A NOTE ON "SOME FURTHER RESULTS IN SIMULTANEOUS CONFIDENCE INTERVAL ESTIMATION"

By S. N. Roy

Institute of Statistics, University of North Carolina

0. Summary. This note gives an explicit proof of a lemma in matrix theory repeatedly used in [2]; the lemma follows easily from other results, but an explicit proof of it may not be trivial. This note also gives closer confidence bounds for one of the several problems discussed in [2].

1. A matrix lemma. A lemma repeatedly used in [2] is the following:

$$c_{\min}(AB^{-1})c_{\min}(BC) \leqq \text{all } c(AC) \leqq c_{\max}(AB^{-1})c_{\max}(BC), \tag{1.1}$$

where A, C, B (and hence B^{-1}) are symmetric positive definite matrices of order, say p, each. This follows easily from

$$c_{\min}(M_1)c_{\min}(M_2) \leqq \text{all } c(M_1M_2) \leqq c_{\max}(M_1)c_{\max}(M_2), \tag{1.2}$$

where M_1 and M_2 are symmetric positive definite matrices.

(a) If $M(p \times p)$ is a symmetric positive definite matrix, then there exists a nonsingular triangular matrix $\tilde{T}$ such that $M = \tilde{T}\tilde{T}'$.

(b) Any nonzero characteristic root of $A(p \times q) \times B(q \times p)$ is a characteristic root of $B(q \times p) \times A(p \times q)$, and vice versa.

(c) If $M(p \times p)$ is symmetric positive definite and $Q(p \times p)$ is any nonsingular matrix, then QMQ' is symmetric positive definite.

(1.2) is proved in [1], (a), (b), and (c) are well-known matrix theorems, and (b) and (c) are also proved in [3].

Turning now to the proof of (1.1) and using (1.2) and (a), (b), and (c), we put $B = \tilde{T}\tilde{T}'$ and note that

$$\begin{aligned} c_{\max}(AB^{-1})c_{\max}(BC) &= c_{\max}(A\tilde{T}'^{-1}\tilde{T}^{-1})c_{\max}(\tilde{T}\tilde{T}'C) \\ &= c_{\max}(\tilde{T}^{-1}A\tilde{T}'^{-1})c_{\max}(\tilde{T}'C\tilde{T}) \geqq c_{\max}(\tilde{T}^{-1}AC\tilde{T}), \end{aligned} \tag{1.3}$$

i.e., $\geqq c_{\max}(AC)$. The other side of the inequality in (1.1) follows in a similar fashion, and this completes the proof of (1.1).

2. Closer bounds on the $c(\Sigma_1\Sigma_2^{-1})$'s than those given in [2]. If S_1 and S_2 stand for the dispersion matrices of random samples of sizes n_1 and n_2 from $N(\xi_i, \Sigma_i)$ (with $i = 1, 2$), the constants $c_{1\alpha}(p, n_1 - 1, n_2 - 1) = c_{1\alpha}$ and

$$c_{2\alpha}(p, n_1 - 1, n_2 - 1) = c_{2\alpha},$$

say, are defined in [2] such that

$$P[c_{1\alpha} \leqq \text{all } c(S_1S_2^{-1}) \leqq c_{2\alpha} \mid \Sigma_1 = \Sigma_2] = 1 - \alpha. \tag{2.1}$$

It is well known, [2] and [3], that if $\Sigma_1 \neq \Sigma_2$ and if $\gamma_1 \leqq \gamma_2 \cdots \leqq \gamma_p$ stand for $c(\Sigma_1\Sigma_2^{-1})$ and D_γ for a diagonal matrix whose diagonal elements are $\gamma_1, \cdots, \gamma_p$,

Received July 19, 1955.

then the $c(D_{1/\sqrt{\gamma}}\, S_1 D_{1/\sqrt{\gamma}} S_2^{-1})$'s have the same joint distribution as that of the $c(S_1 S_2^{-1})$'s under the null hypothesis: $\Sigma_1 = \Sigma_2$. Thus, we have

$$(2.2)\quad P[c_{1\alpha} \leqq \text{all } c(D_{1/\sqrt{\gamma}} S_1 D_{1/\sqrt{\gamma}} S_2^{-1}) \leqq c_{2\alpha} \mid \Sigma_1 \neq \Sigma_2] = 1 - \alpha.$$

The statement under the probability symbol is equivalent to

$$(2.3)\qquad \frac{1}{c_{1\alpha}} \geqq \text{all } c(S_2 D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}) \geqq \frac{1}{c_{2\alpha}}.$$

Now, noting that S_1, S_2 (and hence S_2^{-1}), and $D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}$ are symmetric positive matrices and using (1.1), we have

$$(2.4)\qquad c_{\max}(S_1 S_2^{-1}) c_{\max}(S_2 D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}) \leqq c_{\max}(S_1 D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}).$$

Now, putting $S_1 = \tilde{U}\tilde{U}'$ and remembering [1] that if $A(p \times p)$ is a matrix with real roots, then

$$(2.5)\qquad c_{\min}(AA') \leqq c^2(A) \leqq c_{\max}(AA'),$$

we have

$$(2.6)\quad c_{\max}(\tilde{U}\tilde{U}' D_{\sqrt{\gamma}} \tilde{U}'^{-1} \tilde{U}^{-1} D_{\sqrt{\gamma}}) = c_{\max}[(\tilde{U}' D_{\sqrt{\gamma}} \tilde{U}'^{-1})(\tilde{U}' D_{\sqrt{\gamma}} \tilde{U}'^{-1})'] \geqq c_{\max}^2(\tilde{U}' D_{\sqrt{\gamma}} \tilde{U}'^{-1}),$$

i.e., $\geqq c_{\max}^2(D_{\sqrt{\gamma}})$; $\geqq \gamma_p$.

Combining (2.4) and (2.6), we have

$$(2.7)\qquad c_{\max}(S_2 D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}) \geqq \gamma_p / c_{\max}(S_1 S_2^{-1}),$$

and, in a similar fashion, we also have

$$(2.8)\qquad c_{\min}(S_2 D_{\sqrt{\gamma}} S_1^{-1} D_{\sqrt{\gamma}}) \leqq \gamma_1 / c_{\min}(S_1 S_2^{-1}).$$

Thus, it is easy to check that (2.3) implies

$$(2.9)\qquad \frac{c_{\max}(S_1 S_2^{-1})}{c_{1\alpha}} \geqq \text{all } c(\Sigma_1 \Sigma_2^{-1}) \geqq \frac{c_{\min}(S_1 S_2^{-1})}{c_{2\alpha}},$$

which is therefore a confidence statement with a probability greater than or equal to $1 - \alpha$.

Now, as to the closeness of these bounds compared to those of [2], we note the following: Using (1.2), we have

$$(2.10)\qquad c_{\max}(S_1 S_2^{-1}) \leqq c_{\max}(S_1)\, c_{\max}(S_2^{-1}),$$

i.e., $\leqq c_{\max}(S_1)/c_{\min}(S_2)$, and

$$c_{\min}(S_1 S_2^{-1}) \geqq c_{\min}(S_1)/c_{\max}(S_2).$$

Thus, (2.9) implies

$$(2.11)\qquad \frac{1}{c_{1\alpha}} \cdot \frac{c_{\max}(S_1)}{c_{\min}(S_2)} \geqq \text{all } c(\Sigma_1 \Sigma_2^{-1}) \geqq \frac{1}{c_{2\alpha}} \cdot \frac{c_{\min}(S_1)}{c_{\max}(S_2)},$$

which is therefore a confidence statement with a confidence coefficient greater than or equal to the confidence coefficient of (2.9). Thus, if (2.3) has a probability $1 - \alpha$, (2.9) has a probability $1 - \beta \geqq 1 - \alpha$, and if (2.9) has a probability $1 - \beta$, then (2.11) has a probability $1 - \gamma \geqq 1 - \beta$. The bounds in (2.11) are the ones obtained in [2] in a different way.

REFERENCES

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A NOTE ON THE NORMAL DISTRIBUTION

By Seymour Geisser[1]

National Bureau of Standards

1. It is well known that a necessary and sufficient condition for the independence of the sample mean and variance is that the parent population be normal. This was first shown by R. C. Geary [2], and later Lukacs [3] gave a somewhat simpler proof using characteristics functions.

By using the method of Lukacs one can derive a similar theorem concerning the sample mean and the mean square successive difference.

2. Let $x_1, \cdots, x_n$ be independent and identically distributed with density $f(x)$ and mean μ and variance σ^2.

Let

$$\bar{x} = n^{-1} \sum_{j=1}^{n} x_j,$$

$$\delta_k^2 = 2^{-1}(n-k)^{-1} \sum_{j=1}^{n-k} (x_{j+k} - x_j)^2 \qquad k = 1, 2, \cdots, n-1.$$

The following theorem can be proved:

Theorem: *A necessary and sufficient condition that $f(x)$ be the normal density is that δ_k^2 and $\bar{x}$ be independent.*

Proof: If δ_k^2 and $\bar{x}$ are independent, then we follow Lukacs [3] step for step, replacing

$$s^2 = n^{-2}[(n-1) \sum x_\alpha^2 - 2 \sum \sum x_\alpha x_{\beta+1}]$$

Received on June 14, 1955.

[1] Now at the National Institute of Mental Health.