A NOTE ON BHATTACHARYYA BOUNDS FOR THE NEGATIVE BINOMIAL DISTRIBUTION

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In the lecture notes of Professor Lehmann on the theory of estimation [1], the first two Bhattacharyya lower bounds for the variance of an unbiased estimate of p for the negative binomial have been calculated. It is of some interest to know how the successive bounds turn out, and whether they tend to pq, which we know to be attainable. The object of the present note is to give an explicit expression for the k-th lower bound and show that it tends to pq.

If X has a negative binomial distribution, then we know that

$$P(X = x) = qp^x \qquad x = 0, 1, 2, \cdots, \tag{1}$$

where $q = 1 - p$. Let

$$S_n = \frac{1}{P(x)} \cdot \frac{\partial^n P(x)}{\partial p^n}. \tag{2}$$

Then it is easily verified that

$$S_n = \frac{(-1)^n X^{(n)}}{q^n} + \frac{(-1)^{n-1} n X^{(n-1)}}{pq^{n-1}}, \tag{3}$$

where

$$X^{(m)} = x(x-1) \cdots (x - m + 1).$$

Therefore,

$$S_m S_n = \frac{(-1)^{m+n}}{q^{m+n}} \left[X^{(m)} X^{(n)} - \left(\frac{q}{p}\right) m X^{(m-1)} X^{(n)} - \left(\frac{q}{p}\right) n X^{(n-1)} X^{(m)} + \left(\frac{q}{p}\right)^2 mn X^{(m-1)} X^{(n-1)} \right]. \tag{4}$$

It is well known that

$$E[X^{(m)}] = m! \left(\frac{q}{p}\right)^m, \tag{5}$$

and we have the algebraic identity

$$X^{(m)} X^{(n)} \equiv \sum_{r=0}^{m} \binom{m}{r} n^{(r)} X^{(n+m-r)}, \tag{6}$$

where $n^{(r)} = n(n-1) \cdots (n - r + 1)$. Therefore,

$$E[X^{(m)} X^{n)}] = \sum_{r=0}^{m} \binom{m}{r} n^{(r)} (n + m - r)! \left(\frac{q}{p}\right)^{n+m-r}. \tag{7}$$

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Using (7) in (4), after some simplification we have

$$E(S_mS_n) = \frac{(-1)^{m+n}}{q^m p^{n+m}} n! \sum_{r=0}^{m} \frac{(n+m-r-2)!}{(n-r)!} (mn - r) \binom{m}{r} p^r q^{m-r}. \tag{8}$$

Let $\lambda_{mn} = E(S_mS_n)$. Putting $m = 1, 2, 3$, and 4 in (8), we have in particular,

$$\begin{aligned}
\lambda_{1n} &= \frac{(-1)^{n+1} \cdot n!}{qp^{n+1}}, \\
\lambda_{2n} &= \frac{(-1)^{n+2} \cdot n!}{q^2p^{n+2}} [2q + 2(n-1)], \\
\lambda_{3n} &= \frac{(-1)^{n+3} \cdot n!}{q^3p^{n+3}} [6q^2 + 12q(n-1) + 3n(n-1) + 6], \\
\lambda_{4n} &= \frac{(-1)^{n+4}}{q^4p^{n+4}} \cdot n! [24q^3 + 72q^2(n-1) + 36q(n-1)(n-2) \\
&\qquad + 4n(n+1)(n-7) + 24(3n-1)].
\end{aligned} \tag{9}$$

The k-th lower bound is given by

$$L_k = \begin{vmatrix} \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2k} \\ \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3k} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k2} & \lambda_{k3} & \cdots & \lambda_{kk} \end{vmatrix} \div \begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{kk} \end{vmatrix}. \tag{10}$$

To evaluate the denominator, we multiply the first row by $2/p$ and add the second row to it; the second row is multiplied by $3/p$, and the third row is added to it; and so on. A successive application of this procedure will reduce the determinant to a triangular one, the value of which is easily computed. We thus have,

$$\begin{aligned}
\begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1} & \cdots & \cdots & \lambda_{kk} \end{vmatrix} &= \frac{[k!\,(k-1)! \cdots 1!]^2}{q^{k(k+1)/2}p^{k(k+1)}}, \\
\begin{vmatrix} \lambda_{22} & \cdots & \lambda_{2k} \\ \lambda_{32} & \cdots & \lambda_{3k} \\ \cdots & \cdots & \cdots \\ \lambda_{k2} & \cdots & \lambda_{kk} \end{vmatrix} &= \frac{[k!\,(k-1)! \cdots 1!]^2}{q^{k^2+k-2/2}p^{k^2+k-2}} \{q^{k-1} + q^{k-2} + \cdots + 1\}.
\end{aligned} \tag{11}$$

From (11) we have

$$L_k = p^2q(q^{k-1} + a^{k-2} + \cdots + 1).$$

Therefore,

$$\lim_{k\to\infty} L_k = p^2q/(1-q) = pq.$$

REFERENCES

[1] E. L. Lehmann, "Notes on the theory of estimation," University of California, 1950 (mimeo).