SYMMETRIC FUNCTIONS OF A TWO-WAY ARRAY[1]

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1. Summary. A family of polynomials in the elements of a two-way array, or matrix, is introduced. This family is an extension, from sets to matrices, of the family of symmetric polynomials k_1, k_2, k_{11}, k_3, k_{12}, etc., defined by Tukey [6], christened "polykays" in [7], and which are a generalization of the family k_1, k_2, etc., defined by R. A. Fisher [1]. The polynomials of the present paper, called "bipolykays," are symmetric functions in the sense that they are invariant under permutation of rows and/or columns of the matrix. This paper defines the bipolykays, shows that they are inherited on the average, develops the formulas for use in random pairing, and provides tables for conversion and for multiplication. A description of applications (see [2], [3], and [4]) will be postponed until a later paper. These applications include (a) finding expressions for sampling moments of functions of the elements of a matrix which is a "bi-sample" from a larger matrix, (b) finding expressions for sampling moments of functions (such as estimates of variance components) associated with the analysis of variance of a two-way table with systematic interactions, and (c) finding unbiased estimators for the variances and covariances of estimated variance components in a two-way table without interactions.

2. Introduction. Let $x_I(I = 1, 2, \cdots, N)$ be any population of N numbers, and let $x_i(i = 1, 2, \cdots, n)$ represent elements of a sample of size n from this population. Let $f(n; x_1, \cdots, x_n)$ be a polynomial which is symmetric in the x_i and has coefficients which are functions of n. Such a function extends obviously to a polynomial $f(N; x_1, \cdots, x_N)$, the corresponding symmetric polynomial in the x_I, with the coefficients changed only by replacing n by N. Writing "ave" for the operation of averaging over all $\binom{N}{n}$ distinct samples of size n from the population, we say that $f(n; x_1, \cdots, x_n)$ is "inherited on the average" [6] if

$$\text{ave} f(n; x_1, \cdots, x_n) = f(N; x_1, \cdots, x_N). \tag{1}$$

The functions k_1, k_2, k_{11}, etc., defined in [6] and now called polykays, are symmetric polynomials that are inherited on the average. Any symmetric polynomial can be expressed as a linear combination of polykays, so that the average value (or expected value, if the $\binom{N}{n}$ distinct samples are assigned equal probabilities) of the polynomial can be found simply by replacing each polykay in

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this linear combination by the corresponding population polykay, i.e., by applying (1) to each term.

In order to use polykays in connection with a linear model, say

$$y_{ij} = m_i + e_{ij},$$

one needs to find the polykays of the y's in terms of those of the m's and e's. The rules for doing this are called "pairing formulas" (Section 3), and an important advantage of polykays over most other symmetric polynomial functions inherited on the average is the simplicity of their pairing formulas.

In this paper we shall consider a matrix population of numbers $x_{IJ}(I = 1, 2, \cdots, R; J = 1, 2, \cdots, C)$ from which a bisample (sample matrix) is selected by taking a sample of r of the R rows and another sample of c of the C columns and forming the matrix whose elements are at the intersections of these selected rows and columns. Symmetric polynomial functions of such matrices (i.e., polynomial functions of the elements of a matrix which are invariant under permutation of rows and/or columns) will be considered. It will be shown that any such function can be expressed as a linear combination of bipolykays, which will be defined as a special family of functions that are inherited on the average and have simple pairing formulas. These properties make the bipolykays useful in the determination of moments of moments, for example, associated with a two-way classification.

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3. Polykays. Polykays are defined by examples in [6], and a general definition may be found in [7], [8], or [9]. Since a different, though equivalent, definition appears to be more suited to the extension to bipolykays, this section will be devoted to a general definition of polykays and to the derivation of those properties which will be required in this paper. We begin with some notation and terminology of [6] which will be used throughout.

The symbol $\sum^{\neq}$ will mean a sum over all subscripts that follow, but such that subscripts represented by different letters must remain unequal throughout the summation. For example,

$$\sum_{i,j=1}^{2}{}^{\neq} x_i x_j = x_1 x_2 + x_2 x_1.$$

A *symmetric mean* is a polynomial

$$\frac{1}{M} \sum{}^{\neq} x_i^a x_j^b \cdots x_m^d,$$

where the subscripts are summed from 1 to n (for samples) or from 1 to N (for populations), the exponents are positive integers, and M is the number of terms in the summation. When the sample (or population) size is given, the symmetric mean is specified by the exponents, and so is abbreviated by writing the exponents within brackets, as in

$$\langle a\ b\ d\rangle = \frac{1}{n(n-1)(n-2)} \sum^{\neq} x_i^a x_j^b x_k^d .$$

When a symmetric mean is defined over a population, this fact is indicated by a prime, as in

$$\langle a\ b\ d\rangle' = \frac{1}{N(N-1)(N-2)} \sum^{\neq} x_I^a x_J^b x_K^d .$$

It is obvious that any symmetric polynomial function can be expressed as a linear combination of symmetric means. Since symmetric means are inherited on the average [6], they are sufficient for the problem of finding expressions for sampling moments of moments of a single sample. However, in dealing with an additive model, one works with numbers which are sums of numbers sampled from different populations. To provide for this case, Tukey uses the notion of "random pairing": this means taking two samples, $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$, the order within each having been independently randomized, and adding the two to obtain a new sample $(z_1, \cdots, z_n)$, where $z_i = x_i + y_i$. For symmetric functions of the z's one wants the average value (where the average is taken with respect both to sampling and to randomization of order within samples) expressed, by means of a "pairing formula", in terms of symmetric functions of the two original populations. Using "ave" as before, together with "aver", meaning "average over randomization", and using one and two primes, respectively, for the populations of x's and y's, we have the following example of a pairing formula as applied to the symmetric mean $\langle 12\rangle$ taken over the z's:

$$\begin{aligned} \text{ave aver } \langle 12\rangle &= \langle 12\rangle' + \langle 1\rangle' \langle 2\rangle'' + \langle 2\rangle' \langle 1\rangle'' \\ &\quad + 2\langle 1\rangle' \langle 11\rangle'' + 2\langle 11\rangle' \langle 1\rangle'' + \langle 12\rangle''. \end{aligned}$$

The polykays are linear combinations of symmetric means chosen, among other reasons, because of their having simple pairing formulas. Those of degree 3 or less are defined as follows:

$$\begin{aligned} k_1 &= \langle 1\rangle, & k_{111} &= \langle 111\rangle, \\ k_{11} &= \langle 11\rangle, & k_{12} &= \langle 12\rangle - \langle 111\rangle, \\ k_2 &= \langle 2\rangle - \langle 11\rangle, & k_3 &= \langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle. \end{aligned} \tag{2}$$

The pairing formula for k_{12} becomes, for example,

$$\text{ave aver } k_{12} = k_{12}' + k_1'k_2'' + k_2'k_1'' + k_{12}''.$$

The remainder of this section consists of a general definition of the polykays and a derivation of the pairing formulas for symmetric means and polykays. A new notation for symmetric means (to be extended later to polykays) will first be introduced. Henceforth the notation used in (2), above, will be referred to as the *primary* notation, and that used in (3), below, as the *secondary* notation.

DEFINITION. The entries $a, b, \cdots, d$ of a symmetric mean $\langle ab \cdots d\rangle$ of degree m form a partition of the integer m. It will be convenient to represent such a partition in terms of m distinct symbols, so that the secondary notation for $\langle ab \cdots d\rangle$ will be

$$(3) \qquad \langle q_1q_2 \cdots q_a, r_1r_2 \cdots r_b, \cdots, s_1s_2 \cdots s_d\rangle,$$

where commas are used to separate the *parts* of the partition, and the *lengths* of the parts are the positive integers $a, b, \cdots, d$, whose sum is m. Any use of the word partition below will refer to an expression such as that enclosed in $\langle\ \rangle$'s in (3). Two partitions are *equivalent* (not distinct) if they are identical, except possibly for the order of parts and the order of symbols within a part. Greek letters will be used to represent arbitrary partitions. A partition β is a *subpartition* of a partition α if α can be made equivalent to β merely by the insertion of one or more commas. A *dichotomy* of a partition α is an ordered set $\{\alpha_1, \alpha_2\}$ of two partitions, α_1 and α_2, such that α_1 consists of some of the symbols comprising α, and α_2 of the remaining ones, and such that any two symbols which both occur in α_1 or both in α_2 belong there to the same part if and only if they belonged to the same part of α. The null partition will be denoted by ϕ, so that $\{\phi, \alpha\}$ and $\{\alpha, \phi\}$ are dichotomies of α. A *simple* dichotomy of α into $\{\alpha_1, \alpha_2\}$ has the property that each part of α belongs entirely to α_1 or to α_2. The *join* of partitions α_1 and α_2, having no symbols in common, is that partition α such that $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_1\}$ are simple dichotomies of α. An expression such as $\langle\alpha\rangle$, with brackets enclosing a Greek letter, will denote a symmetric mean, not with just one entry, but with entries which are the lengths of the parts of the partition α. The symmetric mean $\langle\phi\rangle$ is defined to be 1.

THEOREM 1. *The pairing formula for a symmetric mean* $\langle\alpha\rangle$ *is*

$$\text{ave aver } \langle\alpha\rangle = \sum \langle\beta\rangle' \langle\gamma\rangle'',$$

where the summation extends over all distinct dichotomies $\{\beta, \gamma\}$ *of* α.

PROOF. We recall that $\langle\alpha\rangle$ is a symmetric mean for a sample of numbers of the form $x_i + y_i$. Hence if $\langle\alpha\rangle$ is of the form (3), we have

$$\langle\alpha\rangle = \frac{1}{M} \sum\nolimits^{\neq} [(x_i + y_i) \cdots (x_i + y_i)][(x_j + y_j) \cdots (x_j + y_j)] \cdots [(x_k + y_k) \cdots (x_k + y_k)],$$

where there are a, b, and d equal factors within the first, second, and last pair of square brackets, respectively. For a fixed choice of $i, j, \cdots, k$, the product following the $\sum^{\neq}$ symbol expands to a sum of $2^{a+b+\cdots+d}$ terms, each of the form

$$(4) \qquad X_{i,j,\cdots,k}\, Y_{i,j,\cdots,k},$$

where

$$X_{i,j,\cdots,k} = x_i^A x_j^B \cdots x_k^D,$$

and

$$Y_{i,j,\cdots,k} = y_i^{a-A} y_j^{b-B} \cdots y_k^{d-D}.$$

Each term of the form (4) must be summed over the allowable sets of values of $i, j, \cdots, k$, averaged over randomization, and divided by M; aver $\langle\alpha\rangle$ is the sum of these individual results, one for each split of $a, b, \cdots, d$ into $A, B, \cdots, D$ and $a - A, b - B, \cdots, d - D$. From the independence of the two randomizations, we have

$$\text{(5)} \qquad \frac{1}{M} \text{aver} \sum\nolimits^{\neq} X_{i,j,\cdots,k} Y_{i,j,\cdots,k} = \frac{1}{M} \sum\nolimits^{\neq} \text{aver } X_{i,j,\cdots,k} \text{ aver } Y_{i,j,\cdots,k}.$$

But aver $X_{i,j,\cdots,k}$ is simply

$$\langle q_1 \cdots q_A, r_1 \cdots r_B, \cdots, s_1 \cdots s_C \rangle^* = \langle\beta\rangle^*,$$

where $\langle\beta\rangle$ is a symmetric mean of the type mentioned in the statement of the theorem, the asterisk indicating that it at present refers only to the sample of x's in question. Similarly, aver $Y_{i,j,\cdots,k}$ is $\langle\gamma\rangle^{**}$, γ and β being related as in the statement of the theorem. Hence

$$\begin{aligned} \text{aver } \langle\alpha\rangle &= \frac{1}{M} \sum \sum\nolimits^{\neq} \text{aver } X_{i,j,\cdots,k} \text{ aver } Y_{i,j,\cdots,k} \\ &= \frac{1}{M} \sum \sum\nolimits^{\neq} \langle\beta\rangle^* \langle\gamma\rangle^{**}. \end{aligned}$$

The M terms in the $\sum^{\neq}$ summation being equivalent, this reduces to

$$\text{aver } \langle\alpha\rangle = \sum \langle\beta\rangle^* \langle\gamma\rangle^{**},$$

this last summation being as defined in the statement of the theorem. The final step is to average over samples. Since the samples are chosen independently from different populations, and since the symmetric means are inherited on the average, we have the theorem, namely,

$$\text{ave aver } \langle\alpha\rangle = \sum \langle\beta\rangle' \langle\gamma\rangle''.$$

Definition. For partitions of a fixed number, m, of symbols, we say that

$$\text{rank } \alpha < \text{rank } \beta$$

if (a) the number of parts in α exceeds the number of parts in γ, or if (b) α and β have the same number of parts; but when the parts are arranged in order of increasing length, the first $i - 1$ parts of α are equal in length to their corresponding parts in β, while the ith part of α is shorter than the ith part of β.

Our definition of polykays will be in terms of the secondary notation, a polykay being represented by (α) and distinguishable from a symmetric mean (in this notation) only by the use of parentheses in place of $\langle\ \rangle$'s.

Definition. The polykays of degree m are defined by the equations

$$\text{(6)} \qquad \langle\alpha\rangle = (\alpha) + \sum (\beta_\alpha),$$

where there is one equation for each symmetric mean $\langle\alpha\rangle$ of degree m, and where the summation is over all distinct subpartitions β_α of α. [If there are $S(m)$ symmetric means of degree m, the $S(m)$ equations (6) of course define the $S(m)$ polykays that occur on the right if and only if the determinant of the coefficients of the distinct polykays does not vanish. (Two polykays, or two symmetric means, are equivalent, or not distinct, if the partitions representing them can be made equivalent by renaming the symbols.) Since, in each equation of (6), the rank of (α) is greater than that of any of the (β_α), then when those (β_α) which may be equivalent are collected and results are ordered by descending rank, the determinant has ones down the main diagonal and zeros below, so that its value is 1.]

Since any symmetric polynomial function can be expressed as a linear combination of symmetric means, it follows from the definition just given that it can also be expressed as a linear combination of polykays.

EXAMPLE ($m = 3$). The symmetric means are $\langle 111\rangle$, $\langle 12\rangle$, and $\langle 3\rangle$, expressed in the primary notation, or $\langle p, q, s\rangle$, $\langle p, q\ s\rangle$, and $\langle p\ q\ s\rangle$ in the secondary notation, in order of ascending rank. The polykays are then defined by the equations

$$\begin{aligned}\langle p, q, s\rangle &= (p, q, s),\\ \langle p, q\ s\rangle &= (p, q\ s) + (p, q, s),\\ \langle p\ q\ s\rangle &= (p\ q\ s) + (p, q\ s) + (q, p\ s) + (s, p\ q) + (p, q, s).\end{aligned}$$

These may be solved to give

$$\begin{aligned}(p, q, s) &= \langle p, q, s\rangle,\\ (p, q\ s) &= \langle p, q\ s\rangle - \langle p, q, s\rangle,\\ (p\ q\ s) &= \langle p\ q\ s\rangle - \langle p, q\ s\rangle - \langle q, p\ s\rangle - \langle s, p\ q\rangle + 2\langle p, q, s\rangle,\end{aligned}$$

or, in the primary notation,

$$\begin{aligned}k_{111} &= \langle 111\rangle,\\ k_{12} &= \langle 12\rangle - \langle 111\rangle,\\ k_3 &= \langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle.\end{aligned}$$

THEOREM 2. *The pairing formula for a polykay* (α) *is*

$$\text{ave aver } (\alpha) = \sum(\beta)'\ (\gamma)'',$$

where the summation extends now (in contrast with the similarly written summation of Theorem 1) *only over the distinct* simple *dichotomies* $\{\beta, \gamma\}$ *of the partition* α.

PROOF. We obtain the result by induction on rank for a fixed degree m. For the lowest rank, i.e., for $\langle 11 \cdots 1\rangle$ and $k_{11\cdots 1}$ (in the primary notation), the symmetric mean and polykay are identical; since all dichotomies in this case are simple, Theorem 2 holds for this rank by virtue of Theorem 1. For other ranks, we observe in equation (6) that

$$\text{ave aver } \langle\alpha\rangle = \text{ave aver } (\alpha) + \sum \text{ave aver } (\beta_\alpha) \tag{7}$$

and recall that the rank of each of the β_α is less than that of α. The induction assumption then will be that the theorem has been proved for the (β_α). Applying the theorem to any particular ave aver (β_α) gives us the sum of

$$(\gamma)'(\delta)''$$

over all distinct simple dichotomies $\{\gamma, \delta\}$ of β_α. None of these dichotomies can arise from any of the other β_α, since any simple dichotomy determines the partition from which it comes. Hence, since the various β_α are all the distinct subpartitions of α, it follows that

$$\sum \text{ave aver } (\beta_\alpha) = \sum (\gamma)'(\delta)'',$$

where the last sum extends over all $\sum (\gamma)'(\delta)''$ such that $\{\gamma, \delta\}$ is a simple dichotomy of some subpartition of α. This is the same as saying that the sum extends over all $(\gamma)'(\delta)''$ such that the join of γ and δ is a subpartition of α.

Going to the left side of (7), we have, from Theorem 1,

$$\text{ave aver } \langle\alpha\rangle = \sum \langle\lambda\rangle'\langle\mu\rangle'',$$

where the sum extends over all distinct dichotomies $\{\lambda, \mu\}$ of α. Each $\langle\lambda\rangle'$ and $\langle\mu\rangle''$ can be expressed in terms of polykays by equations (6). Since two λ's arising from distinct dichotomies of α cannot contain the same symbols, $\sum \langle\lambda\rangle'\langle\mu\rangle''$ must be equal to

$$\sum (\xi)'(\eta)''$$

where this sum extends over all terms where ξ and η are subpartitions of some λ and μ (or $\xi = \lambda$ or $\eta = \mu$ or both), respectively, $\{\lambda, \mu\}$ being a dichotomy of α. This is evidently the same as the sum of all $(\xi)'(\eta)''$ such that the join of ξ and η is α or a subpartition of α.

The first and third of the three terms of (6) have now been specified, and ave aver (α) is equal to their difference, which is the sum of all $(\xi)'(\eta)''$ such that the join of ξ and η is α.

EXAMPLE. Consider the polykay k_{12}, or $(p,\ q\ s)$. The simple dichotomies of $p,\ q\ s$ are

$$\begin{array}{rcl} p,\ q\ s & \text{and} & \phi, \\ p & \text{and} & q\ s, \\ q\ s & \text{and} & p, \\ \phi & \text{and} & p,\ q\ s, \end{array}$$

so that

$$\begin{aligned} \text{ave aver } k_{12} &= \text{ave aver } (p,\ q\ s) \\ &= (p,\ q\ s)'(\phi)'' + (p)'(q\ s)'' + (q\ s)'(p)'' + (\phi)'(p,\ q\ s)'' \\ &= k'_{12} + k'_1k''_2 + k'_2k''_1 + k''_{12}, \end{aligned}$$

(ϕ) being 1.

4. Bisamples and generalized symmetric means. We turn now to the problem of the present paper. We suppose a population matrix

$$\|x_{IJ}\|, \qquad I = 1, 2, \cdots, R; \; J = 1, 2, \cdots, C$$

from which a bisample

$$\|x_{ij}\|, \qquad i = 1, 2, \cdots, r; j = 1, 2, \cdots, c$$

is selected as described in Section 2. Any polynomial symmetric in the x_{ij} (in the sense defined in section 2) is a linear combination of sums of the type

$$\sum\nolimits^{\neq} x_{pq}^{a_{pq}} \cdots x_{st}^{a_{st}},$$

where the symbol $\sum^{\neq}$, for two-way arrays, will mean summation over all subsequent subscripts, with the restriction that row subscripts represented by different letters must remain different throughout the summation, and the same for column subscripts.

We define *generalized symmetric means* to be averages of monomial functions over a matrix; i.e., a g.s.m. is a polynomial

$$(8) \qquad \frac{1}{M}\Big(\sum_{p,q,\cdots,s,t}^{\neq} x_{pq}^{a_{pq}} \cdots x_{st}^{a_{st}} \Big),$$

where M is the number of terms in the summation. A g.s.m. is specified by the exponents, together with information which tells which ones correspond to elements that lie in the same row, and which ones correspond to elements that lie in the same column. A convenient notation for g.s.m.'s is thus provided by placing the exponents in a matrix within brackets in such a way that exponents which affect elements in the same row of the matrix $\|x_{ij}\|$ are entered in the same row, and similarly for columns. Thus,[3]

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & d \end{bmatrix} = \frac{1}{rc(r-1)(c-1)(c-2)} \sum\nolimits^{\neq} x_{ij}^{a} x_{ik}^{b} x_{mn}^{d}.$$

Ordinarily the zeros will be replaced by dashes. Dashes will also be used to extend every matrix of entries to at least two rows and two columns to avoid confusion with symmetric means and, when parentheses are later introduced, to avoid confusion with binomial coefficients. Thus,

$$\begin{bmatrix} 2 & - \\ - & - \end{bmatrix} = \frac{1}{rc} \sum\nolimits^{\neq} x_{ij}^{2}$$

$$\begin{bmatrix} 3 & - \\ 2 & - \end{bmatrix} = \frac{1}{rc(r-1)} \sum\nolimits^{\neq} x_{ij}^{3} x_{kj}^{2}.$$

Evidently two g.s.m.'s are identical if the matrix of entries of one can be obtained from that of the other by permuting rows and/or columns. The distinct g.s.m.'s of degrees 1 and 2 are as follows:

[3] Square brackets are used, for convenience in printing, in place of ⟨ ⟩'s for g.s.m.'s having more than one row.

Degree 1: $\begin{bmatrix} 1 & - \\ - & - \end{bmatrix}$.

Degree 2: $\begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix}, \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix}, \begin{bmatrix} 2 & - \\ - & - \end{bmatrix}$.

The idea of random pairing for bisamples is a straightforward extension of that described in Section 3 for samples: Given two $r \times c$ matrices $\|x_{ij}\|$ and $\|y_{ij}\|$, the order of rows and of columns is randomized in each, and a new $r \times c$ matrix $\|z_{ij}\|$ is formed by matrix addition of the results.

The general term

$$x_{pq}^{a_{pq}} \cdots x_{st}^{a_{st}}$$

(which specifies, as in (8), a g.s.m. of degree m) contains m factors, a_{pq} of which are equal to x_{pq}, etc. To each of these factors we assign a different symbol, and the resulting set of symbols may be partitioned in two ways—once by rows, and once by columns. The secondary notation for the g.s.m. will then be an ordered pair

$$\langle \alpha/\beta \rangle$$

of partitions α and β, each on the same set of symbols. Each part of α will consist of those symbols which correspond to factors having a particular row subscript, and the parts of β are similarly determined by column subscripts. For example,

$$\begin{bmatrix} 2 & 1 \\ - & 1 \end{bmatrix} = \frac{1}{rc(r-1)(c-1)} \sum^{\neq} x_{pq}^2 x_{ps} x_{ts}$$

becomes, in the secondary notation,

$$\langle a\ b\ d,\ e\ /\ a\ b,\ d\ e \rangle.$$

To establish the property of inheritance on the average, let

$$\langle \alpha/\beta \rangle' = \frac{1}{M} \sum^{\neq} x_{PQ}^{a_{PQ}} \cdots x_{ST}^{a_{ST}}$$

represent any g.s.m. for an $R \times C$ population. This is the average, with equal weights, of all terms $B = x_{PQ}^{a_{PQ}} \cdots x_{ST}^{a_{ST}}$. If $\langle \alpha/\beta \rangle$ represents the same g.s.m. for an $r \times c$ bisample, one or more of the expansions of $\langle \alpha/\beta \rangle$ over various bisamples will contain any given term B. Hence ave $\langle \alpha/\beta \rangle$ is a weighted average of all terms B, and it follows from the symmetry of the set of all $r \times c$ bisamples that the weights in this average are also equal, so that ave $\langle \alpha/\beta \rangle = \langle \alpha/\beta \rangle'$.

THEOREM 3. *For any g.s.m.* $\langle \alpha/\beta \rangle$, *the pairing formula is*

$$\text{ave aver } \langle \alpha/\beta \rangle = \sum \langle \gamma/\delta \rangle' \langle \lambda/\mu \rangle'',$$

where the summation extends over all distinct dichotomies $\{\gamma, \lambda\}$ *of* α *and* $\{\delta, \mu\}$ *of* β, γ *and* δ *consisting of the same symbols.*

The proof of this theorem, being virtually identical with that of Theorem 1, will be omitted.

5. Definition of the bipolykays. In order to make the general definition of bipolykays, we define a "dot-multiplication" for symmetric means as follows:

$$\begin{aligned} \langle\alpha\rangle\cdot\langle\beta\rangle &= \langle\alpha/\beta\rangle \quad \text{if } \alpha \text{ and } \beta \text{ consist of the same symbols} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

This noncommutative multiplication can be extended by distributivity to provide dot-products of linear combinations of symmetric means.

DEFINITION. The *bipolykay* (α/β), where α and β are partitions of the same set of symbols, is

$$(\alpha/\beta) = (\alpha)\cdot(\beta),$$

it being understood that (α) and (β) are expressed as sums of symmetric means (as in the example just before Theorem 2, Section 3) before the dot-product is taken.

EXAMPLE. Consider the bipolykay $\begin{pmatrix}1 & 1\\ - & 1\end{pmatrix}$. (The primary notation for a bipolykay (α/β) is the same as that for the g.s.m. $\langle\alpha/\beta\rangle$, with $\langle\ \rangle$'s replaced by parentheses.) This becomes, in the secondary notation,

$$\begin{aligned} \begin{pmatrix}1 & 1\\ - & 1\end{pmatrix} &= (p\ q,\ s\ /\ p,\ q\ s) \\ &= (p\ q,\ s)\cdot(p,\ q\ s) \qquad \text{by the definition above} \\ &= [\langle s,\ p\ q\rangle - \langle s,\ p,\ q\rangle]\cdot[\langle p,\ q\ s\rangle - \langle p,\ q,\ s\rangle] \\ &\qquad \text{by the example preceding Theorem 2} \\ &= \langle s,\ p\ q\ /\ p,\ q\ s\rangle - \langle s,\ p,\ q\ /\ p,\ q\ s\rangle - \langle s,\ p\ q\ /\ p,\ q,\ s\rangle + \langle s,\ p,q\ /\ p,q,\ s\rangle \\ &= \begin{bmatrix}1 & 1\\ - & 1\end{bmatrix} - \begin{bmatrix}1 & -\\ 1 & -\\ - & 1\end{bmatrix} - \begin{bmatrix}1 & 1 & -\\ - & - & 1\end{bmatrix} + \begin{bmatrix}1 & - & -\\ - & 1 & -\\ - & - & 1\end{bmatrix}. \end{aligned}$$

Since bipolykays are linear combinations (with constant coefficients) of the g.s.m.'s, the bipolykays must also be inherited on the average. By means of the device of ranking (as was done for polykays in Section 3), one can show that the g.s.m.'s can in turn be expressed as linear combinations of bipolykays. (This is done explicitly through degree 4 in Section 8.) Hence any polynomial symmetric function of elements of a bisample can be expressed as a linear combination of bipolykays.

6. Pairing formulas for bipolykays. The statement of pairing formulas for bipolykays requires the following terminology:

DEFINITION. The bipolykay (α/β) is said to be *decomposable* if there exist simple dichotomies $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ of α and β, respectively, such that α_1 and β_1 consist of the same symbols, and neither α_1 nor α_2 is null. In this case, (α/β) may be written as a product

$$(\alpha/\beta) = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2),$$

where the commutative operation denoted by $\times$ is defined by this equation, and (α_1, β_1) and (α_2, β_2) are called *components* of (α/β). If any component is similarly decomposable, the original bipolykay can be written as the $\times$-product of at least three components, and clearly any decomposable bipolykay can finally be written as the $\times$-product of indecomposable components where the set of indecomposable components is unique except for order.

THEOREM 4. *If a bipolykay (α/β) is indecomposable, its pairing formula is simply*

$$\text{ave aver } (\alpha/\beta) = (\alpha/\beta)' + (\alpha/\beta)''.$$

If (α, β) is decomposable and is the $\times$-product of indecomposable components (α_i/β_i), $i = 1, 2, \cdots, d$, *the pairing formula is*

$$\text{ave aver } (\alpha/\beta) = (\alpha/\beta)' + (\alpha/\beta)'' + \sum (\gamma/\delta)'(\lambda/\mu)'',$$

where the summation extends over all expressions for which (γ/δ) is the $\times$-product of $1, 2, \cdots$, or $d - 1$ of the (α_i/β_i) and (λ/μ) is the $\times$-product of the remaining ones.

EXAMPLES:

$$\text{ave aver} \begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix} = \text{ave aver } (p\ q,\ s\ /\ p,\ q\ s)$$

$$= (p\ q,\ s\ /\ p,\ q\ s)' + (p\ q,\ s\ /\ p,\ q\ s)'' \text{ since this is indecomposable}$$

$$= \begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix}' + \begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix}''$$

$$\text{ave aver} \begin{pmatrix} 1 & 1 & - \\ - & - & 1 \end{pmatrix} = \text{ave aver } (p\ q,\ s\ /\ p,\ q,\ s)$$

$$= \text{ave aver } [(p\ q\ /\ p,\ q) \times (s/s)]$$

$$= (p\ q,\ s\ /\ p,\ q,\ s)' + (p\ q,\ s\ /\ p,\ q,\ s)''$$

$$+ (p\ q\ /\ p,\ q)'(s/s)'' + (s/s)'(p\ q\ /\ p,\ q)''$$

$$= \begin{pmatrix} 1 & 1 & - \\ - & - & 1 \end{pmatrix}' + \begin{pmatrix} 1 & 1 & - \\ - & - & 1 \end{pmatrix}'' + \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}' \begin{pmatrix} 1 & - \\ - & - \end{pmatrix}''$$

$$+ \begin{pmatrix} 1 & - \\ - & - \end{pmatrix}' \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}''.$$

(Note: A decomposable bipolykay in primary notation can easily be recognized, as its matrix of entries can be put in the form

$$\left\| \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right\|$$

where A, B, C, D are matrices, with all elements of B and C zero.)

The remainder of this section will be given over to the proof of Theorem 4. As before, asterisks will indicate bipolykays (or g.s.m.'s) for bisamples, and primes will denote population values; if a certain population is indicated by n primes, a bisample from that population will be indicated by n asterisks.

DEFINITION (EXTENDING THE DOT-MULTIPLICATION). In dealing with two different populations, we define

$$[\langle\alpha\rangle^*\langle\beta\rangle^{**}]\cdot[\langle\gamma\rangle^*\langle\delta\rangle^{**}] = [\langle\alpha\rangle\cdot\langle\gamma\rangle]^*[\langle\beta\rangle\cdot\langle\delta\rangle]^{**},$$

and by extension this provides a meaning for any expression which is formally written as a dot product of linear combinations of terms of the type $\langle\alpha\rangle^*\langle\beta\rangle^{**}$. Asterisks may be replaced by primes. (Note: Since we are dealing with matrices, the terms $\langle\alpha\rangle^*\langle\beta\rangle^{**}$ themselves have no meaning.)

LEMMA 1. ave aver $[\langle\alpha\rangle\cdot\langle\beta\rangle]$ = [ave aver $\langle\alpha\rangle$]$\cdot$[ave aver $\langle\beta\rangle$], and ave aver $[(\alpha)\cdot(\beta)]$ = [ave aver (α)]$\cdot$[ave aver (β)]. (Here $\langle\alpha\rangle\cdot\langle\beta\rangle = \langle\alpha/\beta\rangle$ is a g.s.m. for a sample formed by random pairing of two bisamples. The expressions ave aver $\langle\alpha\rangle$ and ave aver $\langle\beta\rangle$ are formal expressions of Theorem 1, their dot product having meaning only from the definition just above. Similarly for the polykays, which must be expressed as sums of g.s.m.'s before the above definition gives them meaning.)

PROOF. If α and β do not consist of the same symbols, the result is trivial. If they do, then $\langle\alpha\rangle\cdot\langle\beta\rangle = \langle\alpha/\beta\rangle$, and so

$$\text{ave aver } [\langle\alpha\rangle\cdot\langle\beta\rangle] = \sum \langle\gamma/\delta\rangle'\langle\lambda/\mu\rangle'',$$

where γ, δ, λ, μ are as described in Theorem 3. Clearly, from Theorem 1, [ave aver $\langle\alpha\rangle$]$\cdot$[ave aver $\langle\beta\rangle$] gives the same sum, so we may now go to the second part of this lemma. In the case of polykays, we have, by their definition,

$$(\alpha) = \textstyle\sum_i a_i\langle\alpha_i\rangle,$$
$$(\beta) = \textstyle\sum_j b_j\langle\beta_j\rangle,$$

where α_i is α or a subpartition of α, and the β_j bear the same relation to β. Hence

$$\text{ave aver } [(\alpha)\cdot(\beta)] = \textstyle\sum_i \sum_j a_i b_j \text{ ave aver } [\langle\alpha_i\rangle\cdot\langle\beta_j\rangle],$$

and

$$[\text{ave aver } (\alpha)]\cdot[\text{ave aver } (\beta)] = \textstyle\sum_i \sum_j a_i b_j \, [\text{ave aver } \langle\alpha_i\rangle]\cdot[\text{ave aver } \langle\beta_j\rangle].$$

By the first part of the lemma these are the same, and so the second part is proved.

LEMMA 2. $(\alpha)'(\beta)''\cdot(\gamma)'(\delta)'' = [(\alpha)\cdot(\gamma)]'[(\beta)\cdot(\delta)]''$. *(Each side of this equation has the meaning that is provided by the above conventions after each polykay has been written as a linear combination of symmetric means.)*

PROOF. As in Lemma 1, we write

$$\begin{aligned}(\alpha)' &= \textstyle\sum_i a_i\langle\alpha_i\rangle',\\ (\beta)'' &= \textstyle\sum_j b_j\langle\beta_j\rangle'',\\ (\gamma)' &= \textstyle\sum_k c_k\langle\gamma_k\rangle',\\ (\delta)'' &= \textstyle\sum_m d_m\langle\delta_m\rangle''.\end{aligned}$$

Then

$$\begin{aligned}(\alpha)'(\beta)''\cdot(\gamma)'(\delta)'' &= \textstyle\sum_i\sum_j a_ib_j\langle\alpha_i\rangle'\langle\beta_j\rangle''\cdot\sum_k\sum_m c_k\,d_m\langle\gamma_k\rangle'\langle\delta_m\rangle''\\ &= \textstyle\sum_i\sum_j\sum_k\sum_m a_ib_jc_k\,d_m\langle\alpha_i\rangle'\langle\beta_j\rangle''\cdot\langle\gamma_k\rangle'\langle\delta_m\rangle'' \\ &\qquad\text{by definition}\\ &= \textstyle\sum_i\sum_j a_ib_j[\langle\alpha_i\rangle\cdot\langle\gamma_k\rangle]'\sum_k\sum_m[\langle\beta_j\rangle\cdot\langle\delta_m\rangle]''\\ &= [(\alpha)\cdot(\gamma)]'[(\beta)\cdot(\delta)]'',\end{aligned}$$

proving Lemma 2.

To prove Theorem 4, we write

$$\begin{aligned}\text{ave aver }(\alpha/\beta) &= \text{ave aver }[(\alpha)\cdot(\beta)] &&\text{by definition}\\ &= [\text{ave aver }(\alpha)]\cdot[\text{ave aver }(\beta)] &&\text{by Lemma 1}\\ &= \textstyle\sum(\lambda_\alpha)'(\mu_\alpha)''\cdot\sum(\lambda_\beta)'(\mu_\beta)'' &&\text{by Theorem 2,}\end{aligned}$$

where the first sum extends over all simple dichotomies $\{\lambda_\alpha, \mu_\alpha\}$ of α, and similarly for the second sum. Hence

$$\text{ave aver }(\alpha/\beta) = \textstyle\sum\sum[(\lambda_\alpha)'\cdot(\lambda_\beta)'][(\mu_\alpha)''\cdot(\mu_\beta)''],$$

by Lemma 2. Now λ_α is a partition consisting of some of the parts of α, with no other changes made, since $\{\lambda_\alpha, \mu_\alpha\}$ is a simple dichotomy of α. Similarly λ_β consists of some of the parts of β. The expression $(\lambda_\alpha)'\cdot(\lambda_\beta)'$ vanishes unless λ_α and λ_β comprise exactly the same symbols. Thus the only nonvanishing terms arise when

(a) $\lambda_\alpha = \alpha$ and $\lambda_\beta = \beta$, producing the term

$$[(\lambda_\alpha)'\cdot(\lambda_\beta)'][(\phi)''\cdot(\phi)''] = (\alpha/\beta)',$$

ϕ being null; or

(b) $\lambda_\alpha = \lambda_\beta = \phi$, producing the term $(\alpha/\beta)''$; or

(c) $\lambda_\alpha = \lambda_\beta$ and $\mu_\alpha = \mu_\beta$ and none of these is null.

The last case cannot happen to an indecomposable bipolykay. If the bipolykay is decomposable, case (c) gives exactly the various terms that correspond to the splitting of the bipolykay into indecomposable components, and Theorem 4 is established.

7. Pairing formulas for certain special cases. Various special cases and degenerate cases arise in connection with pairing when applied to the analysis of variance. In order to deal with some of these, we need first a lemma about polykays and then a theorem:

LEMMA 3. *If $k_{mn\cdots p}$ is any polykay, the coefficients in its expression as a linear combination of symmetric means add to* 0 *unless* $m = n = \cdots = p = 1$.

PROOF. Consider a population, or sample, all of whose elements are equal to 1. Then clearly every symmetric mean has the value 1, and the value of $k_{mn\cdots p}$ is the sum of the coefficients in its expression as a linear combination of symmetric means. We have only to show that in this case $k_{mn\cdots p} = 0$ unless

$$m = n = \cdots = p = 1.$$

Looking at equation (6), we see that, when all parts of α are of length 1, $\langle\alpha\rangle = (\alpha)$, i.e.,

$$k_{11\cdots 1} = \langle 11 \cdots 1\rangle,$$

and, in the present case, each of these equals 1. If α has one part of length 2, and all others are of length 1, then

$$\langle 11 \cdots 12\rangle = (11 \cdots 12) + (11 \cdots 1);$$

or $1 = (11 \cdots 12) + 1$ in the present case, so that $(11 \cdots 12)$ is 0. We can now prove the theorem by induction on rank, supposing that, for a given equation of type (6),

$$\langle\alpha\rangle = (\alpha) + \sum (\beta_\alpha),$$

all polykays of rank less than that of (α) are 0 except for $(11 \cdots 1)$. This equation then becomes

$$1 = (\alpha) + 1,$$

and $(\alpha) = 0$.

THEOREM 5. *Consider a bisample in which all elements in the same row are equal, i.e.,*

$$x_{ij} = x_i \qquad\qquad j = 1, 2, \cdots, c.$$

Over this matrix, a bipolykay (α/β) (a) *is equal to the polykay* (α), *defined over the set of* x_i, *if all parts of the partition* β *are of length* 1 (*i.e., if, in the primary notation, all entries are ones in different columns*); or (b) *is equal to* 0 *otherwise.* Obviously an analogous statement applies to a bisample with constant columns.

PROOF. In this case it is obvious that any g.s.m. $\langle\alpha/\beta\rangle$ is equal to $\langle\alpha\rangle$, the latter being defined over the set of x_i. Now if (α/β) is any polykay, we can write

$$\begin{aligned}(\alpha/\beta) &= (\alpha)\cdot(\beta)\\ &= \textstyle\sum_i a_i\langle\alpha_i\rangle\cdot\sum_j b_j\langle\beta_j\rangle && \text{by definition}\\ &= \textstyle\sum_i\sum_j a_ib_j\langle\alpha_i/\beta_j\rangle\\ &= \textstyle\sum_i\sum_j a_ib_j\langle\alpha_i\rangle, && \text{by remark at beginning of this paragraph.}\end{aligned}$$

This last expression vanishes unless $\sum b_j \neq 0$, i.e., unless (Lemma 3) all parts of the partition β are of length 1. Hence $\sum b_j$ is 1 when it is not 0, and in this case

$$\begin{aligned}(\alpha/\beta) &= \textstyle\sum_i a_i\langle\alpha_i\rangle\\ &= (\alpha_i).\end{aligned}$$

The special cases which we now wish to consider are as follows:

CASE I (CONSTANT ROWS). Theorem 5 shows that in this case a bipolykay (α/β) is given by

$$\begin{aligned}(\alpha/\beta) &= k_{mn\cdots p} && \text{if all parts of } \beta \text{ are of length 1}\\ &= 0 && \text{otherwise,}\end{aligned}$$

$m, n, \cdots, p$ being the lengths of the parts of α.

CASE II (CONSTANT COLUMNS). Here, of course,

$$\begin{aligned}(\alpha/\beta) &= k_{mn\cdots p} && \text{if all parts of } \alpha \text{ are of length 1}\\ &= 0 && \text{otherwise.}\end{aligned}$$

CASE III (CONSTANT ROWS AND COLUMNS). Here all elements of the bisample are equal. It follows that

$$\begin{aligned}(\alpha/\beta) &= k_{11\cdots 1} && \text{if all parts of } \alpha \text{ and } \beta \text{ have length 1}\\ &= 0 && \text{otherwise.}\end{aligned}$$

If d is the common value of the elements of the bisample, and m is the degree of (α/β), then $(\alpha/\beta) = d^m$ when it does not vanish.

These cases might arise, for example, in connection with a linear model such as

$$u_{ij} = m + x_i + y_i + z_{ij},$$

where the x_i can be thought of as a bisample from a matrix with constant rows, the y_j from a matrix with constant columns, m from a matrix with all elements equal, and z_{ij} from an arbitrary matrix representing "cell effects". Using 0, 1, 2, 3, and 4 primes (asterisks) for the populations (bisamples) respectively associated with u, m, x, y, and z, we find the pairing formula for any indecomposable bipolykay (α/β), for example, to be

$$\text{ave aver } (\alpha/\beta) = (\alpha/\beta)' + (\alpha/\beta)'' + (\alpha/\beta)''' + (\alpha/\beta)'''',$$

and the cases above would tell us that some of these terms are zero and others are equivalent to polykays, depending on what bipolykay (α/β) represents.

Instead of sampling from a matrix, one may wish to consider a degenerate case in which the population consists only of rows, with no column designations; i.e., an $r \times c$ bisample is chosen by a selection of r rows followed by a selection of c elements from each of the r rows chosen. There is also the completely degenerate case, with no rows or columns, so that an $r \times c$ bisample is just an ordinary sample, randomly arranged, of rc elements from a set of numbers. This case, which would apply, for example, to the z's in the linear model above if they were regarded as independently sampled "random errors" instead of fixed interactions, is designated as Case IV:

CASE IV. When pairing a bisample with a completely degenerate bisample, we have only to notice that randomization in the latter is not restricted to rows and columns, so that we have, for the completely degenerate case, these results:

(a) All g.s.m.'s with the same entries (primary notation) have equal averages for randomization, e.g.,

$$\text{aver} \begin{bmatrix} 2 & 1 \\ - & - \end{bmatrix} = \text{aver} \begin{bmatrix} 2 & - \\ 1 & - \end{bmatrix} = \text{aver} \begin{bmatrix} 2 & - \\ - & 1 \end{bmatrix}.$$

(b) All bipolykays vanish on the average except those having only diagonal elements (in secondary notation, this means those (α/β) such that α and β are equivalent partitions). This statement can be verified for degrees $\leqq 4$ by observing that, in the relevant conversion formulas (Section 8), the coefficients of g.s.m.'s with the same entries add to zero.

(c) Bipolykays with only diagonal entries are equal, on the average, to the corresponding polykays; e.g.,

$$\text{aver} \begin{pmatrix} 2 & - \\ - & 1 \end{pmatrix} = \text{aver } k_{21}, \qquad \text{etc.}$$

8. Conversion formulas for g.s.m.'s and bipolykays. In this and the next section, tables will be presented which make possible the use of bipolykays up through degree 4, that is, up through variances of variances. The distinct g.s.m.'s of degrees 1 and 2 were listed in Section 4. Those of degrees 3 and 4 require more space (in either notation) and so will be denoted by t's (for degree 3) and f's (for degree 4) with subscripts, even though this notation is less informative, as follows:

Degree 3: $$t_1 = \begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix} \qquad t_6 = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$$

$$t_2 = \begin{bmatrix} 1 & 1 & - \\ - & - & 1 \end{bmatrix}$$

$$t_3 = \begin{bmatrix} 1 & - \\ 1 & - \\ - & 1 \end{bmatrix}$$

$$t_4 = \begin{bmatrix} 1 & 1 & 1 \\ - & - & - \end{bmatrix}$$

$$t_5 = \begin{bmatrix} 1 & - \\ 1 & - \\ 1 & - \end{bmatrix}$$

$$t_7 = \begin{bmatrix} 2 & - \\ - & 1 \end{bmatrix}$$

$$t_8 = \begin{bmatrix} 2 & 1 \\ - & - \end{bmatrix}$$

$$t_9 = \begin{bmatrix} 2 & - \\ 1 & - \end{bmatrix}$$

$$t_{10} = \begin{bmatrix} 3 & - \\ - & - \end{bmatrix}$$

Degree 4:

$$f_1 = \begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

$$f_2 = \begin{bmatrix} 1 & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

$$f_3 = \begin{bmatrix} 1 & - & - \\ 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix}$$

$$f_4 = \begin{bmatrix} 1 & 1 & - & - \\ - & - & 1 & 1 \end{bmatrix}$$

$$f_5 = \begin{bmatrix} 1 & - \\ 1 & - \\ - & 1 \\ - & 1 \end{bmatrix}$$

$$f_6 = \begin{bmatrix} 1 & 1 & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

$$f_7 = \begin{bmatrix} 1 & - \\ 1 & - \\ 1 & - \\ - & 1 \end{bmatrix}$$

$$f_8 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ - & - & - & - \end{bmatrix}$$

$$f_{12} = \begin{bmatrix} 2 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix}$$

$$f_{13} = \begin{bmatrix} 1 & 1 & - \\ 1 & - & 1 \end{bmatrix}$$

$$f_{14} = \begin{bmatrix} 1 & 1 \\ 1 & - \\ - & 1 \end{bmatrix}$$

$$f_{15} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & - \end{bmatrix}$$

$$f_{16} = \begin{bmatrix} 1 & 1 \\ 1 & - \\ 1 & - \end{bmatrix}$$

$$f_{17} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$f_{18} = \begin{bmatrix} 2 & - & - \\ - & 1 & 1 \end{bmatrix}$$

$$f_{19} = \begin{bmatrix} 2 & - \\ - & 1 \\ - & 1 \end{bmatrix}$$

$$f_{23} = \begin{bmatrix} 2 & - \\ 1 & - \\ 1 & - \end{bmatrix}$$

$$f_{24} = \begin{bmatrix} 2 & 1 \\ - & 1 \end{bmatrix}$$

$$f_{25} = \begin{bmatrix} 2 & - \\ 1 & 1 \end{bmatrix}$$

$$f_{26} = \begin{bmatrix} 2 & 1 \\ 1 & - \end{bmatrix}$$

$$f_{27} = \begin{bmatrix} 2 & - \\ - & 2 \end{bmatrix}$$

$$f_{28} = \begin{bmatrix} 3 & - \\ - & 1 \end{bmatrix}$$

$$f_{29} = \begin{bmatrix} 2 & 2 \\ - & - \end{bmatrix}$$

$$f_{30} = \begin{bmatrix} 2 & - \\ 2 & - \end{bmatrix}$$

$$f_9 = \begin{bmatrix} 1 & - \\ 1 & - \\ 1 & - \\ 1 & - \end{bmatrix} \qquad f_{20} = \begin{bmatrix} 2 & 1 & - \\ - & - & 1 \end{bmatrix} \qquad f_{31} = \begin{bmatrix} 3 & 1 \\ - & - \end{bmatrix}$$

$$f_{10} = \begin{bmatrix} 1 & 1 & - \\ 1 & - & - \\ - & - & 1 \end{bmatrix} \qquad f_{21} = \begin{bmatrix} 2 & - \\ 1 & - \\ - & 1 \end{bmatrix} \qquad f_{32} = \begin{bmatrix} 3 & - \\ 1 & - \end{bmatrix}$$

$$f_{11} = \begin{bmatrix} - & 1 & 1 \\ 1 & - & - \\ 1 & - & - \end{bmatrix} \qquad f_{22} = \begin{bmatrix} 2 & 1 & 1 \\ - & - & - \end{bmatrix} \qquad f_{33} = \begin{bmatrix} 4 & - \\ - & - \end{bmatrix}$$

The following conversion formulas apply to the bipolykays of degrees 1 and 2:

Degree 1: $(1) = \langle 1 \rangle$

Degree 2:

$$\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix} = \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} - \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix} = \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix} - \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & - \\ - & - \end{pmatrix} = \begin{bmatrix} 2 & - \\ - & - \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} - \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix} + \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}.$$

The bipolykays of degree 2 have been independently developed by H. Fairfield Smith in [5].

For degrees 3 and 4 we use notation analogous to that used above for g.s.m.'s, letting T's stand for bipolykays of degree 3 and F's for bipolykays of degree 4. Thus

$$T_4 = \begin{pmatrix} 1 & 1 & 1 \\ - & - & - \end{pmatrix}$$

$$F_6 = \begin{pmatrix} 1 & 1 & 1 & - \\ - & - & - & 1 \end{pmatrix}, \qquad \text{etc.}$$

The conversion formulas for bipolykays of degrees 3 and 4 become quite long; but since they are linear, only the coefficients are of interest. These coefficients are found in Table 1 for degree 3 and in Table 2 for degree 4. The nature of the formulas makes it possible for one table to present coefficients for conversion in both directions. The coefficients for any desired expression are found by reading over (or down) to and including the diagonal of ones. For example, in Table 1,

TABLE 1

For conversion of g.s.m.'s and bipolykays of degree 3

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
T_1	1	1	1	1	1	1	1	1	1	1
T_2	−1	1		3		1	1	3	1	3
T_3	−1		1		3	1	1	1	3	3
T_4	2	−3		1				1		1
T_5	2		−3		1				1	1
T_6	1	−1	−1			1		2	2	6
T_7	1	−1	−1				1	1	1	3
T_8	−2	3	2	−1		−2	−1	1		3
T_9	−2	2	3		−1	−2	−1		1	3
T_{10}	4	−6	−6	2	2	6	3	−3	−3	1

$$T_6 = t_1 - t_2 - t_3 + t_6\,,$$

$$t_5 = T_1 + 3T_3 + T_5\,,$$

and similarly in Table 2.

9. Multiplication formulas for bipolykays. The usefulness of the property of inheritance on the average is pretty well limited to the case where functions having this property occur linearly. Any polynomial in bipolykays for one bisample, however, can be expressed as a linear combination of bipolykays for that bisample, given the proper multiplication formulas. We give below the multiplication formulas for bipolykays, up to and including products of degree 4.

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix}^2 = \begin{pmatrix}2 & -\\ - & -\end{pmatrix} + c\begin{pmatrix}1 & 1\\ - & -\end{pmatrix} + r\begin{pmatrix}1 & -\\ 1 & -\end{pmatrix} + rc\begin{pmatrix}1 & -\\ - & 1\end{pmatrix};$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix}\begin{pmatrix}1 & -\\ - & -\end{pmatrix} = 2T_7 + 2T_6 + 2rT_3 + 2cT_2 + rcT_1,$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix}\begin{pmatrix}1 & 1\\ - & -\end{pmatrix} = 2T_8 + 2rT_6 + cT_4 + rcT_2,$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix}\begin{pmatrix}1 & -\\ 1 & -\end{pmatrix} = 2T_9 + 2cT_6 + rT_5 + rcT_3,$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix}\begin{pmatrix}2 & -\\ - & -\end{pmatrix} = T_{10} + rT_9 + cT_8 + rcT_7;$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix} T_1 = 6F_{10} + 3F_{12} + 3rF_3 + 3cF_2 + rcF_1,$$

$$rc\begin{pmatrix}1 & -\\ - & -\end{pmatrix} T_2 = 2F_{13} + F_{15} + F_{18} + 2F_{20} + r(2F_{10} + F_{11}) + c(F_4 + F_6) + rcF_2,$$

TABLE 2

For conversion of g.s.m.'s and bipolykays of degree 4

F	f																																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	−1	1		2		3		6		1	1	1	2	1	3	1	2	2	1	3	1	6	1	3	2	3	2	3	6	2	6	3	6
3	−1		1		2		3		6	1	1	1	1	2	1	3	2	1	2	1	3	1	6	2	3	3	2	3	2	6	3	6	6
4	1	−2		1				3					1				1	1				3			1		1		3	1	3		3
5	1		−2		1				3					1			1		1				3	1			1		1	3		3	3
6	2	−3				1		4							1					1		4		1		1		1	4		4	1	4
7	2		−3				1		4							1					1		4		1	1		1		4	1	4	4
8	−6	12		−3		−4		1														1							1		1		1
9	−6		12		−3		−4		1														1							1		1	1
10	1	−1	−1							1			2	2	2	2	4			2	2	4	4	4	4	6		6	8	8	12	12	24
11	1	−1	−1								1				1	1		1	1			1	1	1	1	2	2		2	2	3	3	6
12	1	−1	−1									1						1	1	1	1	1	1	1	1	1	2	3	2	2	3	3	6
13	−1	2	1	−1						−2			1				2					2			2				4	4	6		12
14	−1	1	2		−1					−2				1			2						2	2					4	4		6	12
15	−2	3	2			−1				−2	−1				1							2		1		2			4		6	3	12
16	−2	2	3				−1			−2	−1					1							2		1	2				4	3	6	12
17	1	−2	−2	1	1					4			−2	−2			1												2	2			6
18	−1	2	1	−1							−1	−1						1				1			1		2		2	2	3		6
19	−1	1	2		−1						−1	−1							1				1	1			2		2	2		3	6
20	−2	3	2			−1				−2		−1								1		2		1		1		3	4		6	3	12
21	−2	2	3				−1			−2		−1									1		2		1	1		3		4	3	6	12
22	6	−12	−6	3		4		−1		8	2	2	−2		−2			−1		−2		1							2		3		6
23	6	−6	−12		3		4		−1	8	2	2		−2		−2			−1		−2		1							2		3	6
24	2	−3	−4		2	1				4	1	1		−2	−1				−1	−1				1					4			3	12
25	2	−4	−3	2			1			4	1	1	−2			−1		−1			−1				1					4	3		12
26	4	−6	−6			2	2			6	2	1			−2	−2				−1	−1										3	3	12
27	1	−2	−2	1	1						2	2						−2	−2							1	1		1	1			3
28	4	−6	−6			2	2			6		3								−3	−3							1			1	1	4
29	−6	12	12	−3	−6	−4		1		−16	−4	−4	4	8	4		−2	2	4	4		−2		−4			−1		1				3
30	−6	12	12	−6	−3		−4		1	−16	−4	−4	8	4		4	−2	4	2		4		−2		−4		−1			1			3
31	−12	24	18	−6		−8	−6	2		−24	−6	−6	6		6	6		3		6	6	−3			−3	−3		−1			1		4
32	−12	18	24		−6	−6	−8		2	−24	−6	−6		6	6	6			3	6	6		−3	−3		−3		−1				1	4
33	36	−72	−72	18	18	24	24	−6	−6	96	24	24	−24	−24	−24	−24	6	−12	−12	−24	−24	12	12	12	12	12	3	4	−3	−3	−4	−4	1

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_3 = 2F_{14} + F_{16} + F_{19} + 2F_{21} + r(F_5 + F_7) + c(2F_{10} + F_{11}) + rcF_8,$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_4 = 3F_{22} + 3rF_{15} + cF_8 + rcF_6,$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_5 = 3F_{23} + rF_9 + 3cF_{16} + rcF_7,$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_6 = F_{17} + F_{24} + F_{25} + F_{26} + r(F_{14} + F_{16}) + c(F_{13} + F_{15}) + rcF_{10},$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_7 = F_{24} + F_{25} + F_{27} + F_{28} + r(F_{19} + F_{21}) + c(F_{18} + F_{20}) + rcF_{12},$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_8 = F_{29} + F_{31} + r(F_{24} + F_{26}) + cF_{22} + rcF_{20},$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_9 = F_{30} + F_{32} + rF_{23} + c(F_{25} + F_{26}) + rcF_{21},$$

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix} T_{10} = F_{33} + rF_{32} + cF_{31} + rcF_{28}.$$

Products of bipolykays of degree 2 are more complicated. The coefficients of the bipolykays of degree 4 in the expressions of these products are tabulated below, using the following abbreviations:

$$a = 2/[rc(c-1)] \qquad g = 1/c$$
$$b = 2/[rc(r-1)] \qquad h = 1/r$$
$$d = 2/[c(c-1)] \qquad k = 1/(rc)$$
$$e = 2/[r(r-1)] \qquad p = 1/[rc(r-1)(c-1)]$$

10. Variances of bipolykays of degree 2. The multiplication table of the preceding section enables us to find the variances, in taking bisamples from a population matrix, of bipolykays of degree 1 or 2. For example, we have

$$\operatorname{var}\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^* = \text{ave aver}\left\{\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^*\right\}^2 - \left\{\text{ave aver}\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^*\right\}^2.$$

From Section 9 we have

$$rc\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^2 = \begin{pmatrix}2 & - \\ - & -\end{pmatrix} + c\begin{pmatrix}1 & 1 \\ - & -\end{pmatrix} + r\begin{pmatrix}1 & - \\ 1 & -\end{pmatrix} + rc\begin{pmatrix}1 & - \\ - & 1\end{pmatrix},$$

and so

$$\text{ave aver}\left\{\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^*\right\}^2 = \text{ave aver}\left\{\frac{1}{rc}\begin{pmatrix}2 & - \\ - & -\end{pmatrix}^* + \frac{1}{r}\begin{pmatrix}1 & 1 \\ - & -\end{pmatrix}^* + \frac{1}{c}\begin{pmatrix}1 & - \\ 1 & -\end{pmatrix}^* + \begin{pmatrix}1 & - \\ - & 1\end{pmatrix}^*\right\}$$

$$= \frac{1}{rc}\begin{pmatrix}2 & - \\ - & -\end{pmatrix}' + \frac{1}{r}\begin{pmatrix}1 & 1 \\ - & -\end{pmatrix}' + \frac{1}{c}\begin{pmatrix}1 & - \\ 1 & -\end{pmatrix}' + \begin{pmatrix}1 & - \\ - & 1\end{pmatrix}',$$

TABLE 3

Products of bipolykays of degree 2

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}	F_{21}	F_{24}	F_{25}	F_{26}	F_{27}	F_{28}
$\begin{pmatrix}1&-\\-&1\end{pmatrix}^2$	1	$4h$	$4g$	e	d	0	0	$8k$	$4k$	$4k$	$2b$	$2a$	0	0	$2p$	$2b$	$2a$	0	0	0	0	0	$2p$	0
$\begin{pmatrix}1&-\\-&1\end{pmatrix}\begin{pmatrix}1&1\\-&-\end{pmatrix}$	0	1	0	$-e$	0	$2h$	0	$4g$	0	0	$-2b$	d	$4k$	0	$-2p$	$-2b$	0	$4k$	0	$2a$	0	0	$-2p$	0
$\begin{pmatrix}1&-\\-&1\end{pmatrix}\begin{pmatrix}1&-\\1&-\end{pmatrix}$	0	0	1	0	$-d$	0	$2g$	$4h$	0	0	e	$-2a$	0	$4k$	$-2p$	0	$-2a$	0	$4k$	0	$2b$	0	$-2p$	0
$\begin{pmatrix}1&-\\-&1\end{pmatrix}\begin{pmatrix}2&-\\-&-\end{pmatrix}$	0	0	0	0	0	0	0	0	0	1	$-e$	$-d$	0	0	$2p$	0	0	$2h$	$2g$	$-2a$	$-2b$	$2k$	$2p$	$2k$
$\begin{pmatrix}1&1\\-&-\end{pmatrix}\begin{pmatrix}1&-\\1&-\end{pmatrix}$	0	0	0	0	0	0	0	0	1	0	$-e$	$-d$	$2h$	$2g$	$2p$	0	0	0	0	$-2a$	$-2b$	$4k$	$2p$	0

	F_4	F_5	F_8	F_9	F_{13}	F_{14}	F_{17}	F_{18}	F_{19}	F_{22}	F_{23}	F_{24}	F_{25}	F_{27}	F_{29}	F_{30}	F_{31}	F_{32}	F_{33}
$\begin{pmatrix}1&1\\-&-\end{pmatrix}^2$	$1+re$	0	h	0	$2r^2b$	0	$2r^2p$	$2rb$	0	$4k$	0	0	0	$2rp$	a	0	0	0	0
$\begin{pmatrix}1&-\\1&-\end{pmatrix}^2$	0	$1+cd$	0	g	0	$2c^2a$	$2c^2p$	0	$2ca$	0	$4k$	0	0	$2cp$	0	b	0	0	0
$\begin{pmatrix}1&1\\-&-\end{pmatrix}\begin{pmatrix}2&-\\-&-\end{pmatrix}$	0	0	0	0	re	0	$-2r^2p$	1	0	h	0	0	$2rb+2g$	$-2rp$	$-a$	0	$2k$	0	0
$\begin{pmatrix}1&-\\1&-\end{pmatrix}\begin{pmatrix}2&-\\-&-\end{pmatrix}$	0	0	0	0	0	cd	$-2c^2p$	0	1	0	g	$2ca+2h$	0	$-2cp$	0	$-b$	0	$2k$	0
$\begin{pmatrix}2&-\\-&-\end{pmatrix}^2$	0	0	0	0	0	0	$2(r^2c^2p-1)$	0	0	0	0	0	0	$2rcp+1$	$ac+h$	$br+g$	0	0	k

and

$$\left\{\text{ave aver}\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^{*}\right\}^{2} = \left\{\begin{pmatrix}1 & - \\ - & -\end{pmatrix}'\right\}^{2}$$

$$= \frac{1}{RC}\begin{pmatrix}2 & - \\ - & -\end{pmatrix}' + \frac{1}{R}\begin{pmatrix}1 & 1 \\ - & -\end{pmatrix}' + \frac{1}{C}\begin{pmatrix}1 & - \\ 1 & -\end{pmatrix}' + \begin{pmatrix}1 & - \\ - & 1\end{pmatrix}'.$$

Hence

$$\operatorname{var}\begin{pmatrix}1 & - \\ - & -\end{pmatrix}^{*} = \left(\frac{1}{rc} - \frac{1}{RC}\right)\begin{pmatrix}2 & - \\ - & -\end{pmatrix}' + \left(\frac{1}{r} - \frac{1}{R}\right)\begin{pmatrix}1 & 1 \\ - & -\end{pmatrix}' + \left(\frac{1}{c} - \frac{1}{C}\right)\begin{pmatrix}1 & - \\ 1 & -\end{pmatrix}'.$$

Proceeding in the same way for the variances of bipolykays of degree 2, we obtain the results given in Table 4, which table provides the coefficients of the expressions of the indicated variances and covariances as linear combinations of population bipolykays. In order to simplify the tabulation, the following expressions are used:

$$A = \frac{1}{r} - \frac{1}{R} \qquad S = \frac{1}{rc} - \frac{1}{RC}$$

$$B = \frac{1}{c} - \frac{1}{C} \qquad T = \frac{2}{(r-1)(c-1)} - \frac{2}{(R-1)(C-1)}$$

$$D = \frac{2}{r-1} - \frac{2}{R-1} \qquad U = \frac{2}{rc(r-1)} - \frac{2}{RC(R-1)}$$

$$E = \frac{2}{c-1} - \frac{2}{C-1} \qquad V = \frac{2}{rc(c-1)} - \frac{2}{RC(C-1)}$$

$$G = \frac{2}{r(r-1)} - \frac{2}{R(R-1)} \qquad W = \frac{2}{c(r-1)(c-1)} - \frac{2}{C(R-1)(C-1)}$$

$$H = \frac{2}{c(c-1)} - \frac{2}{C(C-1)} \qquad Y = \frac{2}{r(r-1)(c-1)} - \frac{2}{R(R-1)(C-1)}$$

$$P = \frac{2}{c(r-1)} - \frac{2}{C(R-1)} \qquad Z = \frac{2}{rc(r-1)(c-1)} - \frac{2}{RC(R-1)(C-1)}.$$

$$Q = \frac{2}{r(c-1)} - \frac{2}{R(C-1)}$$

11. Conclusion. Any symmetric function of elements of a two-way array can be expressed as a linear combination of bipolykays, using the multiplication formulas of Section 9 where necessary. If there is a linear model involved, then the average values of the bipolykays (and hence of the original symmetric function) can be found, by means of pairing formulas, in terms of polykays or bipolykays of the populations from which come the components of the linear model. A later paper will illustrate the use of these procedures in finding unbiased estimators for variance components, as well as the variances of these estimators, etc.

TABLE 4

Variances and covariances of $\begin{pmatrix}1&1\\-&-\end{pmatrix}$, $\begin{pmatrix}1&-\\1&-\end{pmatrix}$, *and* $\begin{pmatrix}2&-\\-&-\end{pmatrix}$

	F_4'	F_5'	F_8'	F_9'	F_{13}'	F_{14}'	F_{17}'	F_{18}'	F_{19}'	F_{22}'	F_{23}'	F_{27}'	F_{29}'	F_{30}'	F_{32}'
var $\begin{pmatrix}1&1\\-&-\end{pmatrix}$	D	0	A	0	$4B+2P$	0	$H+W$	$2P$	0	$4S$	0	W	V	0	0
var $\begin{pmatrix}1&-\\1&-\end{pmatrix}$	0	E	0	B	0	$4A+2Q$	$G+Y$	0	$2Q$	0	$4S$	Y	0	U	0
var $\begin{pmatrix}2&-\\-&-\end{pmatrix}$	0	0	0	0	0	0	$D+E+T$	0	0	0	0	T	$A+Q$	$B+P$	S

	F_{13}'	F_{14}'	F_{15}'	F_{16}'	F_{17}'	F_{22}'	F_{23}'	F_2'	F_{25}'	F_{26}'	F_{27}'	F_{29}'	F_{30}'	F_{31}'	F_{32}'
cov $\begin{pmatrix}1&1\\-&-\end{pmatrix}$, $\begin{pmatrix}1&-\\1&-\end{pmatrix}$	$-G$	$-H$	$2A$	$2B$	Z	0	0	$-2V$	$-2U$	$4S$	Z	0	0	0	0
cov $\begin{pmatrix}1&1\\-&-\end{pmatrix}$, $\begin{pmatrix}2&-\\-&-\end{pmatrix}$	D	0	0	0	$-W-H$	A	0	0	$2B+2P$	0	$-W$	$-V$	0	$2S$	0
cov $\begin{pmatrix}1&-\\1&-\end{pmatrix}$, $\begin{pmatrix}2&-\\-&-\end{pmatrix}$	0	E	0	0	$-Y-G$	0	B	$2A+2Q$	0	0	$-Y$	0	$-U$	0	$2S$

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