KEEPING MOMENT-LIKE SAMPLING COMPUTATIONS SIMPLE[1]

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1. Summary. This is an attempt to present as simply as possible the best tools we know today for keeping computations simple when dealing with samples from general populations. Such computations seem inevitably to be made in terms of quantities related to moments. We develop here the formal structure and interrelations of the two systems of multi-index quantities which seem today to be best adapted to statistical use. The occurrence of two systems is, at least in part, related to the appearance in statistical problems of both multiplication and addition of independent variables. Hence the existence of two systems, whose limiting cases are moments (about a fixed point) and cumulants (or semiinvariants).

We present interconversion formulas, developing definitions and proving the pairing formulas without reference to any infinite populations, and sparing the use of combinatorial techniques as much as we are able. A few multiplication formulas are given, but for a more complete list the reader is referred to Wishart [10]. It is hoped that this paper can be read on its own, with some reference to applications of these techniques to elementary examples [6] and to the sampling properties of estimated variance components in the analysis of variance [7], [8], [9] as motivation.

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2. Introduction. The history of "moments of moments", still the only way we know to attack general sampling distributions, has been long and complicated. Its outstanding feature has been the cutting away of pages and pages of algebra by the introduction of new and sharper tools. The forging of these tools has depended more and more on combinatorial ideas, and while the use of the tools has become simpler, their understanding has become apparently more and more complex. It is the purpose of this paper to show how we can keep almost everything quite simple and still use what today seem to be the best tools. The only useful aspects which we cannot completely handle simply are the actual calculation of certain multiplication formulas, an extensive table of which has been provided by Wishart [10].

The two systems which we shall discuss correspond to moments about a fixed point and to cumulants (semiinvariants). They do not correspond to moments

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about the mean, except insofar as the second and third moments about the mean, happen to be cumulants. From the point of view of practical use, either numerical or algebraic, I am convinced that higher moments *about the mean* are a vermiform appendix of statistical evolution—an evolutionary remnant which will inevitably disappear, though all too slowly.

The two systems involve not a single index, but a set of one or more indices. At first sight this may seem redundant and wasteful, since there are relations which could be used to eliminate the multiple index symbols. But these relations involve sizes of samples or populations, and a great part of the simplicity of the use of the multiple index systems arises from a great reduction in the appearance of these sizes.

We shall work entirely in terms of finite samples or populations, treating infinite populations as special limiting cases. Contrary to the usual view, this does not make matters more complex.

3. The two systems. The first system of polynomial symmetric functions of n numbers $x_1, x_2, \cdots, x_n$ which we shall use are the symmetric means (the mean power products, in combinatorial terminology) which we will denote by angle brackets, as $\langle 3 \rangle$, $\langle 134 \rangle$, etc. and will refer to as symmetric means or brackets. They are defined as the means of products of powers of *different* x_i's, so that, for example,

$$\langle ab \rangle = \frac{\sum^{\neq} x_i^a x_j^b}{n(n-1)}$$

where the sum is over the $n(n-1)$ pairs (i, j) with $i \neq j$. The numerators are a kind of symmetric function of very respectable antiquity, the only modern features being (i) the division by the number of cases to give a mean, which seems natural to the statistician but perhaps not to the combinatorialist, and (ii) the use of multiple subscripts, which the combinatorialist has always done but which the statistician has seemed to resist. This resistance seems to have been due to a feeling that only the simple moments

$$\langle a \rangle = \frac{\sum x_i^a}{n}$$

could be easily calculated from numerical data, and that hence all formulas should be written in terms of moments. This position is tempting rather than irrefutable, and the simplicity of formulas involving the multiple subscripts shows its deficiencies. It is easy to continue the numerical calculation, once the moments are at hand, and find all the symmetric functions of either system. For all of weight ≤ 4 a simple computing form has been presented in [6].

The second, and even more important, system of polynomial symmetric functions is most simply defined in terms of the first system through linear formulas like these

$$k_2 = (2) = \langle 2\rangle - \langle 11\rangle,$$
$$k_{113} = (113) = \langle 113\rangle - 3\langle 1112\rangle + 2\langle 11111\rangle.$$

We shall obtain the general law of formation and supply a compact table of formulas up to weight 8. We shall usually refer to these as "polykays" for the sake of a short, simple term. (While the construction of this term, "kay" for "k" and "poly" for the multiple subscript, seems somewhat revolting to some colleagues, the use of "generalized k-statistics" seems too unhandy to me. In due course, perhaps, someone will find a good, short, terminology.) The polykays with but a single subscript are, of course, Fisher's famous k-statistics, whose introduction was perhaps the largest step in clearing unnecessary algebra out of this field. The multiple index analogs were introduced by Dressel [3] in 1940 in a combinatorial paper which seems to have escaped notice at the time of its appearance. They were introduced independently by the author in 1950 [6] as practical working tools.

4. Random pairing, additive and multiplicative. Let us next consider two sets of n numbers, $x_1^*, x_2^*, \cdots, x_n^*$ and $x_1^{**}, x_2^{**}, \cdots, x_n^{**}$, whose brackets and polykays we shall similarly distinguish with asterisks, as, for example, $\langle 2\rangle^*$ and $\langle 2\rangle^{**}$. We shall be concerned with the results of *pairing* these two sets randomly, more specifically with the results of forming some function of each of the pairs

$$[x_1^*, x_{\pi(1)}^{**}], [x_2^*, x_{\pi(2)}^{**}], \cdots, [x_n^*, x_{\pi(n)}^{**}]$$

where $\pi(1), \pi(2), \cdots, \pi(n)$ is a permutation of the integers $1, 2, \cdots, n$, and where we shall eventually wish to average over all permutations.

The simplest pairing operation is multiplicative pairing, where $x_i = x_i^* x_{\pi(i)}^{**}$. Let us calculate a moment of the resulting x_i, say $\langle a\rangle$, and then average over all pairings [permutations]. We have

$$\langle a\rangle = \frac{\sum x_i^a}{n} = \frac{\sum (x_i^*)^a (x_{\pi(i)}^{**})^a}{n}$$

and when we average, the product of x_i^* and x_j^{**} appears equally often for *all pairs* i and j, so that

$$\text{aver } \{\langle a\rangle\} = \frac{\sum (x_i^*)^a \cdot \sum (x_j^{**})^a}{n^2} = \langle a\rangle^* \langle a\rangle^{**}$$

where we have written "aver" for the *av*erage over *r*andom pairing, as we shall continue to do, and where the denominator of n^2 is easily justified as equal to the number of terms in the numerator when expanded (an average of a mean is again a mean).

The same argument applies to $\langle ab \cdots e\rangle$, as we see if we write

$$y_{ij\cdots m} = x_i^a x_j^b \cdots x_m^e$$
$$y_{ij\cdots m}^* = (x_i^*)^a (x_j^*)^b \cdots (x_m^*)^e$$
$$y_{ij\cdots m}^{**} = (x_i^{**})^a (x_j^{**})^b \cdots (x_m^{**})^e$$

and observe that

$$y_{ij\cdots m} = y^{*}_{ij\cdots m} y^{**}_{\pi(i),\pi(j),\cdots,\pi(m)} .$$

Thus we have

$$\text{aver}\ \{\langle ab \cdots e\rangle\} = \langle ab \cdots e\rangle^{*} \langle ab \cdots e\rangle^{**}.$$

The brackets are ideally suited to multiplicative pairing.

The only other random pairing which we know how to handle at all well statistically is the additive one, where

$$x_i = x_i^{*} + x^{**}_{\pi(i)} .$$

Because of the many statistical problems where additive pairing is taken as the first approximation to reality (classical analysis of variance models, error propagation, etc.), this is the most important case.

The one-index (or as we would say in Section 11, one-part) brackets behave in a manageable, but not simple way under additive random pairing. We have, for example,

$$\begin{aligned}\langle\langle 3\rangle\rangle = \text{aver}\ \{\langle 3\rangle\} &= \text{aver}\ \frac{\sum (x_i^{*} + x^{**}_{\pi(i)})^3}{n} \\ &= \text{aver}\ \frac{\sum (x_j^{*})^3}{n} + 3\ \text{aver}\ \frac{\sum (x_i^{*})^2 (x^{**}_{\pi(i)})}{n} + 3\ \text{aver}\ \frac{\sum (x_i^{*})(x^{**}_{\pi(i)})^2}{n} \\ &\qquad + \text{aver}\ \frac{\sum (x^{**}_{\pi(i)})^3}{n} \\ &= \langle 3\rangle^{*} + 3\langle 2\rangle^{*}\langle 1\rangle^{**} + 3\langle 1\rangle^{*}\langle 2\rangle^{**} + \langle 3\rangle^{**}\end{aligned}$$

and, in general,

$$\begin{aligned}\langle\langle j\rangle\rangle = \text{aver}\ \{\langle j\rangle\} = \langle j\rangle^{*} &+ \binom{j}{1} \langle j-1\rangle^{*}\langle 1\rangle^{**} \\ &+ \binom{j}{2} \langle j-2\rangle^{*}\langle 2\rangle^{**} + \cdots + \langle j\rangle^{**}\end{aligned}$$

where we have introduced a doubling of the brackets to indicate averaging over an *additive* random pairing.

Since the general formula is of binomial type, we can represent it in terms of generating functions. If we define

$$M_{\text{aver}}(t) = 1 + \langle\langle 1\rangle\rangle t + \langle\langle 2\rangle\rangle \frac{t^2}{2!} + \langle\langle 3\rangle\rangle \frac{t^3}{3!} + \cdots ,$$

$$M^{*}(t) = 1 + \langle 1\rangle^{*} t + \langle 2\rangle^{*} \frac{t^2}{2!} + \langle 3\rangle^{*} \frac{t^3}{3!} + \cdots ,$$

$$M^{**}(t) = 1 + \langle 1\rangle^{**} t + \langle 2\rangle^{**} \frac{t^2}{2!} + \langle 3\rangle^{**} \frac{t^3}{3!} + \cdots ,$$

then the general formula becomes

$$M_{\text{aver}}(t) = M^*(t)M^{**}(t).$$

We shall use this relation in Section 11 to obtain general expressions defining the second system of quantities in relation to the brackets.

5. The polykays. This second system will be denoted either by $k_{ab\cdots e}$ or by $(ab \cdots e)$ as may be convenient. We shall use the same double parenthesis convention, so that

$$((ab \cdots e)) = \text{aver}\,\{(ab \cdots e)\}$$

where the averaging is over *additive* random pairing.

Two examples of the second system, beyond the trivial $(1) = \langle 1 \rangle$, are $(2) = \langle 2 \rangle - \langle 11 \rangle$ and $(12) = \langle 12 \rangle - \langle 111 \rangle$. Let us examine the behavior of these quantities under random pairing, using unproven, but formally reasonable, facts about the behavior of multipart brackets. We find

$$\begin{aligned}((2) &= \langle\langle 2 \rangle\rangle - \langle\langle 11 \rangle\rangle = \langle 2 \rangle^* + 2\langle 1 \rangle^*\langle 1 \rangle^{**} + \langle 2 \rangle^{**} - \langle 11 \rangle^* - 2\langle 1 \rangle^*\langle 1 \rangle^{**} - \langle 11 \rangle^{**}\\ &= [\langle 2 \rangle^* - \langle 11 \rangle^*] + [\langle 2 \rangle^{**} - \langle 11 \rangle^{**}] = (2)^* + (2)^{**}\end{aligned}$$

$$\begin{aligned}((12)) &= \langle\langle 12 \rangle\rangle - \langle\langle 111 \rangle\rangle\\ &= \langle 12 \rangle^* + \langle 2 \rangle^*\langle 1 \rangle^{**} + 2\langle 11 \rangle^*\langle 1 \rangle^{**} + \langle 1 \rangle^*\langle 2 \rangle^{**} + 2\langle 1 \rangle^*\langle 11 \rangle^{**} + \langle 12 \rangle^{**}\\ &\qquad - \langle 111 \rangle^* - 3\langle 11 \rangle^*\langle 1 \rangle^{**} - 3\langle 1 \rangle^*\langle 11 \rangle^{**} - \langle 111 \rangle^{**}\\ &= [\langle 12 \rangle^* - \langle 111 \rangle^*] + [\langle 2 \rangle^* - \langle 11 \rangle^*]\langle 1 \rangle^{**} + \langle 1 \rangle^*[\langle 2 \rangle^{**} - \langle 11 \rangle^{**}]\\ &\qquad + [\langle 12 \rangle^{**} - \langle 111 \rangle^{**}]\\ &= (12)^* + (2)^*(1)^{**} + (1)^*(2)^{**} + (12)^{**}\end{aligned}$$

It should now be clear both what the pairing law is likely to be, and that we need some slick trick both to discover the definitions and to prove the result.

Since the trick the author prefers involves symbolic multiplication, we shall postpone its treatment to the last section. We announce here the pairing formula we desire and leave definitions and proofs to Sections 11 and 12. The pairing formula is illustrated by

$$\begin{aligned}\text{ave}\,\{k_{abc}\} = ((abc)) &= (abc)^* + (ab)^*(c)^{**} + (ac)^*(b)^{**} + (bc)^*(a)^{**}\\ &\qquad + (a)^*(bc)^{**} + (b)^*(ac)^{**} + (c)^*(ab)^{**} + (abc)^{**}\end{aligned}$$

In general, we separate the indices into two sets (one of which may be empty) in all possible ways, assigning one set to * and the other to **, and adding up the resulting products. We know that we have the desired definitions when the pairing formula is an identity with $((ab \cdots e))$ the same function of the $\langle\langle fg \cdots j \rangle\rangle$ as $(ab \cdots e)^*$ is of the $\langle fg \cdots j \rangle^*$ and as $(ab \cdots e)^{**}$ is of the $\langle fg \cdots j \rangle^{**}$.

One special case is worthy of notice. If all the x_i^{**} are identically equal to δ, then, (cp. [7])

$$\langle a\rangle^{**} = \delta^a; \qquad \langle ab\rangle^{**} = \delta^{a+b}; \qquad \langle abc\rangle^{**} = \delta^{a+b+c}; \cdots ; (1)^{**} = \delta;$$

$$(a)^{**} = 0, \quad \text{for} \quad a > 1; \qquad 0 = (ab)^{**} = (abc)^{**} = (abcd)^{**} = \cdots .$$

So that the effect of pairing with this population, which is independent of the randomization and exactly equivalent to increasing all the x_i^* by δ, is given by such relations as

$$(1) = (1)^* + \delta, \qquad (2) = (2), \qquad (3) = (3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}
(11) &= (11)^* + 2\delta(1)^* + \delta^2, \\
(12) &= (12)^* + \delta(2)^*, \\
(111) &= (111)^* + 3\delta(111)^* + 3\delta^2(1)^* + \delta^3 \\
(112) &= (112)^* + 2\delta(12)^* + \delta^2(2)^*, \\
(22) &= (22)^*.
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus the effects of increasing all the x_i^* by δ are easily found. We notice for future use that the highest power of δ is the number of 1's in the polykay; which appears with a coefficient found by dropping the 1's from within the parentheses.

6. Inheritance and representation. If $x_1, x_2, \cdots, x_n$ is a sample from x_1', $x_2', \cdots, x_N'$, and if "ave" means a simple average over all samples, then symmetry implies that

$$\text{ave}\,\{\langle ab\cdots e\rangle\} = \frac{\sum \text{aver}\,\{x_i^a x_j^b \cdots x_m^e\}}{\text{number of terms above}} = \frac{\sum^{\neq} (x_i')^a (x_j')^b \cdots (x_m')^e}{\text{number of terms above}}$$

$$= \langle ab \cdots e\rangle'.$$

Since the polykays are expressible linearly in the brackets with integer coefficients the same relation

$$\text{ave}\,\{k_{ab\cdots e}\} = k'_{ab\cdots e}$$

must hold for polykays. We refer to this as inheritance on the average. Some would prefer to say unbiasedness (but there are now so many kinds of unbiasedness!).

Since any polynomial symmetric function can be expressed linearly in terms of the brackets, and since, as we shall see, brackets and polykays can be expressed linearly in terms of one another, every polynomial symmetric function can also be expressed linearly in terms of polykays.

7. Unit parts. Each index "1" which appears in a bracket or a polykay will be called a unit part. These indices play an especially simple role. If we have a linear identity connecting polykays and brackets, we can obtain a new one by adding unit parts to all the terms. Thus from $k_3 = (3) = \langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle$ we derive $k_{13} = (13) = \langle 13\rangle - 3\langle 112\rangle + 2\langle 1111\rangle$, $k_{113} = (113) = \langle 113\rangle - 3\langle 1112\rangle + 2\langle 11111\rangle$ and so on, while from $\langle 2\rangle = (2) + (11)$ we derive $\langle 12\rangle = (12) + (111) = k_{12} + k_{111}$, $\langle 112\rangle = (112) + (1111) = k_{112} + k_{1111}$ and so on. This fact will be proven in Section 12.

Thus it would be convenient to write these formulas in a single shorthand form, such as $(-3) = \langle -3\rangle - 3\langle -2\rangle + 2\langle -\rangle$, $\langle -2\rangle = (-2) + (-)$ where the dashes stand for the number of 1's. This number is usually different in the different appearances in one formula, but these numbers are to be chosen so as to make all terms of the same degree.

Fur further abbreviation, we may drop the dashes themselves (except in $(-)$ and $\langle -\rangle$ where they seem helpful). If we do this, then we can give a single table which presents the coefficients for all the identities connecting brackets and polykays through degree 8. This is done in a compact form (pioneered by David and Kendall [2]) in Table 1, where brackets are expressed in terms of polykays by the coefficients below and on the main diagonal, while polykays are expressed in terms of brackets by the coefficients above and on the main diagonal. (A less convenient table, extending through weight 12, has recently been provided by Abdel-Aty [1].)

More specifically, to express $\langle 1134\rangle$ in terms of polykays, we proceed as follows:

(a) look for $\langle 34\rangle$, which identifies a row, and start with the heavy 1 on the diagonal of that row,

(b) move along that row from the diagonal 1 toward the beginning, writing down the coefficients times the corresponding polykays, (This yields:

$$\langle 34\rangle = (34) + 3(223) + 4(33) + 3(24) + 9(222) + 18(23)$$
$$+ (4) + 21(22) + 5(3) + 9(2) + (-)$$

in shorthand notation.)

(c) note that $1 + 1 + 3 + 4 = 9$ and add 1's to every term to bring the degree of every term up to 9. (This yields:

$$\langle 1134\rangle = (1134) + 3(11223) + 4(11133) + 3(11124) + 9(111222)$$
$$+ 18(111123) + (111114) + 21(1111122)$$
$$+ 5(1111113) + 9(11111112) + (111111111)$$

which is the desired result.)

To expand a polykay in brackets we merely interchange rows and columns, moving upward from the diagonal.

8. Computation modulo unit parts. We see easily, either from the general relations of Sections 11 and 12, or from the nature of the reduced formulas, that when a bracket with g unit parts is written in terms of polykays, only polykays with at least g unit parts appear and vice versa. It is then unequivocal if we write

TABLE I

Conversion Formulae

	(1)	(2)	(3)	(22)	(4)	(23)	(222)	(5)	(24)	(33)	(223)	(2222)	(6)	(25)	(34)	(224)	(233)	(7)	(26)	(35)	(44)	(8)	
⟨−⟩	1	−1	2	1	−6	−2	−1	24	6	4	2	1	−120	−24	−12	−6	−4	720	120	48	36	−5040	
⟨2⟩	1	1	−3	−2	12	5	3	−60	−18	−12	−7	−4	360	84	42	24	16	−2520	−480	−192	−144	20160	⟨2⟩
⟨3⟩	1	3	1		−4	−1		20	4	4	1		−120	−20	−14	−4	−4	840	120	64	48	−6720	⟨3⟩
⟨22⟩	1	2		1	−3	−3	−3	30	15	9	8	6	−270	−90	−42	−33	−21	2520	630	240	180	−25200	⟨22⟩
⟨4⟩	1	6	4	3	1			−5	−1				30	5	2	1		−210	−30	−10	−12	1680	⟨4⟩
⟨23⟩	1	4	1	3		1		−10	−4	−6	−2		120	30	24	8	10	−1260	−240	−140	−96	13440	⟨23⟩
⟨222⟩	1	3		3			1		−3		−3	−4	30	30	9	18	9	−630	−300	−90	−72	10080	⟨222⟩
⟨5⟩	1	10	10	15	5	10		1					−6	−1				42	6	2		−336	⟨5⟩
⟨24⟩	1	7	4	9	1	4	3		1				−15	−5	−3	−2		210	45	15	24	−2520	⟨24⟩
⟨33⟩	1	6	2	9		6				1			−10		−4		−1	140	10	20	16	−1680	⟨33⟩
⟨223⟩	1	5	1	7		2	3				1			−10	−3	−4	−6	210	120	60	24	−5040	⟨223⟩
⟨2222⟩	1	4		6			4					1				−3			30		9	−630	⟨2222⟩
⟨6⟩	1	15	20	45	15	60	15	6	15	10			1					−7	−1			56	⟨6⟩
⟨25⟩	1	11	10	25	5	20	15	1	5		10			1				−21	−6	−3		336	⟨25⟩
⟨34⟩	1	9	5	21	1	18	9		3	4	3				1			−35		−5	−8	560	⟨34⟩
⟨224⟩	1	8	4	16	1	8	12		2		4	3				1			−15		−6	420	⟨224⟩
⟨233⟩	1	7	2	15		8	9			1	6						1		−10	−10		560	⟨233⟩
⟨7⟩	1	21	35	105	35	210	105	21	105	70	105		7	21	35			1				−8	⟨7⟩
⟨26⟩	1	16	20	60	15	80	60	6	30	10	60	15	1	6		15	10		1			−28	⟨26⟩
⟨35⟩	1	13	11	45	5	50	45	1	15	10	45			3	5		10			1		−56	⟨35⟩
⟨44⟩	1	12	8	42	2	48	36		12	16	24	9			8	6					1	−35	⟨44⟩
⟨8⟩	1	28	56	210	70	560	420	56	420	280	840	105	28	168	280	210	280	8	28	56	35	1	⟨8⟩
	(1)	(2)	(3)	(22)	(4)	(23)	(222)	(5)	(24)	(33)	(223)	(2222)	(6)	(25)	(34)	(224)	(233)	(7)	(26)	(35)	(44)	(8)	

(To use pivot on diagonal of ones)

$O(1^g)$ for an arbitrary set of terms each of which, when expanded linearly in brackets or polykays contains at least g unit parts.

If we have an identity which is a polynomial in the polykays, as for example

$$k_1^3 - k_2 k_1 + \frac{1}{n} k_3 + k_{21} \equiv \frac{1}{n^2} k_3 + \frac{3}{n} k_{21} + k_{111}$$

each term has a certain total number of unit parts. In the example, these numbers are 3, 1, 0 and 1 on the left-hand side and 0, 1 and 3 on the right-hand side. The highest total appearing on either side is the unit weight of that side. In the example, the unit weight of each side is 3. If we shift all the x_i by pairing them with a set of values all equal to δ, each term will be replaced by a number of terms involving various powers of δ up to and including a power equal to the number of unit parts. The total coefficients of each power of δ must also give an identity. Thus any identity gives rise to a number of associated identities.

If one side of the initial identity was linear in the polykays, and all obvious cancellations had been made initially (as is the case on the right-hand side of the example), then the coefficient of the highest power of δ appearing on that side after pairing would be linear in the polykays, would have no obvious cancellations, and hence would not vanish identically. The coefficient of the same power of δ on the other side, which is identically equal to this, cannot vanish, and hence

$$\text{(Unit weight on other side)} \geqq \text{(Unit weight on linear side)}$$

Since any polynomial in the polykays is a symmetric polynomial in the x's, it can be written linearly in the polykays. The unit weight will not be increased in this process. In particular, a polynomial in polykays without unit part (and hence of unit weight zero) when written linearly in the polykays involves no polykay with unit part. If we know the linear representation to terms $O(1)$, we know it exactly. Similarly, the unit weight of the left-hand side in the example is 3. If we know the linear representation (the right-hand side) to $O(1^4)$, we know it exactly.

In table 1 we have both heavy and light coefficients. Only the heavy ones need to be used if we adjoin $+O(1)$. Thus, for example,

$$\langle 34 \rangle = (34) + 3(223) + O(1)$$

$$(34) = \langle 34 \rangle - 3(223) + O(1)$$

$$(223) = \langle 223 \rangle + O(1)$$

As this example shows, the formulas are often much simplified by this process.

9. Multiplication of brackets. Both brackets and polykays are chosen so as to remove the inevitable combinatorial difficulties from as many formulas as possible. As a result, combinatorial considerations have been restricted to the formulas for multiplication. For brackets, the resulting formulas are relatively simple. Thus

$$\begin{aligned}\langle a\rangle\langle b\rangle &= \frac{1}{n^2}\sum x_i^a \sum x_j^b \\ &= \frac{1}{n^2}\left[\sum^{\neq} x_i^a x_j^b + \sum x_k^{a+b}\right] \\ &= \frac{1}{n^2}\left[n(n-1)\langle ab\rangle + n\langle a+b\rangle\right] \\ &= \frac{n-1}{n}\langle ab\rangle + \frac{1}{n}\langle a+b\rangle\end{aligned}$$

in a similar way we find

$$\begin{aligned}\langle abc\rangle\langle de\rangle = {}& \frac{(n-3)(n-4)}{n(n-1)}\langle abcde\rangle + \frac{n-3}{n(n-1)}\langle a+d, bce\rangle \\ &+ \frac{n-3}{n(n-1)}\langle a+e, bcd\rangle + \frac{n-3}{n(n-1)}\langle b+c, ace\rangle \\ &+ \frac{n-3}{n(n-1)}\langle b+e, acd\rangle + \frac{n-3}{n(n-1)}\langle c+d, bce\rangle \\ &+ \frac{n-3}{n(n-1)}\langle c+e, abd\rangle + \frac{1}{n(n-1)}\langle a+d, b+e, c\rangle \\ &+ \frac{1}{n(n-1)}\langle a+d, c+e, b\rangle + \frac{1}{n(n-1)}\langle b+d, c+e, a\rangle \\ &+ \frac{1}{n(n-1)}\langle b+d, a+e, c\rangle + \frac{1}{n(n-1)}\langle c+d, a+e, b\rangle \\ &+ \frac{1}{n(n-1)}\langle c+d, b+e, a\rangle.\end{aligned}$$

In general, we obtain all brackets which can be obtained by matching some (including none) of the letters in one bracket with letters in the other and then replacing matched letters by their sum. The coefficient is a simple function of the number of parts in the factors *and* the result, with a simple denominator. It is often convenient to expand coefficients as integer coefficient combinations of

$$p = \frac{1}{n}, \quad q = \frac{1}{n(n-1)}, \quad r = \frac{1}{n(n-1)(n-2)},$$
$$s = \frac{1}{n(n-1)(n-2)(n-3)}, \quad \cdots.$$

These expansions are given for certain products in Table 2. With the aid of this table, multiplication of brackets is merely a matter of exerting moderate patience to be sure that you have all the terms.

10. Multiplication of polykays. The multiplication formulas for polykays are more complex. They can be obtained symbolically (cp. Wishart [10], Kendall

TABLE 2

Expansions of factors for bracket multiplication

Parts of factors	Parts in bracket whose coefficient is sought					
	1	2	3	4	5	6
1×1	p	$1 - p$	—	—	—	—
1×2	—	p	$1 - 2p$	—	—	—
1×3	—	—	p	$1 - 3p$	—	—
1×4	—	—	—	p	$1 - 4p$	—
2×2	—	q	$p - q$	$1 - 4p + 2q$	—	—
2×3	—	—	q	$p - 2q$	$1 - 6p + 6q$	—
2×4	—	—	—	q	$p - 3q$	$1 - 8p + 12q$
3×3	—	—	r	$q - r$	$p - 4q + 2r$	$1 - 9p + 18q - 6r$
3×4	—	—	—	r	$q - 2r$	$p - 6q + 6r$
4×4	—	—	—	s	$r - s$	$q - 4r + 2s$

[5]) or by direct calculation. One way to carry out direct calculation is to express each polykay in brackets, multiply out the brackets and then reconvert the resulting brackets to polykays. For example

$$
\begin{aligned}
k_{12}k_2 &= (12)(2) = [\langle 12\rangle - \langle 111\rangle][\langle 2\rangle - \langle 11\rangle] \\
&= \langle 12\rangle\langle 2\rangle - \langle 111\rangle\langle 2\rangle - \langle 12\rangle\langle 11\rangle + \langle 111\rangle\langle 11\rangle \\
&= [1 - 2p]\langle 122\rangle + p\langle 23\rangle + p\langle 14\rangle - [1 - 3p]\langle 1112\rangle \\
&\qquad - 3p\langle 113\rangle - [1 - 4p + 2q]\langle 1112\rangle - 2[p - q]\langle 122\rangle \\
&\qquad - 2[p - q]\langle 113\rangle - 2q\langle 23\rangle + [1 - 6p + 6q]\langle 11111\rangle \\
&\qquad + 6[p - 2q]\langle 1112\rangle + 6q\langle 122\rangle \\
&= [p - 2q]\langle 23\rangle + p\langle 14\rangle + [1 - 4p + 8q]\langle 122\rangle \\
&\qquad + [-5p + 2q]\langle 113\rangle + [-2 + 13p - 14q]\langle 1112\rangle \\
&\qquad + [1 - 6p + 6q]\langle 11111\rangle \\
&= [p - 2q][(23) + 3(122) + (113) + 4(1112) + (11111)] \\
&\qquad + p[(14) + 3(122) + 4(113) + 6(1112) + (111111)] \\
&\qquad + [1 - 4p + 8q][(122) + 2(1112) + (11111)] \\
&\qquad + [-5p + 2q][(113) + 3(1112) + (11111)] \\
&\qquad + [-2 + 13p - 14q][(1112) + (11111)] + [1 - 6p + 6q](11111) \\
&= [p - 2q](23) + p(14) + [1 + 2p + 2q](122) + O(113) - O(1112) \\
&\qquad + O(11111) \\
&= \frac{n - 3}{n(n - 1)} k_{23} + \frac{1}{n} k_{14} + \frac{n + 1}{n - 1} k_{122}
\end{aligned}
$$

Even for this case, some care in computation is advisable. Clearly direct computation should be avoided to the greatest extent possible.

In some cases, it is possible to obtain a substantial saving in computation by calculating modulo unit parts in a suitable sense. Thus in the example just given we may neglect terms $O(1^2)$. Thus we could have written the products of the brackets as

$$\begin{aligned}[1 - 2p]\langle 122\rangle + p\langle 23\rangle + p\langle 14\rangle - 2(p - q)\langle 122\rangle - 2q\langle 23\rangle + 6q\langle 122\rangle + O(1^2) \\ = [1 - 4p + 8q][(122) + O(1^2)] + [p - 2q][(23) + 3(122) + O(1^2)] \\ + p[(14) + 3(122) + O(1^2)] + O(1^2),\end{aligned}$$

which avoids a certain amount of algebra.

As a more complex example, let us take

$$\begin{aligned}k_{22}k_{22} = (22)(22) &= (\langle 22\rangle - 2\langle 112\rangle + \langle 1111\rangle)(\langle 22 - 2\langle 112\rangle + \langle 1111\rangle) \\ &= \langle 22\rangle\langle 22\rangle - 4\langle 22\rangle\langle 112\rangle + 4\langle 112\rangle\langle 112\rangle - 4\langle 112\rangle\langle 1111\rangle \\ &\quad + 2\langle 22\rangle\langle 1111\rangle + \langle 1111\rangle\langle 1111\rangle\end{aligned}$$

where

$$\begin{aligned}\langle 22\rangle\langle 22\rangle &= [1 - 4p + 2q]\langle 2222\rangle + 4[p - q]\langle 224\rangle + 2q\langle 44\rangle, \\ \langle 22\rangle\langle 112\rangle &= 2q\langle 233\rangle + O(1), \\ \langle 112\rangle\langle 112\rangle &= 2[q - r]\langle 2222\rangle + 2r\langle 224\rangle + 4r\langle 233\rangle + O(1), \\ \langle 22\rangle\langle 1111\rangle &= O(1), \\ \langle 112\rangle\langle 1111\rangle &= O(1), \\ \langle 1111\rangle\langle 1111\rangle &= 24s\langle 2222\rangle + O(1),\end{aligned}$$

so that

$$\begin{aligned}k_{22}k_{22} = [1 - 4p + 10q - 8r + 24s]\langle 2222\rangle + [p - 4q + 8r]\langle 224\rangle \\ + [-8q + 16r]\langle 233\rangle + 2q\langle 44\rangle + O(1).\end{aligned}$$

But

$$\begin{aligned}\langle 44\rangle &= (44) + 6(224) + 9(2222) + O(1), \\ \langle 224\rangle &= (224) + 3(2222) + O(1), \\ \langle 233\rangle &= (233) + O(1), \\ \langle 2222\rangle &= (2222) + O(1),\end{aligned}$$

so that

$$k_{22}k_{22} = [1 + 8p + 16q + 16r + 24s](2222) + [4p + 8q + 8r](224) + [-8q + 16r](233) + 2q(44)$$

$$= \left[1 + \frac{8}{n-2} + \frac{24}{n(n-1)(n-2)(n-3)}\right] k_{2222} + \frac{4}{n-2} k_{224} - \frac{8(n-4)}{n(n-1)(n-2)} k_{233} + \frac{2}{n(n-1)} k_{44}\,.$$

This result agrees with $k_{22}k_{22}$ as obtained from Wishart's formulas in the form

$$k_{22}k_{22} = \left(\frac{n-1}{n+1} k_2^2 - \frac{n-1}{n(n+1)} k_4\right)^2$$

$$= \left(\frac{n-1}{n+1}\right)^2 k_2^4 - 2\frac{(n-1)^2}{n(n+1)^2} k_2^2 k_4 + \frac{(n-1)^2}{n^2(n+1)^2} k_4^2$$

and thus provides an additional check on Wishart's result.

Many of Wishart's formulas were independently obtained by the writer before their publication by Wishart. In most cases agreement was good, and in the others the writer's algebra proved at fault.

A few of the simplest are now given for easy reference:

$$k_2^2 = k_{22} + \frac{1}{n} k_4 + \frac{2}{n-1} k_{22},$$

$$k_1 k_a = k_{1a} + \frac{1}{n} k_{a+1},$$

$$k_1 k_{ab} = k_{ab1} + \frac{1}{n} k_{a+1,b} + \frac{1}{n} k_{b+1,a},$$

$$k_2 k_3 = k_{23} + \frac{1}{n} k_5 + \frac{6}{n-1} k_{23}.$$

For others the reader is referred to Wishart [10]. If products of weights greater than 8 are ever needed, it is very probably that terms in $1/n$, or perhaps through $1/n^2$ will suffice. In such cases, an extension of Table 2, neglecting r, s, t, $\cdots$ (and q if terms in $1/n$ will suffice) forms the basis of a simple method of calculation.

11. The o-multiplication and one-part k's. We know (cp. Section 4) that the generating functions $\{\langle\langle a\rangle\rangle = \text{aver}\ \langle a\rangle\}$, $\{\langle a\rangle^*\}$ and $\{\langle a\rangle^{**}\}$ satisfy the relation

$$M_{\text{aver}}(t) = M^*(t)M^{**}(t)$$

where the $\langle\langle a\rangle\rangle$ are the averages over all pairings of the $\langle a\rangle$ which are defined for all pairings of the sets defining the $\langle a\rangle^*$ and $\langle a\rangle^{**}$. To obtain the one-part k's without reference to the theory of infinite populations, and to prepare the ground work for the introduction of the multipart k's, we introduce a symbolic multiplication among the following quantities: real numbers, all $\langle\langle ab \cdots e\rangle\rangle$, all $\langle ab \cdots e\rangle$, all $\langle ab \cdots e\rangle^*$, all $\langle ab \cdots e\rangle^{**}$, the integer powers of an indeterminate t, and all linear combinations of the above. This multiplication is written "o" and is defined to satisfy:

(1) *except* when a bracket is multiplied by a bracket of the same family, o-multiplication of the elementary quantities is ordinary multiplication,

(2) o-multiplication is distributive with respect to addition, subtraction and multiplication by real numbers,

(3) o-multiplication of brackets from the same family is accomplished by combining indices, as in $\langle 23\rangle\, o\, \langle 14\rangle = \langle 2314\rangle = \langle 1234\rangle$. As examples of rule 3 we have

$$\langle 11\rangle^*\, o\, \langle 34\rangle^* = \langle 1134\rangle^*,$$

$$\langle 2\rangle^{**}\, o\, (\langle 2\rangle^{**} - \langle 11\rangle^{**}) = \langle 22\rangle^{**} - \langle 112\rangle^{**},$$

where rule 2 was used in the latter case, while rule 1 shows that

$$\langle\langle 2\rangle\rangle\, o\, \langle 1\rangle^* = \langle\langle 2\rangle\rangle\langle 1\rangle^*,$$

$$\langle 3\rangle\, o\, \langle 24\rangle^{**} = \langle 3\rangle\langle 24\rangle^{**}.$$

In terms of this symbolic multiplication we have a commutative ring with formal power series in t. We can form formal o-exponentials and o-logarithms of appropriate expressions, and these functions will have the usual formal properties. Thus

$$o\text{-exp}\,(tX) = 1 + tX + \frac{t^2}{2}(X\, o\, X) + \frac{t^3}{6}(X\, o\, X\, o\, X) + \frac{t^4}{24}(X\, o\, X\, o\, X\, o\, X)t \cdots,$$

$$o\text{-log}\,(1 + tX) = tX - \frac{t^2}{2}(X\, o\, X) + \frac{t^3}{3}(X\, o\, X\, o\, X) - \frac{t^4}{4}(X\, o\, X\, o\, X\, o\, X) + \cdots,$$

and, in particular

$$o\text{-log}[(1 + tX_1)\, o\, (1 + tX_2)] = o\text{-log}\,(1 + tX_1) + o\text{-log}\,(1 + tX_2).$$

Now $M^*(t)$ involves brackets with one asterisk, and $M^{**}(t)$ involves brackets with two. Hence

$$M^*(t)\, o\, M^{**}(t) = M^*(t)M^{**}(t) = M_{\text{aver}}(t)$$

and, taking o-logarithms on both sides

$$o\text{-log}\, M^*(t) + o\text{-log}\, M^{**}(t) = o\text{-log}\, M_{\text{aver}}(t).$$

If we write

$$\psi(t) = (1)t + (2)\frac{t^2}{2!} + (3)\frac{t^3}{3!} + \cdots = o\text{-log}\, M(t)$$

and define $\psi_{\text{aver}}(t)$, $\psi^*(t)$ and $\psi^{**}(t)$ similarly, we have

$$\psi_{\text{aver}}(t) = \psi^*(t) + \psi^{**}(t)$$

and, comparing coefficients

$$((j)) = (j)^* + (j)^{**}$$

where $((j))$ is the same function of the $\langle\langle a\rangle\rangle$ as $(j)^*$ is of the $\langle a\rangle^*$ and $(j)^{**}$ is of the $\langle a\rangle^{**}$. Thus we have defined $(j) = k_j$ so as to have the right property. We have only to calculate the relations explicitly.

To do this, we have only to write out

$$\psi(t) = o\text{-}\log M(t)$$

remembering to use o-multiplication on the right. We find

$$\begin{aligned}
tk_1 + \frac{t^2}{2!}k_2 + \frac{t^3}{3!}k_3 + \cdots &= t\langle 1\rangle + \frac{t^2}{2!}\langle 2\rangle + \frac{t^3}{3!}\langle 3\rangle + \cdots \\
&\quad -\frac{1}{2}\left(t\langle 1\rangle + \frac{t^2}{2!}\langle 2\rangle + \cdots\right) o \left(t\langle 1\rangle + \frac{t^2}{2!}\langle 2\rangle + \cdots\right) \\
&\quad + \tfrac{1}{3}(t\langle 1\rangle + \cdots)\, o\, (t\langle 1\rangle + \cdots)\, o\, (t\langle 1\rangle + \cdots) \\
&\quad \cdots \\
&= t\langle 1\rangle + \frac{t^2}{2!}\langle 2\rangle + \frac{t^3}{3!}\langle 3\rangle + \cdots \\
&\quad -\tfrac{1}{2}(t^2\langle 11\rangle + t^3\langle 12\rangle + \cdots) \\
&\quad + \tfrac{1}{3}(t^3\langle 111\rangle + \cdots) + \cdots \\
&= t\langle 1\rangle + \frac{t^2}{2!}(\langle 2\rangle - \langle 11\rangle) + \frac{t^3}{3!}(\langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle) + \cdots + ,
\end{aligned}$$

so that $k_1 = \langle 1\rangle$, $k_2 = \langle 2\rangle - \langle 11\rangle$, $k_3 = \langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle$, $\cdots$. In case the population is infinite, the symmetric means become moment products, the k's become cumulants and the o-multiplication becomes ordinary multiplication. These formulas become the well-known relations connecting cumulants and moments.

$$\begin{aligned}
k_1 &= \mu_1' , \\
k_2 &= \mu_2' - \mu_1'^2, \\
k_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3,
\end{aligned}$$

and the coefficients up to order 12 are given in Kendall ([4], section 3.13).

12. Commutativity. We now wish to show why o-multiplication is commutative with additive pairing. We recall (from Section 4) that

$$\text{aver}\ \{\langle j\rangle\} = \langle\langle j\rangle\rangle = \sum \binom{j}{k}\langle j - k\rangle^*\langle k\rangle^{**} = \sum \binom{j}{k}\langle j - k\rangle^*\, o\, \langle k\rangle^{**}$$

and proceed to find the corresponding formula for a two-part bracket. We have

$$\langle gj\rangle = \frac{1}{n(n-1)} \sum^{\neq} (x_i^* + x_{\pi(i)}^{**})^g (x_j^* + x_{\pi(j)}^{**})^j$$

$$= \frac{1}{n(n-1)} \sum \sum \binom{g}{h} \binom{j}{k} \sum^{\neq} (x_i^*)^{g-h} (x_j^*)^{j-k} (x_{\pi(i)}^{**})^h (x_{\pi(j)}^{**})^k$$

and when we average over a random pairing, we may split the x^* from the x^{**}, just as for one-part brackets, obtaining

$$\langle\langle gj\rangle\rangle = \text{aver } \{\langle gj\rangle\} = \sum \sum \binom{g}{h} \binom{j}{k} \langle g-h, j-k\rangle^* \langle hk\rangle^{**}$$

Now

$$\langle gj\rangle = \langle g\rangle \, o \, \langle j\rangle, \qquad \langle g-h, j-k\rangle^* = \langle g-h\rangle^* \, o \, \langle j-k\rangle^*,$$
$$\langle hk\rangle^{**} = \langle h\rangle^{**} \, o \, \langle k\rangle^{**},$$

and this becomes

$$\langle\langle g\rangle \, o \, \langle j\rangle\rangle = \sum \sum \binom{g}{h} \binom{j}{k} \langle g-h\rangle^* \, o \, \langle h\rangle^{**} \, o \, \langle j-k\rangle^* \, o \, \langle k\rangle^{**}$$

$$= \left[\sum \binom{g}{h} \langle g-h\rangle^* \, o \, \langle h\rangle^{**}\right] o \left[\sum \binom{j}{k} \langle j-k\rangle^* \, o \, \langle k\rangle^{**}\right]$$

$$= \langle\langle g\rangle\rangle \, o \, \langle\langle j\rangle\rangle.$$

Thus averaging over random pairing commutes with o-multiplication for two-part brackets.

An entirely analogous proof holds for brackets with more than two parts, and, since the o-multiplication was defined for brackets and extended by linearity, we have commutativity in general. In particular, we have commutativity for polykays, so that

$$((1) \, o \, (2)) = \text{aver } \{(1) \, o \, (2)\} = ((1)) \, o \, ((2))$$
$$((g) \, o \, (j)) = \text{aver } \{(g) \, o \, (j)\} = ((g)) \, o \, ((j))$$

a result we will use almost at once.

13. The multipart k's. We shall now define the multipart k's by symbolic multiplication, putting

$$(12) = k_{12} = (1) \, o \, (2) = k_1 \, o \, k_2 ,$$

$$(abc \cdots e) = (a) \, o \, (b) \, o \, (c) \, o \cdots o \, (e),$$

this means, of course, that we may find the expressions for the multipart k's by writing out the corresponding single-part k's in terms of brackets and symbolically multiplying out. Thus

$$(22) = (2) \, o \, (2) = [\langle 2\rangle - \langle 11\rangle] \, \text{o} \, [\langle 2\rangle - \langle 11\rangle] = \langle 22\rangle - 2\langle 112\rangle + \langle 1111\rangle.$$

We notice that, for the case of additive random pairing

$$\begin{aligned}((12)) &= ((1))\, o\, ((2)) = [(1)^* + (1)^{**}]\, o\, [(2)^* + (2)^{**}] \\ &= (1)^*\, o\, (2)^* + (1)^*\, o\, (2)^{**} + (1)^{**}\, o\, (2)^* + (1)^{**}\, o\, (2)^{**} \\ &= (12)^* + (1)^*(2)^{**} + (1)^{**}(2)^* + (12)^{**}\end{aligned}$$

and that the formula for $((ab))$ is entirely analogous to this. Indeed, more complex expressions of similar form hold for the more-than-two-part k's and we immediately see that all the multipart k's satisfy the previously announced pairing formulas.

To complete our transformation formulas, we need to express brackets in terms of polykays. To this end, we write out

$$M(t) = o\text{-exp}\,(\psi(t)),$$

we find

$$\begin{aligned}1 + t\langle 1\rangle + \frac{t^2}{2!}\langle 2\rangle + \frac{t^3}{3!}\langle 3\rangle + \cdots\ & 1 + tk_1 + \frac{t^2}{2!}k_2 + \frac{t^3}{3!}k_3 + \cdots \\ & + \frac{1}{2}\left(tk_1 + \frac{t^2}{2!}k_2 + \cdots\right) o \left(tk_1 + \frac{t^2}{2!}k_2 + \cdots\right) \\ & + \tfrac{1}{6}(tk_1 + \cdots)\, o\, (tk_1 + \cdots)\, o\, (tk_1 + \cdots) \\ & + \cdots \\ & = 1 + tk_1 + \frac{t^2}{2!}k_2 + \frac{t^3}{3!}k_3 + \cdots \\ & + \tfrac{1}{2}(t^2k_1\, o\, k_1 + t^3k_1\, o\, k_2 + \cdots) \\ & + \tfrac{1}{6}(t^3k_1\, o\, k_1\, o\, k_1 + \cdots) + \cdots \\ & = 1 + tk_1 + \frac{t^2}{2!}k_2 + \frac{t^3}{3!}k_3 + \cdots \\ & + \tfrac{1}{2}(t^2k_{11} + t^3k_{12} + \cdots) \\ & + \frac{1}{6}\left(t^3k_{111} + \cdots\right) + \cdots \\ & = 1 + tk_1 + \frac{t^2}{2}(k_2 + k_{11}) + \frac{t^3}{3!}(k_3 + 3k_{12} + k_{111}) + \cdots\end{aligned}$$

and comparing coefficients,

$$\langle 1\rangle = k_1, \qquad \langle 2\rangle = k_2 + k_{11}, \qquad \langle 3\rangle = k_3 + 3k_{12} + k_{111} \cdots .$$

For an infinite population, these reduce the familiar formulas expressing moments in terms of cumulants, namely

$$\mu_1 = K_1, \qquad \mu_2 = K_2 + K_{11}, \qquad \mu_3 = K_3 + 3K_{12} + K_{111}, \cdots$$

and again these can be found up to order 12 in Kendall ([4], section 3.13). This time, however, the nature of the exponential function makes it easy to write down the coefficient of

$$k_{\alpha\alpha\cdots\alpha\beta\cdots\beta\cdots\delta} \quad \text{in} \quad \langle \alpha a + \beta b + \cdots \delta\, d \rangle.$$

It is

$$\frac{(\alpha a + \beta b + \cdots + \delta d)!}{(\alpha!)^a(\beta!)^b \cdots (\delta!)^d} \frac{1}{a!b! \cdots d!}$$

For example, the coefficient of k_{12} in $\langle 3 \rangle$ is

$$\frac{3!}{1!2!1!1!} = 3.$$

Thus individual coefficients are easily checked.

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