We shall now render the introductory conditions (A)-(B) exact. First, we write

$$\text{(A)} \qquad \sigma(\zeta) \leqq \epsilon \cdot \sigma(x_i), \qquad i = 1, \cdots, h,$$

where $\epsilon \geqq 0$. Then if ϵ is small, the disturbance ζ is small in the sense that its standard deviation is small relative to the standard deviations of the explanatory variables. To give condition (B) a convenient form we observe that for given $r_1, \cdots, r_h$ there is a point $(r_1^*, \cdots, r_h^*)$ on the boundary of the ellipsoid (4) and a proportionality factor ϵ' with $0 \leqq \epsilon' \leqq 1$ such that

$$\text{(B)} \qquad r_i = \epsilon' \cdot r_i^*, \qquad i = 1, \cdots, h.$$

Then if ϵ' is small, the correlations r_i are small in the sense that the point $(r_1, \cdots, r_h)$ lies near the centre of the ellipsoid.

Thus prepared, we obtain the following

COROLLARY. *On conditions* (a) *and* (b) *of the lemma, we have*

$$| b_i - \beta_i | \leqq \epsilon \cdot \epsilon' / \sqrt{1 - R_i^2}, \qquad i = 1, \cdots, h,$$

where ϵ *and* ϵ' *are defined by* (A)-(B).

Hence if ϵ and ϵ' are of small order the specification error of the regression coefficients b_i will at most be of order $\epsilon\epsilon'$.

In the special case of one explanatory variable, $h = 1$, we have $R_1 = 0$ and $| b_1 - \beta_1 | \leqq \epsilon\epsilon'$. For example, if $\sigma(\zeta) = \frac{1}{5}\sigma(x_1)$ and $r_1 = \rho(x_1, \zeta) = \frac{1}{5}$, the specification error of b_1 cannot exceed 0, 04.

REFERENCES

1. H. WOLD IN ASSOCIATION WITH L. JURÉEN, *Demand Analysis: A Study in Econometrics*, Geber, Stockholm, 1952; and John Wiley and Sons, New York, 1953.
2. H. WOLD, "A theorem on regression coefficients obtained from successively extended sets of variables," *Skand. Aktuarietids.*, Vol. 28 (1945), pp. 181–200.

SETS OF MEASURES NOT ADMITTING NECESSITY AND SUFFICIENT STATISTICS OR SUBFIELDS[1, 2]

BY T. S. PITCHER[3]

Let X be the interval from 0 to 1 and F the field of Borel sets on X. For every $x \leqq \frac{1}{2}$, let m_x be the probability measure assigning probability $\frac{1}{2}$ to the point x and probability $\frac{1}{2}$ to the point $(x + \frac{1}{2})$ and let F_x be the subfield of F consisting of all Borel sets which contain both x and $(x + \frac{1}{2})$ or else neither. Then if M is a set of probability measures consisting of all m_x, $0 \leqq x < \frac{1}{2}$ and some measures assigning probability 0 to every point, the only set of m-measure zero

Received May 4, 1956; revised June 12, 1956.

[1] The research in this document was supported jointly by the Army, Navy, and Air Force, under contract with the Massachusetts Institute of Technology.

[2] For definitions of these concepts, see references [1] and [2].

[3] Staff Member, M.I.T., Lincoln Laboratory.

for every m in M is the empty set. Each F_x is a sufficient subfield for M with conditional expectations defined by

$$\begin{aligned} E(f \mid F_x)(y) &= f(y) && \text{if } y \neq x \text{ or } (x + \tfrac{1}{2}), \\ &= \tfrac{1}{2}[f(x) + f(x + \tfrac{1}{2})] && \text{if } y = x \text{ or } (x + \tfrac{1}{2}). \end{aligned}$$

If M has a best sufficient subfield F^0, then F^0 includes only sets having the property that for every $x < \frac{1}{2}$, either both x and $(x + \frac{1}{2})$ are in the set or neither is, so that $E(f \mid F^0)\ (x) = E(f \mid F^0)\ (x + \frac{1}{2})$. In particular if we write f_x for the characteristic function of the point x, then $E(f_x \mid F^0) = \frac{1}{2}(f_x + f_{x+\frac{1}{2}})$ if $x < \frac{1}{2}$. If f is the characteristic function of the interval from 0 to $\frac{1}{2}$, $E(f \mid F^0) \geqq E(f_x \mid F^0)$ for every $x < \frac{1}{2}$ since $f \geqq f_x$, so that $E(f \mid F^0) \geqq \frac{1}{2}$ everywhere; and since $\int E(f \mid F^0)\, dm_x = \frac{1}{2}[E(f \mid F^0)\ (x) + E(f \mid F^0)\ (x + \frac{1}{2})] = \int f\, dm_x = \frac{1}{2}$ we have $E(f \mid F^0) = \frac{1}{2}$ everywhere. Hence for any other m in M, $\int_0^{\frac{1}{2}} dm = \int_0^1 f\, dm = \int E(f \mid F^0)\, dm = \frac{1}{2}\int_0^1 dm = \frac{1}{2}$, which is clearly not true in general.

Another example of the same type can be made up as follows: let A_1 and A_2 be subsets of the intervals from 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1 respectively, let ϕ be a 1:1 map of A_1 onto A_2, and suppose that A_1 is a Borel set, but A_2 is not.[4] For every x in A_1, let m_x assign probability $\frac{1}{2}$ to x and probability $\frac{1}{2}$ to $\phi(x)$; for every x not in A_1 or A_2, let m_x assign probability 1 to x. If M consists of all these m_x, then, as above, the subfields F_x of sets containing either both x and $\phi(x)$ or neither are sufficient for M. Moreover, if f_1 and f_2 are the characteristic functions of A_1 and A_2 respectively and F^0 is a sufficient subfield contained in all the F_x, then, necessarily, $E(f_1 \mid F^0) = (\frac{1}{2})\ (f_1 + f_2)$, which is not measurable.

These examples do not have necessary and sufficient statistics. Each of the fields F_x is induced by the statistic T_x defined by $T_x(x) = T_x(x + \frac{1}{2}) = x$ and $T_x(y) = y$, otherwise. The T_x are sufficient since the F_x are. Since the only M-null set is the empty set, a necessary and sufficient statistic T^0 would have to be exactly a function of each T_x and hence would induce a sufficient subfield F^0 contained in all the F_x. This has already been shown to be impossible.

REFERENCES

[1] E. L. LEHMANN AND H. SCHEFFE, "Completeness, similar regions, and unbiased estimation, Part I," *Sankhya*, Vol. 10 (1950), pp. 305–340.
[2] R. R. BAHADUR, "Sufficiency and statistical decision functions," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 423–462.
[3] C. KURATOWSKI, *Topologie I*, Deuxieme Edition, Monografie Matematyczne, Warszawa, 1948.

[4] The existence of such a map is proved in [3].