respectively the probability to the left of the origin, the relative distribution on the negative axis, and the relative distribution on the positive axis. Also, for any possible values for the parameters p, $F^{-}(x)$, $F^{+}(x)$ (F^{-}, F^{+} continuous), there corresponds a continuous distribution $F_\theta(x)$.

The hypothesis testing problem (9) can be written equivalently

$$(10) \qquad \begin{aligned} &\text{Hypothesis:} \quad p_\theta = \beta, \qquad \theta \,\varepsilon\, \Omega, \\ &\text{Alternative:} \quad p_\theta < \beta, \qquad \theta \,\varepsilon\, \Omega, \end{aligned}$$

and it is obvious that this is a problem for which a sufficient statistic (p_θ) is appropriate. Let $i(x_1, \cdots, x_n)$ be the number of positive x_i. Obviously the distribution of $i(X_1, \cdots, X_n)$ depends only on p_θ, and the conditional distribution of $(X_1, \cdots, X_n)$ given $i(X_1, \cdots, X_n) = r$ depends only on F_θ^{-}, F_θ^{+}. Hence, $i(x_1, \cdots, x_n)$ is sufficient (p_θ). For the binomial problem of testing $p_\theta = \beta$ against $p_\theta < \beta$, the sign test to reject for large values of $i(x_1, \cdots, x_n)$ is uniformly most powerful. Then by Theorem 2 it is the uniformly most powerful test for the nonparametric location problem.

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A NOTE ON THE BALANCED INCOMPLETE BLOCK DESIGNS

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0. Summary. It is a well-known property of the BIB design that all treatment effects are estimated with the same accuracy, i.e., that the variances of the estimates of the treatment effects are all equal and their covariances are also all equal. We show that the converse is also true. If the estimates of the treatment effects in an incomplete block design all have the same variances and the same covariances, then the design is a BIB.

1. A matrix result. If

$$C = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a \end{pmatrix}$$

is a $v \times v$ matrix, then C has characteristic roots $a + (v - 1)b$ and $a - b$, the latter of multiplicity $v - 1$. We need a second result which is a partial converse:

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LEMMA. *If C is a $v \times v$ matrix such that the sum of all elements in any given row is zero and if $(e \neq 0)$ is a characteristic root of C with multiplicity $(v - 1)$, then*

$$C = \frac{e}{v}\begin{pmatrix} v-1 & -1 & \cdots & -1 \\ -1 & v-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & & v-1 \end{pmatrix}.$$

PROOF.

$$m_i = \begin{pmatrix} m_{i1} \\ m_{i2} \\ \vdots \\ m_{iv} \end{pmatrix}$$

is called a characteristic vector of C corresponding to the characteristic root e_i if

$$(C - e_i I)\begin{pmatrix} m_{i1} \\ m_{i2} \\ \vdots \\ m_{iv} \end{pmatrix} = 0.$$

From

$$C \cdot \begin{pmatrix} v^{-\frac{1}{2}} \\ v^{-\frac{1}{2}} \\ \vdots \\ v^{-\frac{1}{2}} \end{pmatrix} = 0,$$

we see that

$$m_1 = \begin{pmatrix} v^{-\frac{1}{2}} \\ v^{-\frac{1}{2}} \\ \vdots \\ v^{-\frac{1}{2}} \end{pmatrix}$$

is a characteristic vector corresponding to 0.

We may choose v such characteristic vectors $m_1 \cdots m_v$ in such a way that $M = (m_1 \cdots m_v)$ is an orthogonal matrix. We will have

$$M'CM = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline 0 & e_2 & & 0 \\ & & \ddots & \\ & 0 & & e_v \end{array}\right) = \begin{pmatrix} 0 & 0 \\ 0 & eI \end{pmatrix}.$$

Thus,

$$C = M\begin{pmatrix} 0 & 0 \\ 0 & eI \end{pmatrix}M' = (m_1 m_2 \cdots m_v)\begin{pmatrix} 0 & 0 \\ 0 & eI \end{pmatrix}\begin{pmatrix} m_1' \\ m_2' \\ \vdots \\ m_v' \end{pmatrix}$$

$$= e\sum_{i=2}^{v} m_i m_i' = eM_2 M_2',$$

where $M = (m_1 M_2)$.

Now,

$$I = MM' = (m_1 M_2)\begin{pmatrix} m_1' \\ M_2' \end{pmatrix} = m_1 m_1' + M_2 M_2'.$$

$$M_2 M_2' = I - m_1 m_1' = I - \begin{pmatrix} \frac{1}{v} & \cdots & \frac{1}{v} \\ \vdots & & \vdots \\ \frac{1}{v} & \cdots & \frac{1}{v} \end{pmatrix}$$

$$= \frac{1}{v}\begin{pmatrix} v-1 & -1 & \cdots & -1 \\ -1 & v-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & v-1 \end{pmatrix};$$

hence,

$$C = \frac{e}{v}\begin{pmatrix} v-1 & -1 & \cdots & -1 \\ -1 & v-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & v-1 \end{pmatrix}.$$

2. Application to Designs. We deal with an arbitrary incomplete block design with v treatments, b blocks, k plots in a block, and r repetitions of each treatment. τ_i will be the effect of the ith treatment, and β_j, the effect of the jth block. If treatment i appears in block j, then we assume the observation y_{ij} has the form

$$y_{ij} = \tau_i + \beta_j + \epsilon_{ij}. \tag{2.1}$$

$\hat{\tau}_i$, the estimate of τ_i, is computed by equating the adjusted treatment totals (Q_i) to their expectations and solving for the τ's:

$$\begin{aligned} c_{11}\tau_1 + c_{12}\tau_2 + \cdots + c_{1v}\tau_v &= Q_1 \\ \vdots \qquad\qquad\qquad &\quad \vdots \\ c_{v1}\tau_1 + c_{v2}\tau_2 + \cdots + c_{vv}\tau_v &= Q_v, \end{aligned} \tag{2.2}$$

where,

$$c_{ii} = r\left(1 - \frac{1}{k}\right); \qquad c_{ij} = -\frac{\lambda_{ij}}{k}, \qquad i \neq j,$$

and λ_{ij} is the number of blocks in which the treatments i and j occur together. Also,

$$\begin{aligned} \operatorname{cov}(Q_i, Q_j) &= c_{ij}\sigma^2, \\ \operatorname{var} Q_i &= c_{ii}\sigma^2, \end{aligned}$$

or the covariance matrix of $Q_1, \cdots, Q_v$ is

$$C\sigma^2. \tag{2.3}$$

It is well known that for a connected design, the rows and columns of C sum to 0 and its rank is $v - 1$. Hence, 0 is a characteristic root of multiplicity 1.

We now transform from $Q_1, \cdots, Q_v$ to $z_1, \cdots, z_v$:

$$Q = Tz, \tag{2.4}$$

where T is an orthogonal matrix such that

$$T'CT = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e_v \end{pmatrix} = D_e.$$

The covariance matrix of $z_1, \cdots, z_v$ is

$$D_e\sigma^2. \tag{2.5}$$

Also, on $\hat{\tau}_1, \cdots, \hat{\tau}_v$, the solutions of (2.2), we make the substitution $\hat{\tau} = Tn$. These two substitutions and (2.2) then imply

$$CTn = Tz,$$

or

$$D_e n = z. \tag{2.6}$$

If the covariance matrix of $\hat{\tau}_1, \cdots, \hat{\tau}_v$ is $A\sigma^2$, then the covariance matrix of $n_1, \cdots, n_v$ is $M'AM\sigma^2 = D_{e(A)}\sigma^2$. This is diagonal, since the z's are uncorrelated and $e_i n_i = z_i$ ($e_i \neq 0$) for all but one of the n's; we may also verify that this remaining n is zero with probability one.

From (2.5) and (2.6) we have

$$\begin{aligned} D_e &= D_e D_{e(A)} D_e, \\ 1 &= e_i(A) e_i \qquad \text{for } e_i \neq 0, \text{ and finally} \\ e_i(A) &= 1/e_i \qquad \text{for } e_i \neq 0. \end{aligned} \tag{2.7}$$

We are now ready to assemble the parts and exhibit the promised result.

THEOREM. *If the estimates of the treatment effects in an incomplete block design all have the same variances and the same covariances, then the design is* BIB.

PROOF. Let the characteristic roots of C be $e_0 = 0, e_1, e_2, \cdots, e_{v-1}$. By hypothesis, A is of the form

$$\begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a \end{pmatrix}$$

and hence has two different characteristic roots; they are

$$e(A) = 1/e_i, \qquad i = 1, \cdots, v-1, \qquad \text{and } e_0(A);$$

$$e_i = 1/e(A) \text{ for } e_i \neq 0.$$

Therefore, C has roots $e_0 = 0$ and $e = 1/e(A)$, the latter of multiplicity $v-1$.

Using the lemma of Section 1,

$$C = \frac{e}{v}\begin{pmatrix} v-1 & -1 & \cdots & -1 \\ -1 & v-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & v-1 \end{pmatrix},$$

which says that $\lambda_{ij} = \lambda$, and hence our design is BIB.

THE EFFICIENCY FACTOR OF AN INCOMPLETE BLOCK DESIGN[1]

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1. Summary. It is shown that the efficiency factor of a design is r times the harmonic mean of the latent roots of the reduced intrablock normal equations exclusing the root which is always zero.

2. Properties of reduced normal equations. We consider an incomplete block design with the following properties:

(1) There are rt units arranged in b blocks of k units;

(2) Every one of the t treatments occurs r times;

(3) A treatment occurs once or not at all in a block. (This condition is easily seen to be unnecessary for all the results given below.)

Then, it is well known (see for example Kempthorne [1], pp. 541–543) that the reduced normal equations for intrablock estimates of the treatment effects τ_i are of the form

$$\left(rI - \frac{1}{k}\Lambda\right)(\hat{\tau}) = (Q),$$

where Λ_{ii} equals r for all i, $\Lambda_{ij}(= \Lambda_{ji})$ is the number of blocks which contain both treatments i and j, and Q_j is the total for treatment j adjusted for blocks. Also, the interblock estimates are given by the equation

$$(\Lambda)(\tilde{\tau}) = (R),$$

where R_j is equal to the total of blocks containing treatment j minus r/b times the grand total, and the condition $\sum \tilde{\tau}_j = 0$ is imposed for the nonestimable quantity $\sum \tau$.

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