ON THE POWER OF CERTAIN TESTS FOR INDEPENDENCE IN BIVARIATE POPULATIONS[1]

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Summary. Let F_{λ^0} denote the joint distribution of two independent random variables Y_{λ^0} and Z_{λ^0}. The paper investigates properties of the joint distribution F_λ of the linearly transformed random variables Y_λ and Z_λ. Let $\mathfrak{J}_0$ be the Spearman rank correlation test, $\mathfrak{J}_1$ the difference sign correlation test, $\mathfrak{J}_2$ the unbiased grade correlation test (which is asymptotically equivalent to $\mathfrak{J}_0$), $\mathfrak{J}_3$ the medial correlation test, and $\mathfrak{R}$ the ordinary (parametric) correlation test. (Whenever discussing $\mathfrak{R}$ we assume existence of fourth moments.) Properties of the power of these tests are found for alternatives of the above-mentioned form, particularly for alternatives "close" to the hypothesis of independence and for large samples.

Against these alternatives the efficiency of $\mathfrak{J}_3$ is found to depend strongly on local properties of the densities of Y_{λ^0} and Z_{λ^0}, which should invite caution; and the efficiency of $\mathfrak{J}_1$ with respect to $\mathfrak{J}_0$ is often unity.

Incidentally, Pitman's result on efficiency is extended in several directions.

1.1. Introduction. In the investigation, for a class of problems, of operating characteristics of tests of statistical hypotheses, the crucial point is the specification of a class of alternatives which is (i) sufficiently wide to include some approximation to any situation that may arise in this class of problems, and (ii) is manageable mathematically.

For testing the hypothesis that two samples are from the same population, this point has been dealt with—with some measure of success—by specifying as alternatives the cases in which the two populations differ by a location (shift) parameter but otherwise can have any continuous distribution. This seems a satisfactory idealization for a class of problems, and is easy to handle mathematically.

The situation cannot be expected to be so simple for testing of the hypothesis of independence in bivariate populations. As a matter of fact, in many applications it seems that because of the bewildering variety of possible "modes of dependence" it is not feasible to provide a reasonable specification of alternatives satisfying (i) and (ii). This paper makes an attempt to open up this topic by considering a rather narrow class of alternatives for which (ii) is satisfied, though the extent to which (i) is satisfied is much more doubtful. Another class is considered in [8].

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The class of alternatives considered is one under which the two random variables have been obtained by a linear transformation of two independent random variables. Cases of random variables which could have common but unobservable components may conform to this situation.

Thus, suppose that the outcome of the application of a battery $(\mathfrak{A}_1, \mathfrak{A}_2, \cdots)$ of psychological or psychophysical tests to a group of people has been subjected to a factorial analysis, revealing several independent common factors

$$F_1, \cdots, F_c.$$

Suppose that the analysis shows that apparently the outcome A_{i_1} of $\mathfrak{A}_{i_1}$ is practically determined by F_1 and A_{i_2} of $\mathfrak{A}_{i_2}$ by F_2. Psychological tests, especially aptitude tests, are often designed to achieve such "factorial purity" [4]. It may then be reasonable and desirable to identify F_1 operationally with A_{i_1} and F_2 with A_{i_2}. Before doing this, one should make sure that A_{i_1} and A_{i_2} are independent random variables. If $c > 2$, let us assume that for $\mathfrak{A}_{i_1}$ and $\mathfrak{A}_{i_2}$ we can ignore $F_3, \cdots, F_c$, or that those among the latter set which affect A_{i_1} do not affect A_{i_2} and vice versa. Then the above description implies that there exist two independent random variables Y and Z and numbers λ_1, λ_2, λ_3, λ_4, such that

$$A_{i_1} = \lambda_1 Y + \lambda_2 Z,$$

$$A_{i_2} = \lambda_3 Y + \lambda_4 Z,$$

and the hypothesis to be tested can be written

$$\lambda_2 = \lambda_3 = 0.$$

The relative asymptotic efficiencies of the Spearman and medial correlation tests with respect to the ordinary correlation test have been previously considered heuristically by Hotelling and Pabst [6] and Blomqvist [1], respectively, for normal alternatives; as far as is known to the author, no other investigations of the relative efficiencies of the tests discussed here have been published for bivariate distributions which stay constant during the sampling process.

The present paper also contains an extension of Pitman's result on local asymptotic efficiency, which is believed to be of interest in itself.

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1.2. Local asymptotic efficiency according to Pitman and an extension. Any statistical hypothesis and its alternatives can be described by two classes of probability distributions, $\mathfrak{F}_0$ and $\mathfrak{F}_a$, respectively.

We are interested in cases in which there is some natural way of generating the elements of $\mathfrak{F}_a$ from those of $\mathfrak{F}_0$ in such a way that each $F \in \mathfrak{F}_0$ is obtained as a limit of a sequence of elements of the corresponding subset of $\mathfrak{F}_a$. This can be formalized as follows:

Let $\mathfrak{F}_0$ and $\mathfrak{F}_a$ have the property that there is a set $\mathfrak{A}$ of transformations A from one to the other such that (1)

$$\mathfrak{F}_0 \cup \mathfrak{F}_a = \bigcup_{F \varepsilon \mathfrak{F}_0} \mathfrak{A}[F],$$

where $\mathfrak{A}[F] = \bigcup_{A \varepsilon \mathfrak{A}} AF$ and where $F \varepsilon \mathfrak{F}_0$ implies $F \varepsilon \mathfrak{A}[F]$; (2) there exists a K-dimensional metric space Γ of points $\gamma = (\gamma_1, \cdots, \gamma_K)$, which (a) contains a point γ^0 and (b) for each $F \varepsilon \mathfrak{F}_0$ has a subset $\Gamma(F)$ (containing γ^0 as a limit point) whose elements are in one-to-one correspondence with those of $\mathfrak{A}[F]$, with F corresponding to γ^0.

DEFINITION. Given two nonnegative numbers α', α'' for which $0 < \alpha' + \alpha'' < \frac{1}{2}$, let $\mathfrak{I} = \{\mathfrak{I}_n\}$ be a sequence of tests of the hypothesis $\gamma = \gamma^0$, that is, of $F_\gamma = F_{\gamma 0}(\equiv F)$. Let $\mathfrak{I}$ be such that for n observations, $\mathfrak{I}_n$ rejects the hypothesis if and only if the statistic T_n does not exceed a maximal constant (or random variable) t'_{nF} or does not fall below a minimal constant (or random variable) t''_{nF} for which

$$\alpha'_{nF} = P\{T_n \leqq t'_{nF} \mid F\}, \qquad \alpha''_{nF} = P\{T_n \geqq t''_{nF} \mid F\}$$

do not exceed α' and α'', respectively, and converge to these numbers. We then say that $\mathfrak{I}$ is an (α', α'')-*level* test (sequence).

For tests based on ranks, α'_{nF}, α''_{nF}, t'_{nF}, and t''_{nF} are independent of F when F is continuous and hence are "distribution free."

In the following let $F \varepsilon \mathfrak{F}_0$ and α' and α'' be fixed. To obtain at least a certain fixed power $\beta > \alpha' + \alpha''$ for the test sequence $\mathfrak{I}$ under the alternative γ, we have to choose the number n of observations so large that the probability β_n of rejecting the hypothesis with test $\mathfrak{I}_n$ under γ is at least β (and shall otherwise choose n as small as possible). For reaching the same power β with the test sequence $\mathfrak{I}^*$ under the same alternative, we have to choose the minimum number n^* of observations sufficiently large for $\beta^*_{n^*}$, the probability of rejecting the hypothesis with test $\mathfrak{I}^*_{n^*}$ under γ, to be at least β. Now for a fixed β, if we wish to let n increase indefinitely, we have to allow γ to vary with n:

$$\gamma = \gamma(n).$$

In particular, if $\mathfrak{I}$ and $\mathfrak{I}^*$ are consistent (α', α'')-level tests, the sequence $\{\gamma(n)\}$ of alternatives must converge to γ^0. Then, when $\lim_{n\to\infty} n/n^*$ exists, it is reasonable to call it the local asymptotic efficiency of $\mathfrak{I}^*$ with respect to $\mathfrak{I}$ against a sequence $\{\gamma(n)\}$ of alternatives with elements in $\Gamma - \gamma^0$. We shall now enumerate some conditions which suffice for its existence and allow us to calculate it. These slightly generalize those first given by Pitman (see [11]), who examined only cases with α' or α'' equal to 0 and $K = 1$, and possessing certain other simple features.

DEFINITION. Suppose there exist $h > 0$ and functions ψ_n and χ_n over Γ such that for any $k \leqq K$

(i) $\lim_{n\to\infty} P\{[T_n - \psi_n(\gamma^0)]/\chi_n(\gamma^0) \leqq t \mid \gamma^0\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx$;

(ii) $\psi_{nk}(\gamma) = \partial\psi_n(\gamma)/\partial \gamma_k$ exists in a neighborhood of γ^0;

(iii) $n^{-h}\psi_{nk}(\gamma^0)/\chi_n(\gamma^0)$ converges to a constant d_k ;

(iv) the alternatives $\gamma(n)$ are such that $\gamma_k(n) = \gamma_k^0 + n^{-h}\cdot c_k + o(n^{-h})$ (which defines neighborhoods of γ^0);

(v) $\chi_n(\gamma(n))/\chi_n(\gamma^0)$ converges to 1;

(vi) for $\gamma'(n)$ on the ray from γ^0 to $\gamma(n)$, $\psi_{nk}(\gamma'(n))/\psi_{nk}(\gamma^0)$ converges to 1;

(vii) $\lim_{n\to\infty} P\{[T_n - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) \leqq t \mid \gamma(n)\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$

Let

$$\Delta(c) = \sum_{k=1}^{K} c_k d_k .$$

The class of all (α', α'')-level tests, consistent for testing $\gamma = \gamma^0$ against $\gamma \neq \gamma^0$ near γ^0 and such that for a given h there exist functions ψ_n and χ_n over Γ satisfying these properties, will be called $\mathcal{P}^{(1)}_{\alpha',\alpha''}(h, \Gamma)$. The set of vectors c generated by the set of all values $\beta > \alpha' + \alpha''$ will be denoted by $\bar{\Gamma}$.

THEOREM 1.1. *Let $\mathfrak{J} \varepsilon \mathcal{P}^{(1)}_{\alpha',\alpha''}(h, \Gamma)$, $\mathfrak{J}^* \varepsilon \mathcal{P}^{(1)}_{\alpha',\alpha''}(h^*,) \Gamma$, and let $\Delta(c)$ or $\Delta^*(c)$ differ from zero. Then for any sequence of alternatives with elements in $\Gamma - \gamma^0$ for which $c \varepsilon \bar{\Gamma}$ the local asymptotic efficiency of $\mathfrak{J}^*$ with respect to $\mathfrak{J}$ against this sequence exists and equals* 0 *if $h^* < h$, ∞ if $h^* > h$, and*

$$\{\Delta^*(c)/\Delta(c)\}^{1/h}$$

otherwise.

OUTLINE OF PROOF. Write

$$\Phi(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$$

We have

$$\lim_{n\to\infty} P\{[T_n - \psi_n(\gamma^0)]/\chi_n(\gamma^0) \leqq [t_n' - \psi_n(\gamma^0)]/\chi_n(\gamma^0) \mid \gamma^0\} = \alpha',$$

so

$$\lim_{n\to\infty} [t_n' - \psi_n(\gamma^0)]/\chi_n(\gamma_0) = \delta',$$

where $\delta' = \Phi^{-1}(\alpha')$. Similarly, if $\delta'' = \Phi^{-1}(1 - \alpha'')$,

$$\lim_{n\to\infty} [t_n'' - \psi_n(\gamma^0)]/\chi_n(\gamma^0) = \delta''.$$

Now for $0 < \vartheta < 1$

$$\lim_{n\to\infty} [t_n' - \psi_n(\gamma(n))]/\chi_n(\gamma(n))$$
$$= \lim_{n\to\infty}[t_n' - \psi_n(\gamma^0)]/\chi_n(\gamma(n)) - \lim_{n\to\infty} \sum_{k=1}^{K} [c_k + o(1)]n^{-h}$$
$$\psi_{nk}(\vartheta(\gamma(n) - \gamma^0))/\chi_n(\gamma(n))$$
$$= \delta' - \Delta(c),$$

$$\lim_{n\to\infty}[t_n'' - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) = \delta'' - \Delta(c).$$

But $1 - \beta_n$ equals

$$P\{[t_n' - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) \leqq [T_n - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) \leqq [t_n'' - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) \mid \gamma(n)\},$$

so that

$$1 - \beta = \lim_{n\to\infty} (1 - \beta_n) = \Phi(\delta'' - \Delta(c)) - \Phi(\delta' - \Delta(c)) = \Psi_{\alpha',\alpha''}(\Delta(c))$$

(say).

We wish to determine n^* as a function of n in such a way that for the same alternative,

$$\gamma(n) = \gamma^*(n^*),$$

both $\mathfrak{T}$ and $\mathfrak{T}^*$ reach (as closely as possible) the same power β. Let $h^* = h$ (the other cases are handled similarly). By assumption (iv)

$$\gamma(n) \sim \gamma^0 + n^{-h} \cdot c,$$
$$\gamma^*(n^*) \sim \gamma^0 + n^{-h} \cdot c^*,$$

so

$$c^* \sim (n/n^*)^{-h} \cdot c,$$

and

$$\Delta^*(c^*) \sim (n/n^*)^{-h} \sum_{k=1}^{K} c_k d_k^* = (n/n^*)^{-h}\Delta^*(c).$$

In the same manner as above, we obtain for $\mathfrak{T}^*$

$$1 - \beta = \lim_{n\to\infty} (1 - \beta_{n^*}^*) = \Psi_{\alpha',\alpha''}(\Delta^*(c^*)),$$

so that, since $\Psi_{\alpha',\alpha''}$ has an inverse,

$$\Delta^*(c^*) = \Delta(c).$$

Consequently,

$$n/n^* \sim \{\Delta^*(c)/\Delta(c)\}^{1/h}.$$

For sequences $\{\gamma(n)\}$ for which both $\Delta(c)$ and $\Delta^*(c)$ equal zero, Theorem 1.1 yields no result. As noted also in [12], it is desirable to be in a position to expand $\psi_n(\gamma)$ about γ^0, using terms of order higher than the first. We therefore give the following definition:

DEFINITION. Suppose there exist $h > 0$, a smallest integer p, and functions ψ_n and χ_n over Γ, such that for any set $(k_1, \cdots, k_p)$ of p not necessarily different integers $\leqq K$

(i) $\lim_{n\to\infty} P\{[T_n - \psi_n(\gamma^0)]/\chi_n(\gamma^0) \leqq t \mid \gamma^0\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx$;

(ii) $\psi_{nk_1\cdots k_p}(\gamma) = \partial^p\psi_n(\gamma)/\partial\gamma_{k_1}\cdots\partial\gamma_{k_p}$ exists in a neighborhood of γ^0;

(iii) $n^{-hp}\psi_{nk_1\cdots k_p}(\gamma^0)/\chi_n(\gamma^0)$ converges to a constant $d_{k_1\cdots k_p}$;

(iv) the alternatives $\gamma(n)$ are such that $\gamma_k(n) = \gamma_k^0 + n^{-h}c_k + o(n^{-h})$ (which defines neighborhoods of γ^0), and if $c = (c_1, \cdots, c_k)$, then

$$\Delta(c) = (p!)^{-1} \sum_{(k_1 \cdots, k_p)} c_{k_1} \cdots c_{k_p} d_{k_1 \cdots k_p} \neq 0;$$

(v) $\chi_n(\gamma(n))/\chi_n(\gamma^0)$ converges to 1;

(vi) for $\gamma'(n)$ on the ray from γ^0 to $\gamma(n)$, $\psi_{nk_1 \cdots k_p}(\gamma'(n))/\psi_{nk_1 \cdots k_p}(\gamma^0)$ converges to 1;

(vii) $\lim_{n\to\infty} P\{[T_n - \psi_n(\gamma(n))]/\chi_n(\gamma(n)) \leqq t \mid \gamma(n)\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$

The class of all (α', α'')-level tests, consistent for testing $\gamma = \gamma^0$ against $\gamma \neq \gamma^0$ near γ^0 and such that for the same h, p there exist functions ψ_n and χ_n over Γ satisfying these properties, will be called $\mathcal{P}^{(p)}_{\alpha',\alpha''}(h, \Gamma)$. The set of vectors c generated by the set of all values for $\beta > \alpha' + \alpha''$ will be denoted by $\bar{\Gamma}$.

THEOREM 1.2. *If* $\mathfrak{I} \in \mathcal{P}^{(p)}_{\alpha',\alpha''}(h, \Gamma)$ *and* $\mathfrak{I}^* \in \mathcal{P}^{(p^*)}_{\alpha',\alpha''}(h^*, \Gamma)$, *then for any sequence of alternatives for which* $c \in \bar{\Gamma}$,

$$\lim_{n\to\infty} n/n^*$$

exists and equals

$\{\Delta^*(c)/\Delta(c)\}^{1/hp}$ *if* $p^* = p$ *and* $h^* = h$,

0 *if* $p^* \geqq p$ *and* $h^* \leqq h$ *without both being equalities,*

∞ *if* $p^* \leqq p$ *and* $h^* \geqq h$ *without both being equalities.*

We then *define* this as the local asymptotic efficiency $e(\mathfrak{I}^*, \mathfrak{I})$ of $\mathfrak{I}^*$ with respect to $\mathfrak{I}$ against a sequence of alternatives with elements in $\Gamma - \gamma^0$ for which $c \in \bar{\Gamma}$.

REMARKS. (a). When $K = 1$, $\Delta^*(c)/\Delta(c)$ does not involve c, so that the efficiency of $\mathfrak{I}^*$ with respect to $\mathfrak{I}$ does not depend on the particular values taken on by α', α'' or β, being the same against all sequences of alternatives with elements in $\Gamma - \gamma^0$ for which $c \in \bar{\Gamma}$. This is not generally so when $K > 1$. The dependence on the values of α' or α'' is through the sign of $\Delta(c)$, and disappears if either α' or α'' is zero.

(b) The limiting distribution Φ does not have to be normal. It is sufficient if it is continuous, if $\Phi^{-1}(\alpha') = \delta'$ and $\Phi^{-1}(1 - \alpha'') = \delta''$ are uniquely determined, and if there exists $\epsilon > 0$ such that for $x \in (-\epsilon, \infty)$ or $\in (-\infty, \epsilon)$,

$$\Psi_{\alpha',\alpha''}(x) = \Phi(\delta'' - x) - \Phi(\delta' - x)$$

is a monotone function of x, converging to zero as $|x| \to \infty$ (the monotonicity not ceasing to be strict until the function attains the value 0). Note that if either α' or α'' vanishes, all continuous distribution functions Φ which are strictly increasing for the set of t for which $0 < \Phi(t) < 1$ have this property.

(c) Against a fixed alternative $\gamma' \in \Gamma$ near γ^0, for sufficiently large n, the power of $\mathfrak{I}$ is approximately

$$1 - \Psi_{\alpha',\alpha''}[n^{hp}\Delta(\gamma' - \gamma^0)].$$

We may call this expression the *asymptotic power* of γ' (near γ^0). So if, for γ' (near γ^0), $\Delta(\gamma' - \gamma^0) \neq 0$, $\mathfrak{J}$ is *consistent at* γ'; and if $\{\gamma(n)\}$ is such that, as β takes on different values exceeding $\alpha' + \alpha''$, $\Delta(c)$ differs from 0, $\mathfrak{J}$ is *consistent near* γ^0.

1.3. Some tests for independence. Consider a sample $X_1, \cdots, X_n$ $(n \geqq 2)$ from a bivariate population F, where $X_\alpha = (Y_\alpha, Z_\alpha)$, without ties in the Y_α or in the Z_α. Let $R_{(\alpha)}$ be the rank of Y_α, $S_{(\alpha)}$ the rank of Z_α, and $\tilde{Y}$ and $\tilde{Z}$ the sample medians of Y and Z. Define for α, β, γ ranging over $1, \cdots, n$

$$\frac{(n-1)(n+1)}{3n^2} T_{0n} = \frac{1}{n^3} \sum_{\alpha,\beta,\gamma} \operatorname{sgn}(Y_\alpha - Y_\beta) \operatorname{sgn}(Z_\alpha - Z_\gamma)$$

$$= \frac{4}{n^3} \sum_\alpha \left(R_{(\alpha)} - \frac{n+1}{2}\right)\left(S_{(\alpha)} - \frac{n+1}{2}\right);$$

$$T_{1n} = \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \operatorname{sgn}(Y_\alpha - Y_\beta) \operatorname{sgn}(Z_\alpha - Z_\beta);$$

$$\frac{1}{3} T_{2n} = \frac{1}{n(n-1)(n-2)} \sum_{\alpha \neq \beta \neq \gamma \neq \alpha} \operatorname{sgn}(Y_\alpha - Y_\beta) \operatorname{sgn}(Z_\alpha - Z_\gamma), \quad \text{for } n > 2,$$

$$= 0 \text{ for } n = 2;$$

$$T_{3n} = \frac{(N' - N'') + (n' - n'')}{(N' + N'') + (n' + n'')},$$

where N' is the number of α for which $\operatorname{sgn}(Y_\alpha - \tilde{Y}) \operatorname{sgn}(Z_\alpha - \tilde{Z})$ is positive, N'' the number for which it is negative, and $n'[n'']$ is the number of pairs (α, β) for which $\alpha \neq \beta$, $Y_\alpha = \tilde{Y}$, $Z_\beta = \tilde{Z}$, and $\operatorname{sgn}(Y_\beta - \tilde{Y}) \operatorname{sgn}(Z_\alpha - \tilde{Z})$ is positive [negative].

T_{1n} is the difference sign correlation, T_{0n} the (Spearman) rank correlation, T_{2n} the unbiased grade correlation (introduced by Hoeffding [5]), and T_{3n} the medial correlation proposed by Sheppard [15] and discussed by Blomqvist [1]. (The name medial correlation was proposed in [14].)

It may be noted that the formulae for T_{0n} and T_{3n} are obtainable from the formula for the ordinary correlation coefficient

$$R_n = \sum_\alpha (Y_\alpha - \bar{Y})(Z_\alpha - \bar{Z}) / \{\sum_\alpha (Y_\alpha - \bar{Y})^2 \sum_\alpha (Z_\alpha - \bar{Z})^2\}^{1/2}$$

$$(\bar{Y} = \sum_\alpha Y_\alpha / n, \qquad \bar{Z} = \sum_\alpha Z_\alpha / n)$$

by substituting ranks for observations and, in the case of T_{3n} (assuming n even), interpreting ( ) as the signum of the quantities in parentheses. Applications of these operations to the alternative form [3]

$$R_n = \sum_{\alpha,\beta} (Y_\alpha - Y_\beta)(Z_\alpha - Z_\beta) / \{\sum_{\alpha,\beta} (Y_\alpha - Y_\beta)^2 \sum_{\alpha,\beta} (Z_\alpha - Z_\beta)^2\}^{1/2}$$

gives T_{0n} and T_{1n}, and to the alternative form

$$R_n = \frac{\sum_{\alpha\neq\beta\neq\gamma\neq\alpha} (Y_\alpha - Y_\beta)(Z_\alpha - Z_\gamma)}{\{\sum_{\alpha\neq\beta\neq\gamma\neq\alpha} (Y_\alpha - Y_\beta)(Y_\alpha - Y_\gamma) \sum_{\alpha\neq\beta\neq\gamma\neq\alpha} (Z_\alpha - Z_\beta)(Z_\alpha - Z_\gamma)\}^{\frac{1}{2}}}$$

gives T_{0n} and T_{2n}.

These statistics are discussed in [5] and [1], where the following properties are proved for F continuous:

(1) T_{1n} has mean $\tau_1 = \tau_1[F] = E\Phi_{12}(X_\alpha, X_\beta)$, where X_α, X_β is a random sample of size 2 and $\Phi_{12}(x_1, x_2) = \sum_{i\neq j\epsilon(1,2)} \text{sgn}(y_i - y_j)\,\text{sgn}(z_i - z_j)$. Let $\Phi_{11}(x) = E\Phi_{12}(x, X)$, then $\tau_1 = E\Phi_{11}(X)$. One finds $\Phi_{11}(x) = 4F(y, z) - 2F(y, \infty) - 2F(\infty, z) + 1$, so that $\tau_1 = 4\iint F\,dF - 1$. T_{1n} has the variance

$$\sigma_{1n}^2 = \frac{2}{n(n-1)}[2(n-2)\{E\Phi_{11}^2(X) - \tau_1^2\} + 1 - \tau_1^2],$$

which is finite and converges to zero with $1/n$. In case of independence $\tau_1 = 0$, $E\Phi_{11}^2(X) = \frac{1}{9}$, and we have

$$\sigma_{1n}^2 = \frac{2(2n+5)}{9n(n-1)}.$$

The distribution of $(T_{1n} - \tau_1)/\sigma_{1n}$ converges to the normal distribution with 0 mean and unit variance if $E\Phi_{11}^2(X) \neq \tau_1^2$.

(2) T_{2n} has mean $\tau_2 = \tau_2[F] = E\Phi_{23}(X_\alpha, X_\beta, X_\gamma)$, where $X_\alpha, X_\beta, X_\gamma$ is a random sample of size 3 and $\Phi_{23}(x_1, x_2, x_3) = \frac{1}{2}\sum_{i\neq j\neq k\neq i\epsilon(1,2,3)} \text{sgn}(y_i - y_j)\,\text{sgn}(z_i - z_k)$. Let $\Phi_{22}(x_1, x_2) = E\Phi_{23}(x_1, x_2, X)$, $\Phi_{21}(x) = E\Phi_{22}(x, X)$; then $\tau_2 = E\Phi_{21}(X)$. One finds for $n > 2$

$$\begin{aligned}\Phi_{22}(x_1, x_2) &= 1 + 2F(y_1, z_2) + 2F(y_2, z_1)\\ &\quad + \{F(y_1, \infty) - F(y_2, \infty)\}\{\text{sgn}(z_1 - z_2) - 1\}\\ &\quad + \{F(\infty, z_1) - F(\infty, z_2)\}\{\text{sgn}(y_1 - y_2) - 1\},\\ \Phi_{21}(x) &= 1 - 4F(y, \infty) - 4F(\infty, z) + 4F(y, \infty)F(\infty, z)\\ &\quad + 4\int F(y, z)\,dF(\infty, z) + 4\int F(y, z)\,dF(y, \infty),\end{aligned}$$

so that

$$\begin{aligned}\tau_2 &= 12\iint F(y, \infty)F(\infty, z)\,dF(y, z) - 3\\ &= 12\iint \{F(y, \infty) - \tfrac{1}{2}\}\{F(\infty, z) - \tfrac{1}{2}\}\,dF(y, z),\end{aligned}$$

called the "grade correlation." Lemma 4.1 of [8] proves that this equals

$$12\iint F(y, z)\,dF(y, \infty)\,dF(\infty, z) - 3 = 12\iint (F - F_0)\,dF_0.$$

T_{2n} has variance (for $n > 2$)

$$\sigma_{2n}^2 = \frac{6}{n(n-1)(n-2)}\left[3\binom{n-3}{2}\{E\Phi_{21}^2(X) - \tau_2^2\} + 3(n-3)\{E\Phi_{22}^2(X_\alpha, X_\beta) - \tau_2^2\} + 1 - \tau_2^2\right],$$

where X_α and X_β are independent; this is finite and converges to zero with $1/n$. In case of independence $\tau_2 = 0$, $E\Phi_{21}^2(X) = \frac{1}{9}$, $E\Phi_{22}^2(X_\alpha, X_\beta) = \frac{7}{18}$, and we have

$$\sigma_{2n}^2 = \frac{n^2 - 3}{n(n-1)(n-2)};$$

while $\rho(T_{1n}, T_{2n}) \to 1$, ρ representing the correlation coefficient so that the asymptotic functional relation $3T_{1n} = 2T_{2n}$ holds [3]. The latter relation does not hold in general in the case of dependence. $(T_{2n} - \tau_2)/\sigma_{2n}$ converges to the normal distribution with 0 mean and unit variance if $E\Phi_{21}^2(X) \neq \tau_2^2$. All future references to T_{2n}, τ_2, and σ_{2n} will be understood to bear an appropriate qualification for $n = 2$.

(3) Assume that the medians of the Y and Z populations are unique, and denote them by μ and ν, and let $F(\mu, \nu)$ be different from 0 or $\frac{1}{2}$. (The other assumptions in [1] can be shown, by use of the Glivenko-Cantelli theorem, to be superfluous; but the condition on $F(\mu, \nu)$, not given there, is, in fact, essential.) Then the distribution of $(\frac{1}{2})\{(N' - 2n\, F(\mu, \nu)\}[n\, F(\mu, \nu)\{\frac{1}{2} - F(\mu, \nu)\}]^{-1/2}$ converges to the normal distribution with 0 mean and unit variance; so the same holds for $(T_{3n} - \tau_3)/\sigma_{3n}$, where

$$\tau_3 = \tau_3[F] = 4[F(\mu, \nu) - F(\mu, \infty)F(\infty, \nu)],$$

$$\sigma_{3n}^2 = \frac{16}{n} F(\mu, \nu)\{\tfrac{1}{2} - F(\mu, \nu)\} = \frac{1}{n}(1 - \tau_3^2),$$

as $F(\mu, \nu) = \frac{1}{4}(1 + \tau_3)$. In any case under F_0, continuous, the distribution of T_{3n} is symmetric.

(4) $T_{0n} = [(n-2)\, T_{2n} + 3T_{1n}]/(n+1)$. Moreover,

$$\text{cov}\,(T_{1n}, T_{2n}) = \frac{6}{n(n-1)}[(n-3)\{E\Phi_{11}(X)\Phi_{21}(X) - \tau_1\tau_2\} + \{E\Phi_{12}(X_\alpha, X_\beta)\Phi_{22}(X_\alpha, X_\beta) - \tau_1\tau_2\}],$$

where X_α and X_β are independent. In case of independence $E\,\Phi_{11}(X)\,\Phi_{21}(X) = \frac{1}{9}$, $E\,\Phi_{12}(X_\alpha, X_\beta)\,\Phi_{22}(X_\alpha, X_\beta) = \frac{5}{9}$, and we get

$$\sigma_{0n}^2 = \frac{1}{n-1}, \qquad \rho(T_{1n}, T_{2n}) \sim 1.$$

The asymptotic distribution follows from this and the remarks under (2).

In case the variances of Y and Z are finite and positive, one could compare these tests with the (parametric) test $\mathcal{R}$ to see whether the correlation coefficient

ρ vanishes. (We shall see, however, that in cases of interest in this paper, $\rho = 0$ frequently does not imply independence.) According to Cramér ([2], pp. 359 and 366), $ER_n = \rho + 0(1/n)$, and, if the fourth moments are finite,

$$E(R_n - \rho)^2 = \frac{k}{n} + 0(n^{-\frac{3}{2}}),$$

with

$$k = \frac{\rho^2}{4}\left(\frac{\mu_{40}}{\mu_{20}^2} + \frac{\mu_{04}}{\mu_{02}^2} + \frac{2\mu_{22}}{\mu_{20}\,\mu_{02}}\right) - \rho\left(\frac{\mu_{31}}{\mu_{20}} + \frac{\mu_{13}}{\mu_{02}}\right)(\mu_{02}\,\mu_{20})^{-\frac{1}{2}} + \frac{\mu_{22}}{\mu_{20}\,\mu_{02}};$$

and the asymptotic distribution of $(R_n - \rho)(k/n)^{-1/2}$ is normal with zero mean and unit variance when the fourth moments are finite and positive and k differs from 0. If Y and Z are independent, we get $k = 1$; so that (given positive finite fourth moments of X) with respect to classes of alternatives for which (at least in the neighborhood of the null hypothesis) k and ρ differ from 0 and the fourth moments are finite, and for which the convergence to normality is uniform, we have

$$\Delta_{\mathfrak{R}}(c) = \sum_{k=1}^{K} \frac{\partial\rho}{\partial\gamma_k} c_k,$$

provided this $\neq 0$. As a test for independence against a class of alternatives which includes a case in which ρ vanishes, the $\mathfrak{R}$-test is easily seen to be inconsistent. In any case, when finite positive second moments of Y and Z exist, we note that $ER_n = 0$ if Y and Z are independent.

2. General linear transformations. Let $X_{\lambda^0} = (Y_{\lambda^0}, Z_{\lambda^0})$ be a pair of independent random variables with (nondegenerate) marginal distributions G and H. By F_λ we shall denote the joint distributions of the pair $X_\lambda = (Y_\lambda, Z_\lambda)$ of linearly transformed variables

$$Y_\lambda = \lambda_1 Y_{\lambda^0} + \lambda_2 Z_{\lambda^0}$$
$$Z_\lambda = \lambda_3 Y_{\lambda^0} + \lambda_4 Z_{\lambda^0}$$

Let Λ denote the set of those nonsingular linear transformations λ, which do not consist merely of a change of scale or permutation of Y_{λ^0} and Z_{λ^0}, or the identity transformation λ^0. Λ generates a class $\mathfrak{L}$ of transformations of distributions in $\mathfrak{F}^0 = \{GH \mid G, H \text{ nondegenerate}\}$ and a four-dimensional Euclidean space containing $\lambda^0 = (1, 0, 0, 1)$.

A transformation $\lambda \in \Lambda - \{\lambda^0\}$ nearly always makes the resultant Y_λ and Z_λ dependent:

THEOREM 2.1. *If* $\lambda \in \Lambda$, F_λ *is either normal with* $\rho = 0$,

$$\sigma^2(Z_\lambda)/\sigma^2(Y_\lambda) = -(\lambda_3/\lambda_2)/(\lambda_1/\lambda_4),$$

or a distribution of dependent random variables.

PROOF. We have to show that if F_λ is independent, it is normal. In 1948 a proof, by Loève, of a slight extension of the following proposition was published in a treatise of Lévy [10]: For a pair $U = (U_1, U_2)$ of independent random variables to be normally distributed, it suffices that there exists a nonsingular linear transformation λ such that $V_1 = \lambda_1 U_1 + \lambda_2 U_2$ and $V_2 = \lambda_3 U_1 + \lambda_4 U_2$ are independently distributed, and that $\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$. Note that the last condition is implied by the others when $\lambda \in \Lambda - \{\lambda^0\}$. The formula for the ratio of variances follows from this last condition and the vanishing of the correlation of Y_λ and Z_λ.

To arrive at the relative efficiencies, we first derive the following lemmas.

LEMMA 2.1. *Let G and H possess first moments and densities which together with their derivatives are continuous. Let*

$$I_1(\lambda) = \iint F_\lambda \, dF_\lambda,$$

$$I_2(\lambda) = \iint F_\lambda \, dF_\lambda(y, \infty) \, dF_\lambda(\infty, z) = \iint F_\lambda(y, \infty) F_\lambda(\infty, z) \, dF_\lambda(y, z),$$

$$I_3(\lambda) = F_\lambda(\mu, \nu),$$

where $F_\lambda(\mu, \infty) = F_\lambda(\infty, \nu) = \frac{1}{2}$. *(We omit* $\pm\infty$ *from the regions of integration.) Then*

$$\frac{\partial I_1(\lambda^0)}{\partial \lambda_1} = \frac{\partial I_1(\lambda^0)}{\partial \lambda_4} = 0,$$

$$\frac{\partial I_1(\lambda^0)}{\partial \lambda_2} = 2EG'(Y_{\lambda^0}) \operatorname{cov} \{Z_{\lambda^0}, H(Z_{\lambda^0})\} > 0,$$

$$\frac{\partial I_1(\lambda^0)}{\partial \lambda_3} = 2EH'(Z_{\lambda^0}) \operatorname{cov} \{Y_{\lambda^0}, G(Y_{\lambda^0})\} > 0,$$

$$\frac{\partial I_2(\lambda^0)}{\partial \lambda_1} = \frac{\partial I_2(\lambda^0)}{\partial \lambda_4} = 0, \qquad \frac{\partial I_2(\lambda^0)}{\partial \lambda_2} = \frac{1}{2}\frac{\partial I_1(\lambda^0)}{\partial \lambda_2}, \qquad \frac{\partial I_2(\lambda^0)}{\partial \lambda_3} = \frac{1}{2}\frac{\partial I_1(\lambda^0)}{\partial \lambda_3},$$

which expressions are invariant under a change of origin of Y_{λ^0} *and* Z_{λ^0}.

If, moreover, G and H are symmetric about their means and their densities do not vanish there,

$$\frac{\partial I_3(\lambda^0)}{\partial \lambda_1} = \frac{\partial I_3(\lambda^0)}{\partial \lambda_4} = 0,$$

$$\frac{\partial I_3(\lambda^0)}{\partial \lambda_2} = \tfrac{1}{2}G'(EY_{\lambda^0})E \,|\, Z_{\lambda_0} - EZ_{\lambda^0} \,| > 0,$$

$$\frac{\partial I_3(\lambda^0)}{\partial \lambda_3} = \tfrac{1}{2}H'(EZ_{\lambda^0})E \,|\, Y_{\lambda^0} - EY_{\lambda^0} \,| > 0,$$

which expressions are invariant under a change of origin of Y_{λ^0} *and* Z_{λ^0}.

PROOF. Admissibility of differentiation under the integral sign is demonstrated in the proof of Theorem 2.2. If the ranges of increase of G and H have finite bounds, the continuity of the densities implies that they vanish at the bounds, and so the derivatives of these integrals are effectively free of terms involving the values of the integrands at the ends of their ranges. Therefore, we may as well suppose doubly infinite ranges of increase.

We note at once that for $\lambda_2 = \lambda_3 = 0$, and $i = 1, 2,$ or 3, $I_i(\lambda)$ is independent of λ_1 and λ_4, so that

$$\left.\frac{\partial I_i(\lambda)}{\partial \lambda_j}\right|_{\lambda=\lambda^0} = 0 \qquad \text{for } i = 1, 2, 3; j = 1, 4,$$

$$I_1(\lambda) = (\det \lambda)^{-2} \iint \left[G'\left(\frac{\lambda_4 y - \lambda_2 z}{\det \lambda}\right) H'\left(\frac{-\lambda_3 y + \lambda_1 z}{\det \lambda}\right) \int^z \int^y G'\left(\frac{\lambda_4 \bar{y} - \lambda_2 \bar{z}}{\det \lambda}\right) H'\left(\frac{-\lambda_3 \bar{y} + \lambda_1 \bar{z}}{\det \lambda}\right) d\bar{y}\, d\bar{z} \right] dy\, dz.$$

$$\left.\frac{\partial I_1(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = -\int G''(y)G(y)\, dy \int zH'(z)H(z)\, dz - \int G'(y)G'(y)\, dy \int H'(z) \int^z \bar{z}H'(\bar{z})\, d\bar{z}\, dz.$$

Integration by parts gives

$$\int G''(y)G(y)\, dy = G(y)G'(y)\Big]_{-\infty}^{\infty} - \int G'(y)G'(y)\, dy = -EG'(Y_{\lambda^0}),$$

$$\int H'(z) \int^z \bar{z}H'(\bar{z})\, d\bar{z}\, dz = H(z) \int^z \bar{z}H'(\bar{z})\, d\bar{z}\Big]_{-\infty}^{\infty} - \int H(z)zH'(z)\, dz$$

$$= EZ_{\lambda^0} - EZ_{\lambda_0} H(Z_{\lambda^0}).$$

Consequently,

$$\left.\frac{\partial I_1(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = EG'(Y_{\lambda^0})EZ_{\lambda^0} H(Z_{\lambda^0}) - EG'(Y_{\lambda^0})\{EZ_{\lambda^0} - EZ_{\lambda^0}H(Z_{\lambda^0})\}$$

$$= -EG'(Y_{\lambda^0})EZ_{\lambda^0} + 2EG'(Y_{\lambda^0})EZ_{\lambda^0}H(Z_{\lambda^0})$$

$$= 2EG'(Y_{\lambda^0}) \operatorname{cov}\{Z_{\lambda^0}, Y(Z_{\lambda^0})\}.$$

If G_0 and H_0 are the distributions of $Y'_{\lambda^0} = Y_{\lambda^0} - EY_{\lambda^0}$ and $Z'_{\lambda^0} = Z_{\lambda^0} - EZ_{\lambda^0}$,

$$\int H'(z) \int^z \bar{z}H'(\bar{z})\, d\bar{z}\, dz = \int H'_0(z - EZ_{\lambda^0}) \int^z \bar{z}H'_0(\bar{z} - EZ_{\lambda^0})\, d\bar{z}\, dz$$

$$= \int H'_0(z) \int^z \bar{z}H'_0(\bar{z})\, d\bar{z}\, dz + \tfrac{1}{2}EZ_{\lambda^0},$$

which by nondegeneracy of H_0 is less than $\frac{1}{2}EZ_{\lambda^0}$. Since

$$EZ'_{\lambda^0}H_0(Z'_{\lambda^0}) = EZ_{\lambda^0}H(Z_{\lambda^0}) - \tfrac{1}{2}EZ_{\lambda^0}, \qquad EG'_0(Y'_{\lambda^0}) = EG'(Y_{\lambda^0}),$$

it follows that $EZ'_{\lambda^0}H_0(Z'_{\lambda^0}) > 0$ and that

$$\left.\frac{\partial I_1(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = 2EG'_0(Y'_{\lambda^0})EZ'_{\lambda^0}H_0(Z'_{\lambda^0}) = 2EG'_0(Y'_{\lambda^0}) \operatorname{cov}\{Z'_{\lambda^0}, H_0(Z'_{\lambda^0})\} > 0.$$

Similarly,

$$\left.\frac{\partial I_2(\lambda)}{\partial \lambda_3}\right|_{\lambda=\lambda^0} = 2EH'(Z_{\lambda^0}) \operatorname{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\} = 2EH'_0(Z'_{\lambda^0}) \operatorname{cov}\{Y'_{\lambda^0}, G_0(Y'_{\lambda^0})\} > 0.$$

$$I_2(\lambda) = |\det \lambda|^{-3} \iint \left[\left\{\int G'\left(\frac{\lambda_4 y - \lambda_2 z'}{\det \lambda}\right) H'\left(\frac{-\lambda_3 y + \lambda_1 z'}{\det \lambda}\right) dz' \right.\right.$$
$$\left. \cdot \int G'\left(\frac{\lambda_4 y' - \lambda_2 z}{\det \lambda}\right) H'\left(\frac{-\lambda_3 y' + \lambda_1 z}{\det \lambda}\right) dy'\right\}$$
$$\left. \cdot \int^z \int^y G'\left(\frac{\lambda_4 \bar{y} - \lambda_2 \bar{z}}{\det \lambda}\right) H'\left(\frac{-\lambda_3 \bar{y} + \lambda_1 \bar{z}}{\det \lambda}\right) d\bar{y}\, d\bar{z}\right] dy\, dz,$$

$$\left.\frac{\partial I_1(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = -\int G''(y)G(y)\, dy \int z'H'(z')\, dz' \int H'(z)H(z)\, dz$$
$$- \int G'(y)G(y)\, dy \int G''(y')\, dy' \int zH'(z)H(z)\, dz$$
$$- \int G'(h)G'(y)\, dy \int H'(z) \int^z \bar{z}H'(\bar{z})\, d\bar{z}\, dz$$
$$= -\tfrac{1}{2}EG'(Y_{\lambda^0})EZ_{\lambda^0} - EG'(Y_{\lambda^0})\{EZ_{\lambda^0} - EZ_{\lambda^0}H(Z_{\lambda^0})\}$$
$$= \frac{1}{2}\left.\frac{\partial I_1(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0};$$

and similarly,

$$\left.\frac{\partial I_2(\lambda)}{\partial \lambda_3}\right|_{\lambda=\lambda^0} = \frac{1}{2}\left.\frac{\partial I_1(\lambda)}{\partial \lambda_3}\right|_{\lambda=\lambda^0}.$$

Suppose that, moreover, G and H are symmetric about μ' and ν', respectively, and that $G'(\mu')$ and $H'(\nu')$ are different from zero.

$$I_3(\lambda) = |\det \lambda|^{-1} \int^\nu \int^\mu G'\left(\frac{\lambda_4 y - \lambda_2 z}{\det \lambda}\right) H'\left(\frac{-\lambda_3 y + \lambda_1 z}{\det \lambda}\right) dy\, dz.$$

Note that $\mu = \lambda_1\mu' + \lambda_2\nu'$, $\nu = \lambda_3\mu' + \lambda_4\nu'$, $\int^{\mu'} G'(y)\, dy = \frac{1}{2}$, $I_3(\lambda^0) = \frac{1}{4}$;

$$\left.\frac{\partial I_3(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = -\int^{\mu'} G''(y)\, dy \int^{\nu'} zH'(z)\, dz + \gamma' G'(\mu') \int^{\nu'} H'(z)\, dz$$
$$= -G'(\mu') \int^{\nu'} zH'(z)\, dz + \tfrac{1}{2}\nu' G'(\mu') = G'(\mu') \int^{\nu'} (\nu' - z)H'(z)\, dz$$
$$= \tfrac{1}{2}G'(\mu')E\,|Z_{\lambda^0} - \nu'| = \tfrac{1}{2}G'(EY_{\lambda^0})E\,|Z_{\lambda^0} - EZ_{\lambda^0}| > 0,$$

since

$$0 > \int^{\nu'} (z - \nu')H'(z)\, dz = \int^{\nu'} zH'(z)\, dz - \tfrac{1}{2}\nu'.$$

If $Y'_{\lambda^0} = Y_{\lambda^0} - \mu'$ and $Z'_{\lambda^0} = Z_{\lambda^0} - \nu'$ have the distribution G_0 and H_0, this gives

$$\left.\frac{\partial I_3(\lambda)}{\partial \lambda_2}\right|_{\lambda=\lambda^0} = -G_0'(0) \int^0 zH_0'(z)\, dz = \tfrac{1}{2}G_0'(0)E \mid Z'_{\lambda^0} \mid > 0;$$

similarly,

$$\left.\frac{\partial I_3(\lambda)}{\lambda_3}\right|_{\lambda=\lambda^0} = H'(\nu') \int^{\mu'} (\mu' - y)G'(y)\, dy = -H_0'(0) \int^0 yG_0'(y)\, dy > 0.$$

We now obtain the following general theorem (for notations, see Section 1.2):

THEOREM 2.2. *Let* Λ' *denote a set of nonsingular linear transformations of* Y_{λ^0} *and* Z_{λ^0} *which do not consist merely of a change of scale or permutation of* Y_{λ^0} *and* Z_{λ^0} *or the identity transformation* λ^0.

I(a). *Let G and H have first moments and continuously differentiable densities, or be limits of such distributions and possess densities. For sequences* $\{\lambda(n)\}$ *of elements of* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

exist and satisfy[2]

$$c_2EG'(Y_{\lambda^0}) \operatorname{cov}\{Z_{\lambda^0}, H(Z_{\lambda^0})\} + c_3EH'(Z_{\lambda^0}) \operatorname{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\} \neq 0$$

we have[2], *for* $i = 0, 1, 2,$

$$\mathfrak{I}_i, \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_i(c) = 12c_2EG'(Y_{\lambda^0}) \operatorname{cov}\{Z_{\lambda^0}, H(Z_{\lambda^0})\} + 12c_3EH'(Z_{\lambda^0}) \operatorname{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\},$$

which expression is independent of the means of Y_{λ^0} *and* Z_{λ^0}.

(b). *Let G and H be symmetric about their means, have first moments and continuously differentiable densities which do not vanish at the means, or be limits of such distributions and possess densities. For sequences* $\{\lambda(n)\}$ *of elements of*

$$\Lambda' - \{\lambda^0\}$$

converging to λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

[2] If the (continuous) G and H are defined as the pointwise limits of distributions $G^{(\epsilon)}$, $H^{(\epsilon)}$ with continuously differentiable densities, interpret the functionals of G and H in the text as limits with respect to ϵ of the corresponding functionals of $G^{(\epsilon)}$ and $H^{(\epsilon)}$.

exist and satisfy[2]

$$c_2 G'(EY_{\lambda^0})\, E \mid Z_{\lambda^0} - EZ_{\lambda^0} \mid + c_3 H'(EZ_{\lambda^0})\, E \mid Y_{\lambda^0} - EY_{\lambda^0} \mid \neq 0,$$

we have[2]

$$\mathfrak{I}_t, \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha', \alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_3(c) = 2c_2 G'(EY_{\lambda^0})\, E \mid Z_{\lambda^0} - EZ_{\lambda^0} \mid + 2c_3 H'(EZ_{\lambda^0})\, E \mid Y_{\lambda^0} - EY_{\lambda^0} \mid,$$

which expression is independent of the means of Y_{λ^0} *and* Z_{λ^0}.

(c). *Let G, H possess fourth moments and let* $\sigma(Z_{\lambda^0})/\sigma(Y_{\lambda^0}) = b$.

For sequences $\{\lambda(n)\}$ *of elements* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

exist and satisfy

$$bc_2 + \frac{1}{b} c_3 \neq 0,$$

we have

$$\mathfrak{R}, \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha', \alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_{\mathfrak{R}}(c) = bc_2 + \frac{1}{b} c_3,$$

with the latter expression depending on G and H only through b.

II(a). *Let G, H be as in* I(a), *and suppose that no sequence exists as there described. Let there exist, however, sequences* $\{\lambda(n)\}$ *of elements of* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which there is a smallest integer* $p > 1$ *with*

$$\frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda\, dF_\lambda, \qquad \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda\, dF_\lambda(y, \infty)\, dF_\lambda(\infty, z)$$

continuous near $\lambda = \lambda^0$, *with*

$$\lim_{n\to\infty} n^{1/2p}\{\lambda(n) - \lambda^0\} = c$$

existing for each $\beta > \alpha' + \alpha''$, *and with*

$$\sum_{(k_1, \cdots, k_p)} c_{k_1} \cdots c_{k_p} \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda\, dF_\lambda,$$

or

$$\sum_{(k_1, \cdots, k_p)} c_{k_1} \cdots c_{k_p} \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda\, dF_\lambda(y, \infty)\, dF_\lambda(\infty, z)$$

different from zero. Then,

$$\mathfrak{I}_1\text{, applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(p)}_{\alpha',\alpha''}\left(\frac{1}{2p}, \Lambda'\right),$$

$$\Delta_1(c) = \frac{6}{p!} \sum_{(k_1,\cdots,k_p)} c_{k_1} \cdots c_{k_p} \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda \, dF_\lambda$$

or

$$\mathfrak{I}_0 \text{ and } \mathfrak{I}_2\text{, applied as } (\alpha', \alpha'')\text{-level tests, are in } \mathcal{P}^{(p)}_{\alpha',\alpha''}\left(\frac{1}{2p}, \Lambda'\right),$$

$$\begin{aligned}\Delta_0(c) &= \Delta_2(c) \\ &= \frac{12}{p!} \sum_{(k_1\cdots k_p)} c_{k_1} \cdots c_{k_p} \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \iint F_\lambda \, dF_\lambda(y, \infty) \, dF_\lambda(\infty, z).\end{aligned}$$

(b). *Let G, H be as in* I(b), *and suppose that no sequence exists as there described. Let there exist sequences $\{\lambda(n)\}$ of elements of $\Lambda' - \{\lambda^0\}$ converging to λ^0 for which there is a smallest integer $p > 1$ with*

$$\frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} F_\lambda(\lambda_1 EY_{\lambda^0} + \lambda_2 EZ_{\lambda^0}, \lambda_3 EY_{\lambda^0} + \lambda_4 EZ_{\lambda^0})$$

continuous near $\lambda = \lambda^0$, with

$$\lim_{n\to\infty} n^{1/2p}\{\lambda(n) - \lambda^0\} = c$$

existing for each $\beta > \alpha' + \alpha''$, and with

$$\sum_{(k_1\cdots k_p)} c_{k_1} \cdots c_{k_p} \frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\gamma_{k_p}} F_\lambda(\lambda_1 EY_{\lambda^0} + \lambda_2 EZ_{\lambda^0}, \lambda_3 EY_{\lambda^0} + \lambda_4 EZ_{\lambda^0})$$

different from zero. Then

$$\mathfrak{I}_3 \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(p)}_{\alpha',\alpha''}\left(\frac{1}{2p}, \Lambda'\right),$$

$$\begin{aligned}\Delta_3(c) = \frac{4}{p!} \sum_{(k_1\cdots k_p)} c_{k_1} \cdots c_{k_p} &\frac{\partial^p}{\partial\lambda_{k_1} \cdots \partial\lambda_{k_p}} \\ &\cdot F_\lambda(\lambda_1 EY_{\lambda^0} + \lambda_2 EZ_{\lambda^0}, \lambda_3 EY_{\lambda^0} + \lambda_4 EZ_{\lambda_0}).\end{aligned}$$

(c) *Against sequences $\{\lambda(n)\}$ of elements of $\Lambda' - \{\lambda^0\}$ converging to λ^0 for which*

$$\lim_{n\to\infty} \frac{-\lambda_1(n) / \lambda_4(n)}{\lambda_2(n) / \lambda_3(n)} = b^2,$$

$\mathcal{R}$ is not consistent.

PROOF. First we prove (a) and (b), letting G and H satisfy the conditions of Lemma 2.1. Let $R(y_0, z_0) = \{(y, z): y \leqq y_0, z \leqq z_0\}$, and

$$J = |\det \lambda|^{-1} \iint_{R(y_0, z_0)} G''\left(\frac{\lambda_4 y - \lambda_2 z}{\det \lambda}\right) H'\left(\frac{-\lambda_3 y + \lambda_1 z}{\det \lambda}\right) dy\, dz$$
$$= \iint_{R_\lambda(y_0, z_0)} G''(y) H'(z)\, dy\, dz,$$

where $R_\lambda(y_0, z_0)$ is the corresponding continuous transform of $R(y_0, z_0)$. Since $\int^z \int^y G''(y)H'(z)\, dy\, dz$ exists for all y, z, including ∞, there exist $y'' < y'$, $z'' < z'$ such that

$$\iint_{R(y_0, z'') \cup R(y'', z_0)} G''(y)H'(z)\, dy\, dz$$

is arbitrarily small. Let $\bar{R}(y, z) = R(y, z) - R(y, z'') \cup R(y'', z)$, $\bar{R}_\lambda(y, z) = R_\lambda(y, z) - R(y, z'') \cup R(y'', z)$. Since, moreover, for λ close to λ^0 there exist $y' \geqq y_0$, $z' \geqq z_0$ close to y_0 and z_0 such that

$$\bar{R}_\lambda(y_0, z_0) \subset R(y', z'),$$

convergence of J at infinity is uniform in λ near λ^0. Since the integrand is also continuous uniformly in λ over $R(y', z') - \bar{R}(y'', z'')$, the integral over

$$R(y', z') - \bar{R}_\lambda(y_0, z_0)$$

can be made arbitrarily small, and so also the absolute continuity of J is uniform in λ near λ^0. For the other integrals arising in the partial derivatives of I_1, I_2, and I_3, we also get these uniform properties, so that the generalized Lebesgue convergence theorem is applicable, and the partial derivatives are continuous functions of λ near λ^0 and can be obtained by differentiation under the integral sign. The continuity in λ of I_1, I_2, and I_3, and of $E\Phi_{11}^2(X_\lambda)$ and $E\Phi_{21}^2(X_\lambda)$ in which occur expressions such as

$$\iint F_\lambda(y, \infty) F_\lambda(\infty, z) \int F_\lambda(\bar{y}, z)\, dF_\lambda(\bar{y}, \infty)\, dF_\lambda(y, z),$$

follows likewise. The continuity of $E\Phi_{11}^2(x_\lambda)$ and $E\Phi_{11}^2(x_\lambda)$ implies (utilizing the results quoted in Section 1.3) that

$$\sigma_{1n}^2(\lambda) \sim \{E\Phi_{11}^{(\lambda)^2}(X_\lambda) - \tau_1^2(\lambda)\} \sigma_{1n}^2(\lambda^0)$$

and

$$\sigma_{2n}^2(\lambda) \sim \{E\Phi_{21}^{(\lambda)^2}(X_\lambda) - \tau_2^2(\lambda)\} \sigma_{2n}^2(\lambda^0)$$

are positive in the neighborhood of $\lambda = \lambda^0$, so that the asymptotic distribution of $(T_{in} - \tau_i)/\sigma_{in}$ is normal with zero mean and unit variance in that neighborhood for $i = 0, 1, 2$. For $i = 3$, this follows at once from the easily verified fact that $F_\lambda(\mu, \nu)$ is different from both 0 and $\frac{1}{2}$ when λ is near λ^0.

The approach to normality of T_{in} is uniform. For $i = 1, 2$ (and so for $i = 0$) this follows from the method of proof of the approach to normality, which uses the central limit theorem for the identically distributed random variables $\Phi_{i1}^{(\lambda)}(X_{\lambda,\alpha}) - \tau_i(\lambda)(\alpha = 1, \cdots, n)$, and the fact that the asymptotic distributions of two random variables $V_\lambda = n^{-1/2}(i + 1)\sum_\alpha \Phi_{i1}^{(\lambda)}(X_{\lambda,\alpha})$ and

$$V'_\lambda = \sqrt{n}\,\{T_{in}(X_\lambda) - \tau_i(\lambda)\}$$

are the same when the expectation of the squared difference converges to zero, by noting the above continuity property. In fact, $E(V_\lambda - V'_\lambda)^2$ converges to zero uniformly and EV'^2_λ is continuous in λ. For $i = 3$, the continuity in λ of F_λ and its first partial and cross derivatives at (μ, ν) is also sufficient for the uniform approach to normality. This is so because the asymptotic normality proof involves the Glivenko-Cantelli theorem, which holds uniformly when the distribution functions are continuous uniformly with respect to the parameter (see [13]), and an argument analogous to that of the de Moivre-Laplace theorem for binomial random variables with constants equal to the first partial and cross derivatives of F_λ at (μ, ν).

It remains to consider continuous distribution functions $G = G^{(0)}$, $H = H^{(0)}$ which are limits of distribution functions $G^{(\epsilon)}$, $H^{(\epsilon)}$, as mentioned in the theorem. Since by Pólya's theorem the convergence is uniform, given any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ there is a sphere of $F_\lambda^{(\epsilon)}$-measure bigger than $1 - \eta$ with the property that on its complement C we have (if $\bar{F}^{(\epsilon)} = F^{(\epsilon)} - F^{(0)}$)

$$\int_C\int |\bar{F}_\lambda^{(\epsilon)}(y, z)|\, dF_\lambda^{(\epsilon)}(y, z) \leqq \int_C\int dF_\lambda^{(\epsilon)}(y, z),$$

$$\int_C\int |\bar{F}_\lambda^{(\epsilon)}(y, \infty)| F_\lambda^{(\epsilon)}(\infty, z)\, dF_\lambda^{(\epsilon)}(y, z) \leqq \iint_C dF_\lambda^{(\epsilon)}(y, z),$$

which is less than η, giving the convergence of $I_{i\epsilon}(\lambda)$ to $I_{i0}(\lambda)$ for $i = 1, 2$ (defined in an obvious way) by the Helly-Bray theorem; we easily obtain it for $i = 3$ as well. Since $I_{i\epsilon}(\lambda)$ and $I_{i0}(\lambda)$ are continuous in λ, this convergence holds uniformly in a neighborhood of λ^0 by a slight extension of the reasoning in [7]. Similarly we get uniform convergence of $EV_{\lambda\epsilon}^2$ and $EV'^2_{\lambda\epsilon}$ to $EV_{\lambda 0}^2$ and $EV'^2_{\lambda 0}$ and of $E(V_{\lambda\epsilon} - V'_{\lambda\epsilon})^2$ to $E(V_{\lambda 0} - V'_{\lambda 0})^2$, so that the limits are continuous in λ near λ^0. Let $\lambda^{(k)}$ coincide with λ^0 except at the kth coordinate, where it equals λ_k. By the uniformity in λ of the convergence of $I_{i\epsilon}(\lambda)$ as $\epsilon \to 0$, we have, for λ near λ^0,

$$\begin{aligned}\lim_{\epsilon\to 0} \partial I_{i\epsilon}(\lambda^0)/\partial\lambda_k &= \lim_{\epsilon\to 0}\lim_{\lambda_k\to\lambda_k^0} [I_{i\epsilon}(\lambda^{(k)}) - I_{i\epsilon}(\lambda^0)]/(\lambda_k - \lambda_k^0)\\ &= \lim_{\lambda_k\to\lambda_k^0}\lim_{\epsilon\to 0} [I_{i\epsilon}(\lambda^{(k)}) - I_{i\epsilon}(\lambda^0)]/(\lambda_k - \lambda_k^0)\\ &= \lim_{\lambda_k\to\lambda_k^0} [I_{i0}(\lambda^{(k)}) - I_{i0}(\lambda^0)]/(\lambda_k - \lambda_k^0) = \partial I_{i0}(\lambda^0)/\partial\lambda_k .\end{aligned}$$

Finally we have to show that the partial derivative $I_{ik0}(\lambda)$ is continuous near λ^0 for λ approaching λ^0 along any ray: Letting

$$\lambda(t) = \lambda^0 + t(\lambda' - \lambda^0), \qquad \lambda^{(k)}(t) = \lambda^{(k)} + t(\lambda' - \lambda^{(k)}),$$

$$\partial I_{i0}(\lambda^0)/\partial\lambda_k = \lim_{\epsilon\to 0} \partial I_{i\epsilon}(\lambda^0)/\partial\lambda_k = \lim_{\epsilon\to 0} \lim_{t\to 0} \lim_{\lambda_k\to\lambda_k'} A_t(\epsilon)$$

$$= \lim_{t\to 0} \lim_{\lambda_k\to\lambda_k'} \lim_{\epsilon\to 0} A_t(\epsilon) = \lim_{t\to 0} \lim_{\lambda_k\to\lambda_k'} A_t(0)$$

$$= \lim_{t\to 0} \partial I_{i0}(\lambda(t))/\partial\lambda_k,$$

where

$$A_t(\epsilon) = [I_{i\epsilon}(\lambda^{(k)}(t)) - I_{i\epsilon}(\lambda(t))]/(\lambda_k - \lambda_k').$$

Now we prove (c). It is easy to see that if the fourth moments of Y_{λ^0} and Z_{λ^0} are finite, and if

$$-\frac{\lambda_1(n)/\lambda_4(n)}{\lambda_2(n)/\lambda_3(n)} \neq \frac{\sigma^2(Z)}{\sigma^2(Y)}$$

for all sufficiently large n, the parametric test $\mathcal{R}$, involving the ordinary correlation coefficient R_n, can be compared with the above tests. It has (see last part of Section 1.3)

$$\Delta_{\mathcal{R}}(c) = \frac{\sigma(Z_{\lambda^0})}{\sigma(Y_{\lambda^0})} c_2 + \frac{\sigma(Y_{\lambda^0})}{\sigma(Z_{\lambda^0})} c_3,$$

whatever be G and H. In fact, one finds $k \sim 1$ near $\lambda = \lambda^0$, and on the subset S_ϵ of the sample space on which the sample moments involved in the definition of R_n differ from the corresponding population moments by less than $\epsilon > 0$, Tchebycheff inequalities on $P_\lambda(S_\epsilon)$ are satisfied uniformly when the fourth moments of Y_{λ^0} and Z_{λ^0} are finite, so that (compare [2], p. 366) we can approximate uniformly $n^{1/2}(R_n - \rho)$ by a linear function of the differences. Moreover, by the continuity in λ of $EY_\lambda^2 Z_\lambda^2$, the distribution of these differences is seen to be uniformly asymptotically normal. This concludes the proof of the theorem.

Since the efficiency of $\mathfrak{J}_3$ depends so strongly on local properties of the density, we must conclude that this test is not to be recommended generally.

In the case of most interest where, except for a change of scale, and/or origin, Y_{λ^0} and Z_{λ^0} have the same distribution, the condition on the sequences $\{\lambda(n)\}$ does not explicitly depend on the form of this distribution. It is therefore worthwhile to particularize our result to that case.

THEOREM 2.3. *Let Λ' denote a set of nonsingular linear transformations of Y_{λ^0} and Z_{λ^0} which do not consist merely of a change of scale or permutation of Y_{λ^0} and Z_{λ^0} or of the identity transformation λ^0. Let there exist a and b such that*

$$P\{Z \leqq t\} = G\{(t - a)/b\}.$$

I(a). *Let G have a first moment and a continuously differentiable density or be the limit of a sequence of such distributions and possess a density.*

For sequences $\{\lambda(n)\}$ *of elements of* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

exist and satisfy[3]

$$bc_2 + \frac{1}{b} c_3 \neq 0,$$

we have,[3] for $i = 0, 1, 2$

$$\mathfrak{I}_i, \text{ applied as an } (\alpha', \alpha'')\text{-level test is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_i(c) = 12\left(bc_2 + \frac{1}{b} c_3\right) EG'(Y_{\lambda^0}) \operatorname{cov} \{Y_{\lambda^0}, G(Y_{\lambda^0})\},$$

which is independent of the mean of Y_{λ^0}.

(b). *Let G be symmetric about its mean, have a first moment and a continuously differentiable density which does not vanish at the mean, or be the limit of such distributions and possess a density.*

For sequences $\{\lambda(n)\}$ *of elements of* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

exist and satisfy[3]

$$bc_2 + \frac{1}{b} c_3 \neq 0,$$

we have[3]

$$\mathfrak{I}_3, \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_3(c) = 2\left(bc_2 + \frac{1}{b} c_3\right) G'(EY_{\lambda^0}) E|Y_{\lambda^0} - EY_{\lambda^0}|,$$

which is independent of the mean of Y_{λ^0}.

(c) *Let G possess a fourth moment and let* $\sigma(Z_{\lambda^0})/\sigma(Y_{\lambda^0}) = b$.

For sequences $\{\lambda(n)\}$ *of elements of* $\Lambda' - \{\lambda^0\}$ *converging to* λ^0 *for which for each* $\beta > \alpha' + \alpha''$ *the numbers*

$$\lim_{n\to\infty} \sqrt{n}\, \lambda_2(n) = c_2, \qquad \lim_{n\to\infty} \sqrt{n}\, \lambda_3(n) = c_3$$

---

[3] If the (continuous) G is defined as the pointwise limit of distributions $G^{(\epsilon)}$ with a continuously differentiable density, interpret the functionals of G in the text as limits with respect to ϵ of the corresponding functionals of $G^{(\epsilon)}$.

exist and satisfy

$$bc_2 + \frac{1}{b} c_3 \neq 0,$$

we have

$$\mathcal{R}, \text{ applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha', \alpha''}(\tfrac{1}{2}, \Lambda'),$$

$$\Delta_R(c) = bc_2 + \frac{1}{b} c_3,$$

whatever be G.

II. *For sequences* $\{\lambda(n)\}$ *for which*

$$\lim_{n \to \infty} \frac{-\lambda_1(n)/\lambda_4(n)}{\lambda_2(n)/\lambda_3(n)} = b^2,$$

such as occurs in sequences of rotations (discussed below), there may still exist $p > 1$ *for which* $\Delta_i(c)$ *is well defined for* $i = 0, 1, 2,$ or 3 *following Theorem* 2.2II. (*See Lemmas* 2.2, 2.3, *and* 2.4, *below, which also deal with other cases where sequences as in* I *do not exist.*) *For such sequences* R *is not consistent; and for* G *possessing a symmetric density and* $b = 1$, *the tests* $\mathfrak{J}_i$ $(i = 0, 1, 2, 3)$ *are not consistent.*

PROOF. We only need to prove the absence of consistency of the $\mathfrak{J}_i$, and therefore the vanishing of the τ_i, under the conditions mentioned. For this purpose the location of the center of symmetry is immaterial, and we shall suppose it and a to be zero. Let f be the joint density of Y_λ and Z_λ, and for simplicity suppose that

$$\lambda_1(n) = \lambda_4(n) = \lambda_1, \qquad \lambda_2(n) = -\lambda_3(n) = \lambda_2$$

exactly. Then, since $G'(t) = G'(-t)$,

$$f(y, z) = (\lambda_1^2 + \lambda_2^2)^{-1} G'\left(\frac{\lambda_1 y - \lambda_2 z}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}\right) G'\left(\frac{\lambda_2 y + \lambda_1 z}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}\right)$$

$$= (\lambda_1^2 + \lambda_2^2)^{-1} G'\left(\frac{y_2 z - \lambda_1 y}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}\right) G'\left(\frac{\lambda_1 z + \lambda_2 y}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}\right) = f(z, -y),$$

which identity is easily seen to imply $\tau_i = 0$ for $i = 0, 1, 2, 3$. (In the case of rotations one can prove this result even for the case where no densities exist, using Theorem 4.1 of [9].)

REMARK. It may be noted that under the assumptions made on G in the preceding theorem under I(b),

$$0 < \text{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\} < \tfrac{1}{2} E \mid Y_{\lambda^0} - EY_{\lambda^0} \mid < \tfrac{1}{2}\sigma(Y_{\lambda^0}),$$

while, of course,

$$\text{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\} \leqq \tfrac{1}{2}\sigma(Y_{\lambda^0})/\sqrt{3}$$

($\sigma(Y_{\lambda^0})$ need not be finite).

EXAMPLES. The following are some numerical examples of the application of

I of Theorem 2.3. Here $bc_2 + (1/b)c_3 \neq 0$ and $\Delta' = \Delta(c)[bc_2 + (1/b)c_3]^{-1}$. Thus the asymptotic power of any of these tests against λ' (near λ^0) is

$$1 - (2\pi)^{-1/2} \int_{\delta' - \Delta(\lambda' - \lambda^0)n^{1/2}}^{\delta'' - \Delta(\lambda' - \lambda^0)n^{1/2}} e^{-1/2x^2}\, dx,$$

with the Δ corresponding to that test. The relative efficiency of two tests is found by squaring the quotient of the entries in the corresponding columns:

G	Δ_0', Δ_1', Δ_2'	Δ_3'	Δ_R'
normal	$3/\pi$	$2/\pi$	1
uniform	1	1/2	1
parabolic	162/175	9/16	1
Laplace	36/32	1	1

The above theorem depends on development in a Taylor series expansion up to the first term of $I_i(\lambda)$ about λ^0 for $i = 1, 2, 3$. In case G and H are the same distribution except for a scale or location factor, but $bc_2 + (1/b)c_3 = 0$, this term vanishes, and we have to obtain second- or higher-order terms. This is done in the following three lemmas, the calculations for which are interesting but exceedingly laborious. They were carried out by Mr. Arnold Kaplan under the author's general direction; his contribution is here gratefully acknowledged. In the derivations, conditions allowing differentiations under the sign of integration were freely assumed, and in Lemma 2.2 (2.3, 2.4) the existence and continuity of the second (third, fourth) derivative of the density of Y_{λ^0} and the existence of its moments up to the same order was assumed (although this may not be necessary).

LEMMA 2.2. *Suppose there exist numbers $b > 0$ and a such that*

$$H(t) = G\left(\frac{t-a}{b}\right), \qquad b\lambda_2 + \frac{1}{b}\lambda_3 = 0.$$

For $i = 0, 1, 2, 3$, let Δ_i' equal the coefficient of $(bc_2 + \frac{1}{b}c_3)$ in the expression for $\Delta_i(c)$ in Theorem 2.3I, and let $1/q = bc_2(c_4 - c_1)$. With the notations and conditions of the previous lemma and further conditions on G implied by the remarks immediately above, we obtain, neglecting third-order terms,

$$I_1(\lambda) - I_1(\lambda^0) = 2b\,\lambda_2(\lambda_4 - \lambda_1)EG'(Y_{\lambda^0})\operatorname{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\},$$

$$I_2(\lambda) - I_2(\lambda^0) = b\,\lambda_2(\lambda_4 - \lambda_1)EG'(Y_{\lambda^0})\operatorname{cov}\{Y_{\lambda^0}, G(Y_{\lambda^0})\} - b^2\lambda_2^2[\operatorname{cov}\{Y_{\lambda^0}, G'(Y_{\lambda^0})\}]^2,$$

$$I_3(\lambda) - I_3(\lambda^0) = \tfrac{1}{2}b\lambda_2(\lambda_4 - \lambda_1)G'(EY_{\lambda^0})E|\,Y_{\lambda^0} - EY_{\lambda^0}\,|,$$

$$\rho(\lambda) - \rho(\lambda^0) = b\lambda_2(\lambda_4 - \lambda_1),$$

so that $q\Delta_1(c)$, $q\Delta_3(c)$, and $q\Delta_R(c)$ are the same as Δ_1', Δ_3' and Δ_R' respectively, while $q\Delta_2(c) = q\Delta_0(c)$ is generally different from $\Delta_2' = \Delta_0'$ if G is asymmetric.

If the distribution of Y_{λ^0} is symmetric about its mean, the covariance Y_{λ^0} and

$G'(Y_{\lambda^0})$ vanishes, and the development of $I_1(\lambda)$ and of $2I_2(\lambda)$ coincide. If $\lambda_4 = \lambda_1$, the first and third expressions vanish, and $\rho(\lambda) \equiv 0$. Moreover, if the distribution of Y_{λ^0} is symmetric about its mean, the second expression vanishes as well. We therefore investigate third-order terms in

LEMMA 2.3. *With the notations and conditions of the previous lemma, we obtain, neglecting fourth-order terms, when* $\lambda_1 = \lambda_4$:

$$I_1(\lambda) - I_1(\lambda^0) = -8b^2\lambda_2^2(\lambda_1 - 1)\{EY_{\lambda^0}G'(Y_{\lambda^0})\}\left\{EY_{\lambda^0}G'(Y_{\lambda^0}) + \frac{a}{b}EG'(Y_{\lambda^0})\right\},$$
$$I_2(\lambda) - I_2(\lambda^0) = b^2\lambda_2^2(2\lambda_1 - 3)[\operatorname{cov}\{Y_{\lambda^0}, G'(Y_{\lambda^0})\}]^2,$$
$$I_3(\lambda) - I_3(\lambda^0) = \tfrac{1}{2}b^2\lambda_2^2(\lambda_1 - 1).$$

If the distribution of Y_{λ^0} is symmetric about zero, the first two expressions vanish. Further development shows, however, that (with $\lambda_1 \neq 1$) $I_1(\lambda) - I_1(\lambda^0)$ does not vanish identically in that case:

LEMMA 2.4. *With the notations and conditions of the previous lemma, and symmetry of the distribution of* Y_{λ^0} *about zero, we have, neglecting fifth-order terms,*

$$I_1(\lambda) - I_1(\lambda^0) = -\frac{a}{b}\lambda_2^3(\lambda_1 - 1)\int\{G''(y)\}^2\,dy\{3b^2EY_{\lambda^0}^2 + a^2\} + b^2\lambda_2^2(\lambda_1 - 1)^2,$$

where the first expression on the right-hand side vanishes for $a = 0$.

3. Rotations. As a special case of linear transformations, we consider the class of rotations. For $|\theta| \leqq \pi/4$, write F_θ for the distribution obtained from F_0 by a rotation

$$\lambda_1 = \lambda_4 = \cos\theta, \qquad \lambda_2 = -\lambda_3 = \sin\theta.$$

An immediate application of Theorem 2.2 yields:

THEOREM 3.1. *Let* G, H *have first moments and continuously differentiable densities, or be limits of such distributions and possess densities; and consider null sequences* $\{\theta(n)\}$ *of nonzero angles of rotation for which for each* $\beta > \alpha' + \alpha''$

$$\lim_{n\to\infty}\sqrt{n}\sin\theta(n) = c$$

exists and differs from 0.

(a). *If*[2]

$$EG'(Y_0)\operatorname{cov}\{Z_0, H(Z_0)\} \neq EH'(Z_0)\operatorname{cov}\{Y_0, G(Y_0)\},$$

then,[2] *for* $i = 0, 1, 2,$

$$\mathfrak{I}_i, \textit{ applied as an } (\alpha', \alpha'')\textit{-level test, is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Theta),$$
$$\Delta_i(c) = 12c[EG'(Y_0)\operatorname{cov}\{Z_0, H(Z_0)\} - EH'(Z_0)\operatorname{cov}\{Y_0, G(Y_0)\}].$$

(b). *Let* G *and* H *also have nonvanishing densities at the means and be symmetric about the means.*

If[2]

$$G'(EY_0)E\,|Z_0 - EZ_0| \neq H'(EZ_0)E\,|Y_0 - EY_0|,$$

then[2]

$$\mathfrak{I}_3\text{, applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Theta),$$

$$\Delta_3(c) = 2c[G'(EY_0)E \mid Z_0 - EZ_0 \mid - H'(EZ_0)E \mid Y_0 - EY_0 \mid].$$

(c). *If* $EY_0^4 < \infty$ *and* $EZ_0^4 < \infty$ *and* $\sigma(Z_0)/\sigma(Y_0) = b \neq 1$, *but otherwise* G *and* H *are arbitrary,*

$$\mathfrak{R}\text{, applied as an } (\alpha', \alpha'')\text{-level test, is in } \mathcal{P}^{(1)}_{\alpha',\alpha''}(\tfrac{1}{2}, \Theta),$$

$$\Delta_R(c) = c\left\{ b - \frac{1}{b} \right\}.$$

If (a) $$EG'(Y_0) \operatorname{cov}\{Z_0, H(Z_0)\} = EH'(Z_0) \operatorname{cov}\{Y_0, G(Y_0)\}$$

or (b) $$G'(EY_0)E \mid Z_0 - EZ_0 \mid = H'(EZ_0)E \mid Y_0 - EY_0 \mid$$

(due, for example, to $G = H$), one applies II of Theorems 2.2 or 2.3. In general this proves to be laborious. For sufficiently smooth distribution functions one may expect to find p for which $\Delta_i(c) \neq 0$, since, generally, F_θ depends on θ. However, the remark below Lemma 2.2 implies that $\mathfrak{R}$ is not consistent when $b = 1$.

Further properties of the rotations have been studied in [9].

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