

ESTIMATION OF LOCATION AND SCALE PARAMETERS BY
ORDER STATISTICS FROM SINGLY AND DOUBLY
CENSORED SAMPLES

Part I. The Normal Distribution up to Samples of Size 10

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1. Introduction. Type II censored samples [3] are considered, whereby the total number of the sample elements is known but the observations for some of the extreme elements are missing. Singly censored samples are those in which only the smallest r_1 observations or the largest r_2 observations are missing, whereas samples having both r_1 smallest and r_2 largest observations missing are called doubly censored samples. This general case of estimation includes, as special cases, those estimates obtained from singly censored samples as well as those obtained by taking all the sample elements (i.e., r_1 or $r_2 = 0$ and $r_1 = r_2 = 0$).

The approach to the general case in censoring is of value not only for its numerical results. It enables the drawing of inferences concerning interesting and important patterns for the coefficients, variances, and the relative efficiencies of the estimates. These features could not be and were not revealed in the earlier, less general studies. These conclusions will be considered in Section (5).

In Part I, estimation of the mean and standard deviation from singly and doubly censored samples drawn from the normal distribution will be considered for samples $n \leq 10$. A generalization of an alternative estimate previously given by Gupta is also obtained. In future work, it is planned to extend the tables up to samples of size $n \leq 20$ and to include the two- and one-parameter single-exponential distributions.

Estimates of the parameters using the best linear systematic statistics are obtained by arranging the known sample elements in ascending order (i.e., $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n)}$) and applying the method of least squares to get the best linear combination of them. The coefficients provided for these linear estimates of the ordered observations make them unbiased with minimum variance. The method used in calculation is identical with that given by Gupta in [3] with slight modifications.

2. The normal distribution. To advance the study of order statistics for the normal distribution, Hastings *et al.* [4] calculated the means, variances, and covariances of the order statistics up to samples of size 10. Godwin [1] calculated these quantities more accurately as well as extending them to more decimal places. From his tables, he was able to calculate the best linear systematic statistic of the standard deviation [2] using all the sample elements for samples of

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TABLE I

Variances and covariances of order statistics in samples of sizes up to 20 from a normal population

<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value	
2	1	1	.6816901139	7	3	3	.2197215626	9	4	4	.1705588454	
		2	.3183098861			4	.1655598429			5	.1369913669	
2	.6816901139	5	.1296048425			6	.1126671842					
3	.5594672038	6	.2104468615			5	.1661012814					
3	1	2	.2756644477	8	4	4	.2104468615	10	5	6	.1661012814	
		3	.1648683485			1	.3728971434			1	.3443438233	
		2	.4486711046			2	.1863073997			2	.1712629030	
4	.1580080701	3	.1259660300			3	.1162590989					
4	1	2	.2455926930	8	1	4	.0947230277	10	1	4	.0882494247	
		3	.1580080701			5	.0747650242			5	.0707413677	
		4	.1046840000			6	.0602075169			6	.0583987134	
		2	.3604553434			7	.0482985508			7	.0489206279	
5	1	3	.2359438935	8	2	8	.0368353073	10	2	8	.0410844589	
		4	.4475340691			2	.2394010458			9	.0340406470	
		2	.2243309596			3	.1631958727			10	.0266989351	
		3	.148147252			4	.1232633317			2	.2145241430	
		4	.1057719776			5	.0975647193			3	.1466226180	
6	2	5	.0742152685	9	3	6	.0787224662	10	3	4	.1117015961	
		3	.3115189521			7	.0632466118			5	.0897428245	
		4	.2084354440			3	.2007687900			6	.0741995414	
		4	.1499426668			4	.1523584312			7	.0622278486	
		3	.2868336616			5	.1209637555			8	.0523067222	
		1	.4159271090			6	.0978171355			9	.0433711561	
7	3	2	.2085030023	9	4	4	.1871862195	10	3	3	.1750032834	
		3	.1394352565			5	.1491754908			4	.1338022448	
		4	.1024293940			1	.3573533264			5	.1077445336	
		5	.0773637839			2	.1781434240			6	.0892254012	
		6	.0563414544			3	.1207454442			7	.0749183943	
		2	.2795777392			4	.0913071400			8	.0630332449	
		3	.1889859560			5	.0727422354			4	.1579389144	
8	4	4	.1396640604	9	6	6	.0594831125	10	4	5	.1275089295	
		5	.1059054582			7	.0490764061			6	.1057858169	
		3	.2462125354			8	.0400936927			7	.0889462026	
		4	.1832727978			9	.0310552188			5	.1510539039	
		1	.3919177761			2	.2256968778			6	.1255989678	
		2	.1961990246			3	.1541163526			11	1	.3332474428
		3	.1321155811			4	.1170056918			2	.1653647712	
		4	.0984868607			5	.0934477394			3	.1123584351	
9	5	5	.0765598346	9	7	6	.0765461431	10	5	4	.0855170596	
		6	.0599187124			7	.0632354695			5	.0688483064	
		7	.0448022105			8	.0517146091			6	.0572007586	
		2	.2567328862			3	.1863826133			7	.0483754063	
		3	.1744833274			4	.1420779776			8	.0412423472	
		4	.1307298656			5	.1137680176			9	.0351103357	
		5	.1019550089			6	.0933625386			10	.0294198503	
		6	.0799811748			7	.0772351806			11	.0233152868	

TABLE I—Continued

<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value		
11	2	2	.2051975798	12	3	4	.1212063211	13	3	8	.0589221432		
		3	.1403096511			5	.0982605602			9	.0514460445		
		4	.1071492595			6	.0822228461			10	.0449637542		
		5	.0864430257			7	.0701213964			11	.0390643799		
		6	.0719305024			8	.0604384621			4	4	.1330111820	
		7	.0608869662			9	.0522825611				5	.1082512667	
		8	.0519504506			10	10			.0450357615	6	.0909855605	
		9	.0442549455				4			.1398109405	7	.0780173339	
	10	.0371029977	4			.1135687821	8			.0677217143			
	3	3	.1657242880			5	6			.0951645279	5	9	.0591628729
		4	.1269672925	7	.0812419810		10	.0517328050					
		5	.1026407291	8	.0700795832		5	.1232503256					
		6	.0855178832	9	.0606620874		6	.1037367701					
		7	.0724741050	5	5		.1306137359	7	.0890434754				
		8	.0618873278		6		.1096212247	8	.0773552864				
		9	.0527550069	7	.0936951520		9	.0676230994					
		4	4	.1479546565	8		8	.0808972960	6	6		.1183175325	
	5		.1198752861	6		.1266377911	7	.1016824204					
	6		.1000346585	7		.1083945831	8	.0884194610					
	7		.0848765182	6		6	.1266377911	8		.0884194610			
	8		.0725451434			7	.1083945831	7		.1167989950			
	5		5	.1396410804		13	1	.3152053842		14	1	1	.3077301026
			6	.1167449805			2	.1557272904				2	.1517203662
			7	.0991935960			3	.1058908842				3	.1031719531
8		.0848765182	4	.0808649736	4		.0788715916						
9		.0527550069	5	.0654634499	5		.0639657428						
6		6	.1371624335	6	.0548221797		6	.0537064714					
		12	1	.3236363870	7		.0468833088	7	.0460899189				
			2	.1602373762	8		.0406132548	8	.0401141688				
	3		.1089309641	9	.0354226462	9	.0352141760						
	4		.0830686767	10	.0309322744	10	.0310371163						
	5		.0670884464	11	.0268537250	11	.0273362865						
	6		.0559933694	12	.0228858068	12	.0239061001						
	7		.0476620974	13	.0184348220	13	.0205080257						
	8		.0410208554	2	2	.1904130721	14	.0166279801					
	9		.0354439060		3	.1302055829	2	.1844200252					
	10		.0305012591	4	.0997262696	3	.1260791989						
	11		.0257945392	5	.0808785938	4	.0966524633						
12	.0206221233		6	.0678145832	5	.0785202981							
2	2	.1972646039	7	.0580457285	6	.0660028340							
	3	.1349020328	8	.0503167946	7	.0566896715							
	4	.1031959206	9	.0439095087	8	.0493708148							
	5	.0835045822	10	.0383601798	9	.0433617156							
	6	.0697859658	11	.0333147765	10	.0382337404							
	7	.0594590652	12	.0284018130	11	.0336863221							
	8	.0512113198	3	3	.1513917013	12	.0294681314						
	9	.0442747124		4	.1162698131	13	.0252863928						
10	.0381191478	5	.0944566603	3	3	.1457045665							
11	.0322507340	6	.0792922993		4	.1119816877							
3	3	.1579786877	7	.0679282354									

TABLE I—Continued

n	i	j	Value	n	i	j	Value	n	i	j	Value		
14	3	5	.0911181271	15	2	6	.0643390895	16	1	3	.0985009764		
		6	.0766754957			7	.0554074400			4	.0754040023		
		7	.0659084825			8	.0484238833			5	.0613086724		
		8	.0574341188			9	.0427294113			6	.0516624963		
		9	.0504677802			10	.0379177516			7	.0445503705		
		10	.0445169192			11	.0337151721			8	.0390194716		
		11	.0392352316			12	.0299152347			9	.0345378158		
		12	.0343322071			13	.0263303885			10	.0307810093		
		4	4			.1272273070	14			.0227213594	11	.0275353612	
			5			.1036931108	3			3	.1407322502	12	.0246479007
			6			.0873562483				4	.1082138452	13	.0219956755
	7		.0751519909			5				.0881605755	14	.0194585037	
	8		.0655310936			6				.0743268436	15	.0168710289	
	9		.0576120957	7	.0640558183	16				.0138287378			
	10		.0508402240	8	.0560136122	2				2	.1743940788		
	11		.0448243469	9	.0494485109					3	.1191409287		
	5		5	.1171012461	10					.0438960670	4	.0914359918	
		6	.0987747550	11	.0390426915		5	.0744591145					
		7	.0850536546	12	.0346513382		6	.0628093909					
		8	.0742181416	13	.0305060359		7	.0542033941					
6	9	.0652867776	4	4	.1222328270		8	.0475009769					
	10	.0576401464		5	.0997323941		9	.0420638230					
	6	.1115324579		6	.0841705696	10	.0375018250						
	7	.0961405595	7	.0725946869	11	.0335574912							
	8	.0839617110	8	.0635175907	12	.0300461298							
	9	.0739069221	9	.0560990511	13	.0268189579							
	7	.1090269480	10	.0498187836	14	.0237301562							
15	1	8	.0953087256	11	.0443247452	15	.0205785433						
		1	.3010415703	12	.0393501820	3	3	.1363385612					
		2	.1481297708	5	5		.1118698986	4	.1048706756				
		3	.1007223449		6		.0945206004	5	.0855189036				
		4	.0770594060		7		.0815891122	6	.0722075087				
		5	.0625845851		8		.0714331681	7	.0623568515				
		6	.0526530129		9		.0631224388	8	.0546749107				
		7	.0453078886	10	.0560795065		9	.0484366096					
		8	.0395736673	11	.0499127743		10	.0431979377					
		9	.0349035905	6	6	.1058666366	11	.0386652995					
		10	.0309614122		7	.0914683204	12	.0346277256					
		11	.0275211039		8	.0801407559	13	.0309149135					
		12	.0244126313	9	.0708582099	14	.0273595378						
		13	.0214819828	10	.0629824402	4	4	.1178657554					
14	.0185333263	7	7	.1026916923	5		.0962513413						
15	.0151137071		8	.0900499964	6		.0813480448						
2	2	.1791215291	9	.0796738323	7		.0703000911						
	3	.1224176953	8	8	.1016946521		8	.0616728990					
	4	.0939067144		16	1		.2950098090	9	.0546595026				
	5	.0763912337	2		.1448881689	10	.0487647746						

TABLE I—Continued

<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value		
16	4	11	.0436607328	17	2	9	.0413928192	17	6	12	.0478122599		
		12	.0391112669			10	.0370349110			7	7	.0929031780	
		13	.0349253749			11	.0332940892			8	8	.0818194607	
	5	5	.1073517089		12	.0299982825	9		9	.0728154074			
		6	.0908232622		13	.0270170379	10		10	.0652667274			
		7	.0785480532		14	.0242386812	11		11	.0587626219			
		8	.0689488802		15	.0215459396	8		8	.0907361650			
		9	.0611364182		16	.0187658306	9		9	.0808000267			
	6	10	.0545638941		3	3	.1324207975		18	9	10	.0724599963	
		11	.0488684327			4	.1018792434				9	9	.0900465814
		12	.0437882959			5	.0831421716				1	1	.2845301297
		7	6		.1010461906	4	6			.0702850403	2	2	.1392501620
			7		.0874627156		7			.0607964413	3	3	.0946172637
			8		.0768239668		8			.0534208202	4	4	.0724851730
			9		.0681545540		9			.0474555487	5	5	.0590304274
			10		.0608534805		10			.0424726884	6	6	.0498600635
			11		.0545210724		11			.0381925587	7	7	.0431302310
8		7	.0974026613	5	12	.0344194567	8	8		.0379260195			
		8	.0856181916		13	.0310047771	9	9		.0337388141			
		9	.0760015577		14	.0278210708	10	10		.0302610667			
	10	.0678931922	15		.0247342095	11	11	.0272938041					
17	1	8	.0957213007	6	4	4	.1140068197	2	2	12	.0247002471		
		9	.0850291218			5	.0931620339			13	13	.0223801573	
		1	.2895330037			6	.0788266621			14	14	.0202537421	
	2	2	.1419424629		5	7	.0682298909		3	3	15	.0182488619	
		3	.0964748737			8	.0599826092				16	16	.0162850441
		4	.0738849615			9	.0533057575				17	17	.0142368875
		5	.0601272302			10	.0477239973				18	18	.0117719054
		6	.0507326948			11	.0429261816				2	2	.1662929294
		7	.0438236491			12	.0386942630				3	3	.1135058132
		8	.0384672834			13	.0348624030				4	4	.0871597604
		9	.0341441055			14	.0312881041				5	5	.0710825990
3	10	.0305389548	6	5	.1034004377	4	4	6	.0600975754				
	11	.0274465527		6	.0875729930			7	7	.0520217423			
	12	.0247237144		7	.0758534534			8	8	.0457683625			
	13	.0222620771		8	.0667204245			9	9	.0407317967			
	14	.0199690651		9	.0593187706			10	10	.0365451034			
	15	.0177476891		10	.0531257771			11	11	.0329704894			
	16	.0154552071		11	.0477987292			12	12	.0298442464			
	17	.0127264751		12	.0430970793			13	13	.0270462261			
4	2	.1701426762	7	13	.0388375657	5	5	14	.0244806359				
	3	.1161866734		6	.0968824669			15	15	.0220607111			
	4	.0891982557		7	.0839811738			16	16	.0196894667			
	5	.0726970385		8	.0739130260			17	17	.0172154925			
	6	.0613998459		9	.0657442736			3	3	.1288998943			
	7	.0530761573		10	.0589030403			4	4	.0991828539			
8	.0466140918	11	.0530137275	5	5	.0809899792							

TABLE I—Continued

<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value	<i>n</i>	<i>i</i>	<i>j</i>	Value		
18	3	6	.0685324700	18	8	8	.0864960639	19	3	8	.0511541418		
		7	.0593598602			9	.0771762286				9	.0456228816	
		8	.0522488413			10	.0693891332				10	.0410365629	
		9	.0465162123		11	.0627116906	11			.0371346427			
		10	.0417473296		9	9	.0853127880			12	.0337391171		
		11	.0376730987			10	.0767442321			13	.0307215918		
		12	.0341080171			19	1			1	.2799358050	14	.0279835020
		13	.0309157650		2					.1367768168	15	.0254424108	
		14	.0279875014		3					.0929061763	16	.0230195063	
		15	.0252244786		4	.0711902425	17			.0206214645			
		16	.0225161109		5	.0580094835	4			4	4	.1074740839	
		4	4		.1105660331	6					.0490405678	5	.0879051965
			5		.0903973787	7					.0424705246	6	.0745033878
			6		.0765579277	8				.0374006329	7	.0646406188	
			7		.0663522086	9				.0333319395	8	.0570032284	
	8		.0584310521	10	.0299634144	9		.0508572608					
	9		.0520394281	11	.0271011338	10		.0457576598					
	10		.0467183404	12	.0246129452	11		.0414165091					
	11		.0421694861	13	.0224037540	12		.0376368753					
	12		.0381869632	14	.0204007370	13		.0342765540					
	13		.0346192645	15	.0185431530	14		.0312262549					
	14		.0313452497	16	.0167731147	15		.0283944527					
	15		.0282548286	17	.0150223067	16		.0256935148					
	5		5	5	.0999084321	18		.0131789994	5	5	.0967944745		
				6	.0846879168	19		.0109382527			6	.0821055695	
				7	.0734460811	2	2	7			.0712796742		
		8	.0647101858	3	.1110590145			8		.0628870095			
		9	.0576543520	4	.0852931053			9		.0561272025			
		10	.0517756675	5	.0695970759		10	.0505141639					
		11	.0467468133	6	.0588910196		11	.0457330144					
		12	.0423415563	7	.0510351093		12	.0415681234					
		13	.0383932046	8	.0449652247		13	.0378636088					
		14	.0347682770	9	.0400891754		14	.0344995261					
		6	6	6	.0932407331		10	.0360490040		6	15	.0313752928	
				7	.0809202644		11	.0326137544				6	.0900218693
				8	.0713338046		12	.0296258236				7	.0782029063
9			.0635829688	13	.0269716592		8	.0690294360					
10			.0571197288	14	.0245641909		9	.0616336896					
11	.0515868552		15	.0223306885	10		.0554877905						
12	.0467370896		16	.0202017247	11		.0502493169						
13	.0423879846		17	.0180952193	12	.0456834841							
7	7		7	.0890167025	18	.0158767294	7	13	.0416203596				
			8	.0785179677	3	3			14		.0379290224		
			9	.0700199026					4		.0967367097	7	.0856172981
	10		.0629269074	5				.0790298792	8		.0756153413		
	11		.0568501034	6	.0669273696	9		.0675433161					
	12		.0515199092	7	.0580336124	10		.0608297030					

TABLE I—*Concluded*

n	i	j	Value	n	i	j	Value	n	i	j	Value
19	7	11	.0551032224	20	2	11	.0322405467	20	5	7	.0693175756
		12	.0501089625			12	.0293684960			8	.0612251429
		13	.0456621835			13	.0268315105			9	.0547222526
	8	8	.0828339961			14	.0245479493			10	.0493374275
		9	.0740273546			15	.0224526609			11	.0447662310
		10	.0666958229			16	.0204888032			12	.0408014074
		11	.0604372723			17	.0185994024			13	.0372948400
	9	12	.0549752083			18	.0167136502			14	.0341351571
		9	.0812876330			19	.0147107671			15	.0312332040
	20	10	10			.0732703911	3			3	.1228134687
11			.0664202898	4	.0945049010	6		.0871511254			
1		10	.0807909751	5	.0772355098	7	.0757703360				
		1	.2756966156	6	.0654510179	8	.0669555789				
		2	.1344941714	7	.0568056677	9	.0598659769				
		3	.0913234064	8	.0501310269	10	.0539910639				
		4	.0699879991	9	.0447763202	11	.0490008080				
		5	.0570566384	10	.0403482354	12	.0446702771				
		6	.0482701093	11	.0365934287	13	.0408385549				
		7	.0418437826	12	.0333397949	14	.0373845194				
	8	.0368937058	13	.0304645792	15	.0342111024					
	9	.0329296302	14	.0278756579	7	7	.0826123955				
10	.0296562523	15	.0254994381	8		.0730383676					
11	.0268838808	16	.0232716371	9	.0653307665						
12	.0244839567	17	.0211277373	10	.0589387428						
13	.0223649803	18	.0189874448	11	.0535056766						
14	.0204584277	4	4	.1046766243	12	.0487882257					
15	.0187096782		5	.0856442356	13	.0446121090					
16	.0170711408	6	.0726321560	14	.0408459989						
17	.0154951854	7	.0630731775	8	8	.0796309757					
18	.0139227072	8	.0556855081		9	.0712591607					
19	.0122530117	9	.0497539273	10	.0643103375						
20	.0102047204	10	.0448455403	11	.0583997310						
2	2	2	.1595731636	11	.0406811669	12	.0532644495				
		3	.1088143707	12	.0370709493	13	.0487159834				
	4	.0835758044	13	.0338793392	9	9	.0778118317				
	5	.0682247554	14	.0310045146		10	.0702526464				
	6	.0577699656	15	.0283650517	11	.0638176734					
	7	.0501109523	16	.0258897454	12	.0582229133					
	8	.0442041191	17	.0235070343	10	10	.0769474356				
	9	.0394693443	5	5		.0939960007	11	.0699266198			
	10	.0355565554		6	.0797773755						

TABLE II—Continued

n	r_1	r_2	$\gamma^{(1)}$	$\gamma^{(2)}$	$\gamma^{(3)}$	$\gamma^{(4)}$	$\gamma^{(5)}$	$\gamma^{(6)}$	$\gamma^{(7)}$	$\gamma^{(8)}$	$\gamma^{(9)}$	$\gamma^{(10)}$	
7	0	4	-.34744564	-.01345544	1.36090107								
			-.86817366	-.32689877	1.19507242								
	0	5	-1.27331716	2.27331716									
			-1.68122579	1.68122579									
	1	1	.27183155	-.61077842	.15198954	.15235782	.15198954	.27183155					
					-.10607146	.00000000	.10607146	.61077842					
	1	2	.17480153	.14322942	.14322942	.16388504	.51558402						
			-.82879521	-.12575459	-.12575459	.02477706	.92877274						
	1	3	-.05916442	-.12704486	.12704486	.93211956							
			-1.24827431	-.15477199	-.15477199	1.40304629							
1	4	-.87159736	-.87159736	1.87159736									
		-2.47116575	-2.47116575	2.47116575									
2	2			.41569461	.16861078	.41569461							
				-1.41760741	.00000000	1.41760741							
2	3			.00000000	1.00000000								
				-2.83521483	2.83521483								
0	0		.12500000	.12500000	.12500000	.12500000	.12500000	.12500000	.12500000	.12500000	.12500000	.12500000	
			-.24758623	-.12944776	-.07130849	-.02295726	.02295726	.07130849	.12944776	.24758623			
0	1		.09969946	.1188201	.12079601	.12649213	.13176484	.13698715	.13898715	.13898715	.13898715	.13898715	
			-.29775817	-.15150866	-.07963530	-.02000181	.03635632	.09505131	.27042840	.41749631			
0	2		.05691876	.09621316	.11531993	.13090112	.14512418	.15522384	.15522384	.15522384	.15522384	.15522384	
			-.36375811	-.17875554	-.08308945	-.01319507	.05693091	.14512418	.15522384	.15522384	.15522384	.15522384	
0	3		-.01672011	.06765099	.10840499	.14131770	.16984643	.16984643	.16984643	.16984643	.16984643	.16984643	
			-.45862177	-.21555013	-.09699747	.00022386	.77094550						
0	4		-.15491146	.01760383	.10013416	1.08717346							
			-.61096114	-.27072110	-.10611506	.98779730							

TABLE II—Continued

n	r_1	r_2	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	$y^{(4)}$	$y^{(5)}$	$y^{(6)}$	$y^{(7)}$	$y^{(8)}$	$y^{(9)}$	$y^{(10)}$
9	0	3	.01040388 -.37968567	.06597348 -.19359128	.09230224 -.10482346	.11331998 -.03323390	.13204377 .03166420	.58595666 .67968924				
	0	4	-.07313367 -.47658635	.03155502 -.23351551	.08087391 -.11807995	.11994592 -.02556713	.84075879 .85374893					
	0	5	-.22717960 -.63301232	-.02842070 -.29441786	.06443680 -.13477100	1.19116347 1.06220121						
	0	6	-.5662662 -.93552052	-.15208218 -.40469106	1.71850879 1.34022157							
	0	7	-1.68675768 -1.80924841	2.68675768 1.80924841								
	1	1		.20970437 -.45274773	.11588689 -.11065463	.11624355 -.05323076	.11633036 .00000000	.11624355 .05323076	.11588689 .11065463	.20970437 -.45274773		
	1	2		.16260603 -.55443214	.10736149 -.12906081	.11488307 -.05627598	.12137684 .01089528	.12749454 .07752140	.36632802 .65135225			
	1	3		.07989097 -.70150427	.09363515 -.15346705	.11399182 -.05777177	.13209087 .02994545	.58039119 .88279764				
	1	4		-.07684209 -.93990137	.06990479 -.18956185	.11531222 -.05376340	.89162509 1.18522662					
	1	5		-.42723635 -1.40567629	.02184604 -.25344578	1.40539081 1.65912206						
	1	6		-1.58736731 -2.77525943	2.58736731 2.77525943							
	2	2			.31343459 -.83170114	.12427928 -.08848435	.12457225 .00000000	.12427928 .08848435	.31343459 .83170114			
	2	3			.20395900 -1.12219549	.11906520 -.10231994	.13296497 .02227016	.54401083 1.20224528				

TABLE II—*Concluded*

n	r_1	r_2	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$	$\mathcal{Y}(t)$
10	1	3	.09423044 -.58415474	.08464767 -.14396103	-.09790987 -.07337756	.10654691 -.00974899	.12039960 .05136065	.48326580 .75988167					
	1	4	-.00426639 -.73588101	.06648054 -.17194771	-.09386130 -.07966018	.11787039 .00307342	.72607816 .98441548						
	1	5	-.18661330 -.98311476	.03511935 -.21446715	.08919507 -.08594598	1.06229888 1.28352789							
	1	6	-.58772619 -1.46776140	-.02885426 -.29176358	1.01658046 1.75952498								
	1	7	-1.8997978 -2.89604971	2.8997978 2.89604971									
	2	2	.27977552 -.70208441	.27977552 -.70208441	.10994078 -.09470422	.11028371 -.03101584	.11028371 .03101584	.10994078 -.09470422	.27977552 .70208441				
	2	3	.20496319 -.88982286	.20496319 -.88982286	.10323533 -.11005067	.11220127 -.02620385	.11982080 .05494874	.45918042 .97112842					
	2	4	.06055627 -1.19522527	.06055627 -1.19522527	.09352928 -.13182552	.11776906 -.01442865	.72814539 1.34147944						
	2	5	-.26483993 -1.79471038	-.26483993 -1.79471038	.07347216 -.16877724	1.19136777 1.96348761							
	2	6	-1.34060718 -3.56767731	-1.34060718 -3.56767731	2.34060718 3.56767731								
	3	3			.39067840 -1.28320587	.11932160 -.05586351	.11932160 .05586351	.39067840 1.28320587					
	3	4			.18711959 -1.97908791	.11984110 -.05532554	.69303931 2.08436845						
	3	5			-.48466706 -3.95105520	1.48466706 3.95105520							
	4	4				.50000000 -4.07605089	.50000000 4.07605089						

sizes ≤ 10 . Owing to the fact that the sum of the rows of the variance matrix of the order statistics (and by symmetry the sum of the columns) is equal for any sample size, the best linear systematic statistic of the population mean is, of course, the sample mean [5].

Later, Gupta [3] used Godwin's tables of means, variances, and covariances to find the best linear systematic statistic of the population mean and standard deviation from singly censored samples. Although Gupta considered Godwin's results as being a special case of his own, the more general case of estimation from doubly censored samples considered here includes the other two cases (r_1 or $r_2 = 0$ and $r_1 = r_2 = 0$).

Recently, Rosser [6], at the National Bureau of Standard's Numerical Research Analysis in Los Angeles, tabulated up to nineteen decimal places the expected values of the standardized i th order statistics X_i , where $y_i = \mu + \sigma X_i$. Following this, Teichroew [7], under the same sponsorship, calculated the expected values of the product of the i th and the j th order statistics in samples of sizes ≤ 20 , drawn from a normal population. These valuable accomplishments make possible an advance in the study of systematic statistics.

Using these tables, the variances and covariances of the order statistics in samples from a normal distribution up to ten decimal places were calculated. The variances and the covariances for order statistics of sample sizes up to 20, to ten decimal places are given in Table I. The missing entries may be obtained by

$$\text{Cov}[X_{(i)}X_{(j)}] = \text{Cov}[X_{(n-i+1)}X_{(n-j+1)}].$$

In comparing the variances and covariances for $n \leq 10$ with those tabulated (to five decimals) by Godwin [1], it can be seen that several of the latter values are in error by more than one unit in the fifth decimal place, and one value $\text{Cov}[X_{(1)}X_{(10)}]$ when $n = 10$ is in error by as much as 5 units.

Table II gives the coefficients α_{1i} and α_{2i} for the best linear systematic statistics of the mean and standard deviation, respectively, from singly and doubly censored samples of sizes ≤ 10 drawn from a normal population. For the case when $r_1 = r_2 = 0$, these tabulated results are more accurate than Godwin's [2]. For the singly censored sample, i.e., r_1 or $r_2 = 0$, the tabulated results are more accurate than those of Gupta because the latter based his calculations on the earlier tabulations of Godwin. The tabulated values in Table II are correct except for the last place, which may be in error by one or two units.

In comparing these tabulated values with those of the cited references, the results of Gupta [3] can be compared directly, whereas Godwin's [2] values are obtained in terms of rank differences. Consequently, Godwin's first coefficient should be exactly equivalent, except for sign, to that herein, and the remainder can be obtained by subtraction.

If the coefficients of an estimate are sought for a value of r_1 not given in the table, these can be obtained by interchanging the values of r_1 and r_2 and rearranging the observations in descending order. In such an event, the coefficients

6	2	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.21474267 .77471679 0	.14942352 .11233100 .01276945	.16600214 .14927902 .03751914	.20713842 .21136168 .08805478	.32479943 .33752616 .20989335	.82636223 .72430317 .65033968		
7	0	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.14285714 .08749856 0	.15346296 .15269771 0	.16680538 .22005773 .02997909	.20946061 .35717357 .10645593	.39543079 .77852463 .38638223			
8	2	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.17312828 .36219907 0	.12954025 .09242321 .00899283	.17312828 .36219907 0	.21044686 .79619765 .12726434	.21375841 .21678723 .11055562	.35409951 .34408411 .24421558	.93099884 .73416689 .71859780	
	0	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.12500000 .07461121 0	.13255044 .11928888 0	.13988098 .11706951 .02495718	.16225105 .15419334 .05377493	.22331839 .36548926 .14825957	.47112520 .79042118 .47276076		
	1	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.14091823 .15943388 .01832825	.16234808 .22720134 .05643660	.22331839 .36548926 .14825957			
	2	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.14567896 .22999555 0	.16269076 .37323003 .04936270	.23922424 .81130952 .23246854			
	3	$V(\mu^*)$ $V(\sigma^*)$ $Cov(\mu^*, \sigma^*)$.16818086 .81706103 0					

for the best linear systematic statistic of the mean will be identical with those given in the table, whereas those for the standard deviation will be numerically the same but with opposite sign.

The variances of the estimates and their covariances are given in Table III in terms of σ^2 . These values (for $r_1 = 0$) may also be compared with those of Gupta. Their percentage efficiencies relative to the best linear systematic statistic based on the complete sample are tabulated in Table IV.

3. Alternative estimates. Gupta suggested alternative linear estimates for the mean and standard deviation of a normal population to be used for samples of size > 10 because of the tediousness of calculating the exact estimate and because the variances and covariances of the order statistics for larger samples were unavailable. He calculated the variances and relative efficiencies of these alternative estimates for a sample of size 10. Gupta's alternative linear estimates were intended for the special case of singly censored samples. In the present work, this has been extended to permit its use in the general case of doubly censored samples.

This alternative estimate is based on the assumption that the variance matrix of the order statistics is a unit matrix. Therefore, the alternative estimates for doubly censored samples with r_1 and r_2 missing observations will be

$$(3.1) \quad \mu^{*'} = \sum_{i=r_1+1}^{n-r_2} b_i y_{(i)},$$

where $\mu^{*'}$ is the alternative estimate of the population mean, and

$$(3.2) \quad \sigma^{*' } = \sum_{i=r_1+1}^{n-r_2} c_i y_{(i)},$$

where $\sigma^{*'}$ is the alternative estimate of the population standard deviation. The values of b_i and c_i in $\mu^{*'}$ and $\sigma^{*'}$ are determined by

$$(3.3) \quad b_i = \frac{1}{n - r_1 - r_2} - \frac{\bar{u}_k(u_i - \bar{u}_k)}{\sum_{j=r_1+1}^{n-r_2} (u_j - \bar{u}_k)^2}$$

and

$$(3.4) \quad c_i = \frac{(u_i - \bar{u}_k)}{\sum_{j=r_1+1}^{n-r_2} (u_j - \bar{u}_k)^2},$$

where

$$\bar{u}_k = \frac{1}{n - r_1 - r_2} \sum_{j=r_1+1}^{n-r_2} u_j,$$

or the arithmetic mean of the expected values of the uncensored sample elements.

The estimates (3.1) and (3.2) are unbiased estimates of the mean and stand-

TABLE IV

Percentage efficiencies of estimates of the mean (μ^*) and standard deviation (σ^*) for censored samples relative to uncensored samples in a normal population up to size 10

n	r ₁	r ₂								
		0	1	2	3	4	5	6	7	8
2	0	100.00								
		100.00								
3	0	100.00	74.29							
		100.00	43.19							
4	0	100.00	87.10	48.73						
		100.00	59.60	26.75						
	1		83.84							
			25.51							
5	0	100.00	91.86	70.44	32.72					
		100.00	68.45	41.91	19.16					
	1		88.58	69.73						
			40.44	18.00						
6	0	100.00	94.23	80.58	55.58	23.19				
		100.00	74.03	51.72	32.11	14.85				
	1		91.32	80.52	50.57					
			50.29	30.55	13.86					
	2			77.61						
				13.64						
7	0	100.00	95.61	86.06	68.97	43.98	17.29			
		100.00	77.89	58.61	41.40	25.92	12.08			
	1		93.09	85.64	68.20	36.13				
			57.30	39.76	24.50	11.24				
	2			82.52	67.88					
				24.16	10.99					
8	0	100.00	96.50	89.36	77.04	58.48	35.30	13.43		
		100.00	80.73	63.73	48.39	34.42	21.68	10.16		
	1		94.30	88.70	77.00	55.97	26.53			
			62.55	46.80	32.84	20.41	9.44			
	2			85.81	76.83	52.25				
				32.44	19.99	9.20				
	3				74.32					
					9.13					

TABLE IV—*Concluded*

n	r ₁	r ₂								
		0	1	2	3	4	5	6	7	8
9	0	100.00	97.11	91.52	82.21	68.20	49.59	28.83	10.77	
		100.00	82.90	67.69	53.84	41.12	29.40	18.61	8.76	
	1		95.18	90.74	82.19	67.48	45.63	20.18		
			66.62	52.34	39.53	27.94	17.48	8.13		
10	0			88.14	81.62	67.07	38.97			
				39.11	27.42	17.05	7.90			
	1				78.81	66.89				
					16.93	7.82				
10	0	100.00	97.56	93.03	85.72	74.85	60.09	42.27	23.96	8.87
		100.00	84.62	70.85	58.23	46.55	35.70	25.62	16.27	7.69
	1		95.85	92.20	85.60	74.69	58.40	37.42	15.86	
			69.88	56.83	45.00	34.19	24.29	15.27	7.13	
2				89.87	84.78	74.69	56.61	29.40		
				44.58	33.62	23.74	14.85	6.93		
3					82.09	74.48	53.60			
					23.59	14.69	6.84			
4						72.29				
						6.81				

ard deviation, respectively. In order to compare these estimates with the best linear systematic statistics of the mean and standard deviation, the coefficients, variances, and relative efficiencies of the alternative estimates were calculated for all samples of size up to and including 10 and for all values of r_1 and r_2 . The variances and relative efficiencies of the alternative estimate are provided only for samples of size 10 in Table V. The variance matrix of the estimates is obtained from $(A'A)^{-1}A'VA(A'A)^{-1}\sigma^2$, where V is the variance matrix of the order statistics and A is the coefficient matrix when their means are expressed in terms of the parameters. The percentage efficiencies are relative to the corresponding best linear systematic statistics.

Examination of Table V shows that for $n = 10$, and for all the values of r_1 and r_2 , the alternative estimates are highly efficient and can replace the best linear systematic statistics without great loss of precision. In fact, use of these estimates for samples greater than 10 may be preferable to awaiting tabulation of coefficients for the most efficient estimates which involve the inversion of matrices of large order. The most efficient linear systematic statistics for the mean and standard deviation of a normal population from censored samples of sizes ≤ 20 will be given in a sequel to this paper (Part II). The authors feel,

TABLE V
Variances and relative efficiencies of alternative estimates of the mean (μ^{**}) and standard deviation (σ^{**}) for censored samples of size 10 from a normal population*

n	r ₁	r ₂								
		0	1	2	3	4	5	6	7	8
10	0	.1000000 100.00 .05766121 99.87	.10308928 99.43 .07022636 96.92 .10534509 99.04 .08489780 97.08	.10961456 98.06 .08642325 94.07 .11034725 98.29 .10545474 96.11 .11330806 98.20 .13414020 96.32	.12148953 96.03 .10748774 92.03 .12008840 97.28 .13383112 95.64 .12042277 97.95 .17683334 96.88 .12376594 98.43 .24837948 96.16	.14281760 93.54 .13638670 90.72 .13904835 96.29 .17586018 95.80 .13682291 97.85 .24748758 98.02 .13558068 99.03 .39221654 99.96 .13832644 100.00 .84582633 100.00	.18261677 91.13 .17888915 90.17 .17855378 95.89 .24533968 96.65 .17922164 98.57 .38944383 99.56 .18655569 100.00 .84259430 100.00	.26336721 89.83 .24796349 90.66 .27531177 97.06 .38411228 98.22 .34008123 100.00 .83163934 100.00	.45614307 91.50 .38065951 92.97 .63041997 100.00 .80752873 100.00	1.12690295 100.00 .74912245 100.00
	1									
	2									
	3									
	4									

* The percentage efficiency is calculated relative to the corresponding best linear systematic statistic.

however, that for samples of sizes >20 , the alternative estimate should be used because of its high relative efficiency to the exact estimate.

4. Numerical example. Ten medical students were learning to measure systolic blood pressure and practicing by taking readings upon each other. Poor technique was present in all observations but the skill of the observers in using the measuring device was limited initially to the central portion of the range. Owing to the relatively larger measurement error known to exist at the extremes, observations believed to be less than 105 mm. and greater than 125 mm. were censored.¹ This practice resulted in censoring two observations on the left and three on the right in the sample of ten.

The data, when arranged in an array, appear as the first column in the following tabulation:

Ordered observations	Coefficients			
	Exact estimate		Alternate estimate	
	μ^*	σ^*	μ^{**}	σ^{**}
1. —	0	0	0	0
2. —	0	0	0	0
3. 108	.20496319	-.88982266	.09515275	-.79906860
4. 111	.10382533	-.11005067	.15114637	-.37232645
5. 119	.11220127	-.02620385	.20170682	.01300816
6. 121	.11982080	.05494874	.25071680	.38652614
7. 125	.45918942	.97112842	.30127725	.77186075
8. —	0	0	0	0
9. —	0	0	0	0
10. —	0	0	0	0
Estimate of parameter by linear systematic statistics	118.9	16.61	119.1	17.17

If one assumes that the sample was drawn from a normal universe, the exact coefficients (α_{1i} and α_{2i}) for the best linear estimate, obtained from Table II for the case $n = 10, r_1 = 2, r_2 = 3$, are those shown in the second and third columns. If the alternative estimate is desired, the coefficients are obtainable from (3.3) and (3.4) in combination with a table for the values of u_i .

The exact and alternative estimates of the mean and standard deviation are provided. They are similar, as they should be according to the relative efficiencies

¹The derivation of coefficients given in Table II for estimating the mean and standard deviation was based upon the assumption that censoring would occur by fixed percentages of the sample. The use of these same coefficients in the present example where censoring is performed according to fixed points on the abscissa raises a question of possible bias. Based upon the results in a sampling investigation, it is felt that this possible bias is probably not an appreciable one and of little concern in practical considerations.

given in Table V, viz., 97.95 percent for the mean and 96.88 percent for the standard deviation.

5. Conclusions. Certain characteristic features can be gleaned from the tables presented herein. These are as follows:

(1). *Table II (Coefficients for the exact estimates)*

(a) By censoring a sample, the coefficients for the estimate of the mean and standard deviation undergoing maximum change are those which are associated with the extreme known sample elements.

(b) For a fixed sample size and for r_1 fixed, as r_2 increases the coefficient of the largest known element increases, whereas that of the smallest known element decreases.

(c) If the sample size is odd, and all the sample elements are censored except the middle one and its neighbor on either side, the central observation will have all the weight in estimating the mean (i.e., the other observation is of no value). If the sample size is even under the same circumstances of censoring, each middle observation has one-half of the weight in estimating the mean.

(2). *Table IV (Relative efficiency of exact estimates)*

(a) Comparing the relative efficiencies of the estimate for σ with the corresponding efficiency for μ in censored samples, the efficiency of the estimate of the former drops more rapidly than that of μ .

(b) Reading the entries for σ^* in diagonal fashion reveals that, for fixed n and fixed uncensored sample size ($r_1 + r_2 = \text{constant}$), the efficiency of the "best" estimate of σ is remarkably constant independently of r_1 and r_2 . That is, it does not matter whether the missing observations are at one end or the other of the sample or divided in any way between the two ends.

(c) Owing to the previous relationship, an interesting simple table is given showing how the efficiency in estimating σ varies with the number of known values for each sample size. This is useful for practical work in censoring.

Rough guide for assessing approximate efficiency (percent) of estimates of σ*

Sample size, n	Number of uncensored observations in sample, or $k = n - r_1 - r_2$								
	2	3	4	5	6	7	8	9	10
2	100								
3	45	100							
4	25	60	100						
5	20	40	70	100					
6	15	30	50	75	100				
7	11	25	40	60	75	100			
8	9	20	35	45	60	80	100		
9	8	17	30	40	50	65	80	100	
10	7	15	25	35	45	55	70	85	100

* These values are only 2 or 3 percent off (or less) in almost all cases.

(d) A simple fact is also evident for the estimate of the mean, μ . Its relative efficiency holds up—about 70 percent or better—so long as the sample median

value (the two middle values, if size n is even) remains known, no matter how many values below (or above) this one are missing. Thus, for $n = 10$, even if only the two values $y_{(5)}$ and $y_{(6)}$ are known, with the other 8 missing ($r_1 = r_2 = 4$), the efficiency is still 72 percent. But if one of these mid-values, e.g., $y_{(6)}$, is lost, then all of the other values (to one side) $y_{(1)}$, $y_{(2)}$, $y_{(3)}$, $y_{(4)}$, $y_{(5)}$ (i.e., $r_1 = 0$, $r_2 = 5$) cannot make up for it. They produce an efficiency no better than 60 percent. In other words, *a single central value is worth more than half the sample in estimating the mean.*

(e) In estimating the standard deviation, the foregoing situation for the mean is the direct opposite. In fact, hardly any censoring is tolerable even in the most favorable case. Thus, for $n = 10$, it can be seen from the above rough table that for as little as two missing observations ($r_1 + r_2 = 2$), whether at the same or opposite ends of the sample, the efficiency barely attains 70 percent. For more missing elements, the efficiency drops rapidly from a value under 60 to as little as 7 percent.

(3). *Table V.* (Relative efficiency and variances of alternative estimates, $n = 10$).

The alternative and the "best" estimate of either μ or σ are identical, as indicated by the 100 percent entry, when $k = n - r_1 - r_2 = 2$, i.e., when only two of the sample values are known. This is because there are then only two coefficients to be estimated, which makes the unbiased estimator unique and therefore the same as the (unbiased) alternative estimator, which is therefore a fortiori of minimum variance. The other case of identity is for a complete sample ($r_1 = r_2 = 0$), where the best estimate of μ is the sample mean, and this is also the alternative estimate. This is because the average, $\bar{u}_n = \sum_{i=1}^n u_i$, of all the means of the sample order statistics is equal to the population mean, which may be taken as zero without loss of generality; by (3.3) this shows that all the coefficients b_i must be equal, giving the sample mean.

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