, Princeton, N. J.

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