SUMS OF INDEPENDENT TRUNCATED RANDOM VARIABLES[1]

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1. Summary and introduction. Let (x_{nk}), $(k = 1, 2, \cdots, k_n; n = 1, 2, \cdots)$ be a double sequence of infinitesimal (i.e. $\lim_{n\to\infty} \max_{1\leq k \leq k_n} P\{|x_{nk}| > \epsilon\} = 0$ for every $\epsilon > 0$) random variables such that for each n, $x_{n1}, \cdots, x_{nk_n}$ are independent. Let $S_n = x_{n1} + \cdots + x_{nk_n}$ and let $F_n(x)$ be the distribution function of S_n. For any $a > 0$ let the random variables x_{nk}^a be defined by

$$x_{nk}^a = \begin{cases} x_{nk}, & \text{if} \quad -a < x_{nk} \leq a, \\ 0, & \text{otherwise,} \end{cases}$$

and let $F_n^a(x)$ be the distribution function of $S_n^a = x_{n1}^a + \cdots + x_{nk_n}^a$. In the next section certain necessary and sufficient conditions are given for $F_n^a(x)$ to converge $(n \to \infty)$ to a limiting distribution and in particular it is shown that if $F_n^a(x)$ converges to $F(x)$, then $F(x)$ has finite moments of all orders. In Sec. 3 it is shown that if $F_n^a(x)$ converges to $F(x)$, then for each positive integer k the kth moment of $F_n^a(x)$ approaches the kth moment of $F(x)$ as $n \to \infty$.

We shall call the random variables (x_{nk}) a truncated system if there exists a $b > 0$ independent of k and n such that $P\{|x_{nk}| > b\} = 0$. We note that if we start with a truncated system we can choose $a > 0$ such that $x_{nk}^a = x_{nk}$.

2. Conditions for convergence. Since the random variables (x_{nk}) are infinitesimal and independent within each row, it is clear that the random variables (x_{nk}^a) are also. From a well-known theorem of Khintchine, (c.f. [1]), it follows that for the weak convergence of $F_n(x)$ (or $F_n^a(x)$) to a limiting distribution $F(x)$, $F(x)$ must be infinitely divisible.

Let $F(x)$ be any infinitely divisible distribution function and let $\varphi(t)$ be its characteristic function. According to the formulas of Levy and Khintchine [1] for the representation of the characteristic function of an infinitely divisible distribution we have

$$\begin{aligned} \log \varphi(t) &= i\gamma t + \int_{\infty}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) \frac{1+u^2}{u^2}\, dG(u) \\ &= i\gamma(\tau)t - b^2t^2/2 + \int_{-\infty}^{-\tau} (e^{iut} - 1)\, dM(u) \\ &\quad + \int_{\tau}^{\infty} (e^{iut} - 1)\, dN(u) + \int_{-\tau}^{0-} (e^{iut} - 1 - iut)\, dM(u) \\ &\quad + \int_{0+}^{\tau} (e^{iut} - 1 - iut)\, dN(u), \end{aligned} \tag{2.1}$$

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where $G(u)$ is bounded nondecreasing function ($G(-\infty) = 0$), γ a real constant,

$$M(u) = \int_{-\infty}^{u} \frac{1+z^2}{z^2}\, dG(z) \quad \text{for} \quad u < 0,$$

$$(2.2) \qquad \begin{aligned} N(u) &= -\int_{u}^{\infty} \frac{1+z^2}{z^2}\, dG(z) \quad \text{for} \quad u > 0, \\ b^2 &= G(+0) - G(-0) \quad \text{and} \quad \gamma(\tau) \\ &= \gamma + \int_{|u|<\tau} u\, dG(u) - \int_{|u|\geqq\tau} \frac{1}{u}\, dG(u), \end{aligned}$$

and where τ and $-\tau$ are continuity points of $N(u)$ and $M(u)$ respectively.

Let $F_{nk}(x)$ and $F_{nk}^{a}(x)$ be the distribution functions of x_{nk} and x_{nk}^{a} respectively. From the definition of x_{nk}^{a} we note

$$(2.3) \qquad F_{nk}^{a}(x) = \begin{cases} 0, & \text{for } x \leqq -a, \\ F_{nk}(x) - F_{nk}(-a), & \text{for } -a \leqq x < 0, \\ F_{nk}(x) + 1 - F_{nk}(a), & \text{for } 0 \leqq x \leqq a, \\ 1, & \text{for } x \geqq a. \end{cases}$$

The following theorem (c.f. [1], p. 124) will be needed.

THEOREM 1. *In order that the distribution functions of the sums* $S_1 = x_{n1} + \cdots + x_{nk_n}$ *of independent infinitesimal, random variables converge to the distribution function* $F(x)$, *it is necessary and sufficient that*:

(1) *At continuity points of* $M(u)$ *and* $N(u)$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x), \quad \text{for} \quad x < 0,$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F_{nk}(x) - 1) = N(x), \quad \text{for} \quad x > 0;$$

(2) $$\lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\} = \lim_{\epsilon\to 0} \underline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\} = b^2;$$

(3) $$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF_{nk}(x) = \gamma(\tau),$$

where $M(u)$, $N(u)$, b^2, *and* $\gamma(\tau)$ *are given by* (2.1) *and* (2.2).

Now using the notation of (2.1) we have the following theorem.

THEOREM 2. *If for some* $a > 0$ $F_n^a(x)$ *converges to* $F(x)$, *then the function* $G(u)$ *is nonincreasing for* $u > a$ *and for* $u < -a$.

Proof. Since $F_n^a(x)$ converges to $F(x)$, according to Theorem 1 we know that at continuity points of $M(u)$ and $N(u)$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F^a_{nk}(x) = M(x) \quad \text{for} \quad x < 0 \quad \text{and}$$

(2.4)

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F^a_{nk}(x) - 1) = N(x) \quad \text{for} \quad x > 0.$$

Thus from (2.3) and (2.4) since $M(u)$ and $N(u)$ are nondecreasing functions, we see that $M(u) = 0$ for $u < -a$ and $N(u) = 0$ for $u > a$. Using (2.2) the conclusion of the theorem follows.

Now given $F(x)$ infinitely divisible define (using the notation of (2.1) and (2.2)) for any $a > 0$, $\pm a$ continuity points of $G(u)$,

$$G^a(u) = \begin{cases} 0, & \text{for } u \leqq -a, \\ G(u) - G(-a), & \text{for } -a \leqq u \leqq a, \\ G(a) - G(-a), & \text{for } u \geqq a, \end{cases} \tag{2.5}$$

$$\gamma^a = \gamma - \int_{|u|>a} \frac{1}{u}\, dG(u),$$

and let $F^a(x)$ be the (infinitely divisible) distribution given by (2.1) using the function $G^a(u)$ and the constant γ^a. We note that $F^a(x)$ is also given by (2.1) using the function $M^a(u)$ and $N^a(u)$ defined by

$$M^a(u) = \begin{cases} 0, & \text{for } -\infty < u < -a, \\ M(u) - M(-a), & \text{for } -a \leqq u < 0, \end{cases}$$

(2.6)

$$N^a(u) = \begin{cases} 0, & \text{for } a < u < \infty, \\ N(u) - N(a), & \text{for } 0 < u \leqq a, \end{cases}$$

(with b^2 unchanged) and

$$\gamma^a(a) = \begin{cases} \gamma(\tau) & \text{for} \quad \tau \leqq a, \\ \gamma(\partial) & \text{for} \quad \tau > a. \end{cases}$$

(With this notation we have the following theorem.

THEOREM 3. *If $F_n(x)$ converges to $F(x)$, then for any $a > 0$ ($\pm a$ continuity points of $G(u)$) $F^a_n(x)$ converges to $F^a(x)$. In particular, if $G(u)$ is nonincreasing outside of the interval $[-a, a]$ then $F^a_n(x)$ converges to $F(x)$.*

Proof. Since $F_n(x)$ converges to $F(x)$, parts (1), (2), and (3) of Theorem 1 hold. We note that continuity points of $M(u)$ and $N(u)$ coincide with those of $G(u)$ so that $-a$ and a are continuity points of $M(u)$ and $N(u)$ respectively. From (2.3) and (2.6) it follows that at continuity points of $M^a(u)$ and $N^a(u)$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F^a_{nk}(x) = M^a(x) \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=1}^{k_n} (F^a_{nk}(x) - 1) = N^a(x),$$

for $x < 0$ and $x > 0$ respectively. Also it is clear that part (2) of Theorem 1 holds with $F_{nk}(x)$ replaced by $F^a_{nk}(x)$ and that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<\tau} x \, dF^a_{nk}(x) = \gamma^a(\tau).$$

Thus from the sufficiency of Theorem 1 we see that $\lim_{n\to\infty} F^a_n(x) = F^a(x)$ (at continuity points of $F^a(x)$). We note that if $G(u)$ is nonincreasing outside of $[-a, a]$ then $F^a(x) = F(x)$. This proves Theorem 3.

Combining Theorems 2 and 3 we can state the following theorem.

THEOREM 4. *If $F_n(x)$ converges to $F(x)$ and if $\pm a$ are continuity points of $G(u)$, then a necessary and sufficient condition for $F^a_n(x)$ to converge to $F(x)$ is that $G(u) = G(+\infty)$ for $u \geqq a$ and $G(u) = G(-\infty) = 0$ for $u \leqq -a$.*

THEOREM 5. *If $F^a_n(x)$ converges to $F(x)$, then $F(x)$ has finite moments of all orders.*

Proof. By Theorem 2 we know that $G(u)$ is nonincreasing outside of the interval $[-a, a]$. In particular it follows that $\int_{-\infty}^{\infty} x^n \, dG(x) < \infty$ for all n. By the result of [2] it follows that $F(x)$ has finite moments of all orders.

We remark that if the system (x_{nk}) is a truncated system we have the following analogues of Theorems 2 and 5.

THEOREM 2a. *If $F_n(x)$ converges to $F(x)$, then the function $G(u)$ is nonincreasing for $u > a$ and for $u < -a$.*

THEOREM 5a. *If $F_n(x)$ converges to $F(x)$, then $F(x)$ has finite moments of all orders.*

3. Convergence of moments. In the remainder of this paper we shall assume that (x_{nk}) is a truncated system. If this is not the case, the following results apply to the system (x^a_{nk}) previously discussed.

In view of Theorem 5a it is natural to consider the question of the convergence of moments of the distribution function $F_n(x)$ of the random variable S_n to the moments of $F(x)$. The principle result of this section is contained in the following theorem.

THEOREM 6. *If (x_{nk}) is a truncated system, and if $F_n(x)$ converges to $F(x)$, then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^k \, dF_n(x) = \int_{-\infty}^{\infty} x^k \, dF(x),$$

for every positive integer k.

The author first proved this theorem in the special case where $F(x)$ was the Poisson distribution (see *Bull. Amer. Math. Soc.*, Vol. 61, Abstract No. 435) and where $k = 2$. This more general form was obtained at a later date (*Bull. Amer. Math. Soc.*, Vol. 62, Abstract No. 264).

The proof of Theorem 6 requires several lemmas which we state and prove below.

Using the same notation as in section 2, according to the result of [2] we know that

$$\int_{-\infty}^{\infty} x^{2k} \, dF(x) < \infty \Leftrightarrow \int_{-\infty}^{\infty} x^{2k} \, dG(x) < \infty,$$

and assuming $F(x)$ has finite moments of all orders that,

$$\chi_1 = \gamma + \int_{-\infty}^{\infty} u \, dG(u) \quad \text{and} \quad \chi_r = \int_{-\infty}^{\infty} (u^{r-2} + u^r) \, dG(u), \tag{3.1}$$

where χ_r is the rth semi-invariant of $F(x)$. In particular letting μ be the mean and σ^2 the variance of $F(x)$, we see

$$\mu = \gamma + \int_{-\infty}^{\infty} u \, dG(u) \quad \text{and} \quad \sigma^2 = G(+\infty) + \int_{-\infty}^{\infty} u^2 \, dG(u). \tag{3.2}$$

LEMMA 1. *Under the hypothesis of Theorem* 6, $\lim_{n\to\infty} \sigma^2(S_n) = \sigma^2$, *where* $\sigma^2(S_n)$ *is the variance of* S_n *and* σ^2 *is the variance of* $F(x)$.

Proof. Since $F_n(x)$ converges to $F(x)$ by Theorem 1, page 112 of [1], we have $G_n(x) \equiv \sum_{k=1}^{k_n} \int_{-\infty}^{x} u^2/(1 + u^2) \, dF_{nk}(u + \alpha_1) \to G(x)$ as $n \to \infty$ at all continuity points of $G(u)$ and also $G_n(+\infty) \to G(+\infty)$, where $\alpha_{nk} = \int_{|x|<\tau} x \, dF_{nk}(x)$, ($\tau > 0$ an arbitrary positive constant). (Remark. By hypothesis $P\{|x_{nk}| > a\} = 0$ for some $a > 0$. We may and do take $\tau > a$ so that $\alpha_{nk} = \mu_{nk} =$ mean of x_{nk}. Hence in the remainder of the proof we assume $\alpha_{nk} = \mu_{nk}$.) Now since $x^2/(1 + x^2) = x^2 - [x^4/(1 + x^2)]$, we see

$$\begin{aligned} G_n(+\infty) &= \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^2 \, dF_{nk}(x + \mu_{nk}) \\ &\quad - \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^4/(1 + x^2) \, dF_{nk}(x + \mu_{nk}) \to G(+\infty) \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 \, dG_n(x) &= \int_{-\infty}^{\infty} x^2 \, d \sum_{k=1}^{k_n} \int_{-\infty}^{x} \frac{u^2}{1 + u^2} \, dF_{nk}(u + \mu_{nk}) \\ &= \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^4/(1 + x^2) \, dF_{nk}(u + \mu_{nk}). \end{aligned} \tag{3.4}$$

By Theorem 2a, $G(x)$ is nonincreasing outside of some interval. Now since the random variables are infinitesimal it follows that $\lim_{n\to\infty} \max_{1 \leq k \leq k_n} |\alpha_{nk}| = 0$. Thus since $P\{|x_{nk}| > a\} = 0$ for some $a > 0$, we know that there exists an $A > 0$ such that $G(x)$ and $G_n(x)$ are nonincreasing for $x < -A$ and $x > A$ ($n = 1, 2, \cdots$). Therefore by Helly's convergence theorem

$$\int_{-\infty}^{\infty} x^k \, dG_n(x) = \int_{-A}^{A} x^k \, dG_n(x) \to \int_{-A}^{A} x^k \, dG(x) = \int_{-\infty}^{\infty} x^k \, dG(x) \tag{3.5}$$

as $n \to \infty$. Letting $k = 2$ and using (3.3) and (3.4) we see

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^2 \, dF_{nk}(x + \alpha_{nk}) = G(+\infty) + \int_{-\infty}^{\infty} x^2 \, dG(x).$$

Now $x_{n1}, x_{n2}, \cdots, x_{nk_n}$ are for each n independent random variables and since $\alpha_{nk} = \mu_{nk}$, by virtue of (3.2) we see $\lim_{n\to\infty} \sigma^2(S_n) = \sigma^2$. This proves Lemma 1.

Having obtained this result we can now prove that the means μ_n of $F_n(x)$ approach the mean μ of $F(x)$.

LEMMA 2. *Under the hypothesis of Theorem* 6,

$$\mu_n = \int_{-\infty}^{\infty} x\, dF_n(x) \rightarrow \int_{-\infty}^{\infty} x\, dF(x) = \mu \quad \text{as} \quad n \rightarrow \infty$$

(*i.e., Theorem* 6 *holds for* $k = 1$.)

Proof. For the proof of this lemma we appeal to Theorem 2 of [1], page 100. Since the random variables (x_{nk}) are infinitesimal and since $\max_{1 \leq k \leq k_n} |\mu_{nk}| = \max_{1 \leq k \leq k_n} |\alpha_{nk}| \rightarrow 0$ as $n \rightarrow \infty$ we see that the random variables $(x_{nk} - \mu_{nk})$ are also infinitesimal. This together with Lemma 1 shows that the hypothesis of Theorem 2, page 100 of [1] is satisfied and hence we may conclude in particular that

$$\mu_n = \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x\, dF_{nk}(x) \rightarrow \gamma' \quad \text{as} \quad n \rightarrow \infty,$$

where γ' is the constant associated with Kolmogorov's formula for the characteristic function of the infinitely divisible distribution $F(x)$. But the constant of Kolmogorov's formula is the mean of the distribution (i.e. $\gamma' = \mu$). This proves Lemma 2.

LEMMA 3. *Under the hypothesis of Theorem* 6

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^r\, dF_{nk}(x + \mu_{nk}) - \chi_{r(n)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

$r = 2, 3, \cdots$, *where* $\chi_{r(n)} =$ r*th semi-invariant of* S_n.

Proof. We note that

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^r\, dF_{nk}(x + \mu_{nk}) - \chi_{r(n)} = 0 \quad \text{for} \quad r = 2, 3$$

and

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x\, dF_{nk}(x) - \chi_{r(1)} = 0.$$

Let

$$\mu_n^{(r)} = \int_{-\infty}^{\infty} (x - \mu_n)^r\, dF_n(x)$$

and let $\mu_{nk}^{(r)} = \int_{-\infty}^{\infty} x^r\, dF_{nk}(x + \mu_{nk})$. Now since (x_{nk}) is a truncated system, and since $\max_{1 \leq k \leq k_n} | \mu_{nk} | \rightarrow 0$ as $n \rightarrow \infty$ we see

$$\max_{1 \leq k \leq k_n} \left| \int_{-\infty}^{\infty} x^r\, dF_{nk}(x + \mu_{nk}) \right| = \max_{1 \leq k \leq k_n} \left| \int_{-A}^{A} x^r\, dF_{nk}(x + \mu_{nk}) \right|$$

for some $A > 0$. Now given $0 < \epsilon < 1$ we see

$$\max_k \left| \int_{-A}^{A} x^r\, dF_{nk}(x + \mu_{nk}) \right| \leq \max_k \int_{-\epsilon}^{\epsilon} | x |^r\, dF_{nk}(x + \mu_{nk})$$

$$+ \max_k \int_{\epsilon \leq |x| \leq A} |x|^r \, dF_{nk}(x + \mu_{nk}) \leq \epsilon^r + A^r \max_k P\{|x - \mu_{nk}| \geq \epsilon\}$$

and since $(x_{nk} - \mu_{nk})$ are infinitesimal we see

$$\lim_{n\to\infty} \max_{1 \leq k \leq k^n} \left| \int_{-\infty}^{\infty} x^r \, dF_{nk}(x + \mu_{nk}) \right| = 0, r = 2, 3, \cdots . \tag{3.6}$$

Also we see (for $r \geq 2$)

$$\sum_{k=1}^{k_n} | \mu_{nk}^{(r)} | = \sum_{k=1}^{k_n} \left| \int_{-\infty}^{\infty} x^r \, dF_{nk}(x + \mu_{nk}) \right|$$

$$\leq \sum_{k=1}^{k_n} \int_{-A}^{A} | x^r | \, dF_{nk}(x + \mu_{nk}) \leq A^{r-2} \sum_{k=1}^{k_n} \int_{-A}^{A} x^2 \, dF_{nk}(x + \mu_{nk})$$

$$= A^{r-2}\sigma^2(S_n) \to A^{r-2}\sigma^2 \text{ as } n \to \infty$$

by Lemma 1. Hence

$$\sum_{k=1}^{k_n} | \mu_{nk}^{(r)} | \tag{3.7}$$

is bounded in n for $r = 2, 3 \cdots$. Let $\chi_r(Z)$ denote the rth semi-invariant of the random variable Z and let $\mu_z^{(r)}$ denote the rth central moment of Z. For $r > 3$ we note

$$\chi_r(Z) = \mu_z^{(r)} + f(\mu_z^{(r-1)}, \cdots , \mu_z^{(2)}), \tag{3.8}$$

where f is a polynomial in $\mu_z^{(r-1)}, \cdots , \mu_z^{(2)}$ each term of which is at least degree 2 (c.f. [1], page 66). Thus $\chi_r(x_{nk}) = \mu_{nk}^{(r)} + f(\mu_{nk}^{(r-1)}, \cdots , \mu_{nk}^{(2)})$. Now if X and Y are independent random variables we note ([1], page 64) $\chi_r(X + Y) = \chi_r(X) + \chi_r(Y)$. Hence since $S_n = x_{n1} + \cdots + x_{nk_n}$ is the sum of independent random variables we see

$$\chi_r(S_n) = \chi_{r(n)} = \sum_{k=1}^{k_n} \mu_{nk}^{(r)} + \left\{ \sum_{k=1}^{k_n} f(\mu_{nk}^{(r-1)} , \cdots , \mu_{nk}^{(2)}) \right\}. \tag{3.9}$$

The general term of the expression in braces may be written as $T = c \sum_{k=1}^{k_n} \prod_{i=1}^{p} \mu_{nk}^{(s_i)}$ where c is a constant, $2 \leq s_i < r$, $p \geq 2$ and where $s_i = s_j$ does not imply $i = j$. But

$$| T | \leq c \max_k | \mu_{nk}^{(s_1)} | \max_k | \mu_{nk}^{(s_2)} | \cdots \max_k | \mu_{nk}^{(s_{p-1})} | \sum_{k=1}^{k_n} | \mu_{nk}^{(s_p)} | .$$

Thus by (3.6) and (3.7) we see that $T \to 0$ as $n \to \infty$. Since the number of terms in f depends only on r this shows that the quantity in braces in (3.9) approaches zero as $n \to \infty$. This proves the Lemma.

Proof of Theorem 6. We note

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x^r \, dF_{nk}(x + \mu_{nk}) = \int_{-\infty}^{\infty} (x^{r-2} + x^r) \, dG_n(x),$$

where $G_n(x) = \sum_{k=1}^{k_n} \int_{-\infty}^{x} [u^2/(1 + u^2)] \, dF_{nk}(u + \mu_{nk})$ as defined in Lemma 1.

Now by (3.5)

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} x^k\, dG_n(x) = \int_{-\infty}^{\infty} x^k\, dG(x)\; k = 0, 1, 2, \cdots$$

and therefore for $r \geqq 2$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} (x^{r-2} + x^r)\, dG_n(x) = \int_{-\infty}^{\infty} (x^{r-2} + x^r)\, dG(x).$$

But $\int_{-\infty}^{\infty} (x^{r-2} + x^r)\, dG(x)$ is by (3.1) the rth semi-invariant of the infinitely divisible distribution $F(x)$. Thus (for $r > 1$)

$$\text{(3.10)}\quad \lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{-\infty}^{\infty} x^r\, dF_{nk}(x + \mu_{nk}) = \chi_r \equiv r\text{th semi-invariant of } F(x).$$

Using (3.10) and Lemma 3 we obtain

$$\text{(3.11)}\qquad \lim_{n\to\infty} \chi_{r(n)} = \chi_r\,,$$

that is the rth semi-invariant of $F_n(x)$ approaches the rth semi-invariant of $F(x)$ as $n \to \infty$. Let $\mu^{(k)} = \int_{-\infty}^{\infty} (x - \mu)^k\, dF(x)$. By Lemmas 1 and 2 we have $\mu_n \to \mu$, $\mu_n^{(2)} \to \mu^{(2)}$ as $n \to \infty$. Now $\chi_3 = \mu^{(3)}$, $\chi_{3(n)} = \mu_n^{(3)}$; $\chi_4 = \mu^{(4)} - 3(\mu^{(2)})^2$, $\chi_{4(n)} = \mu_n^{(4)} - 3(\mu_n^{(2)})^2$ and in general as indicated in (3.8) $\chi_r = \mu^{(r)} + f(\mu^{(r-1)}, \cdots, \mu^{(2)})$, where f is a polynomial and $\chi_{r(n)} = \mu_n^{(r)} + f(\mu_n^{(r-1)}, \cdots, \mu_n^{(2)})$. Using (3.11) and an induction argument we see $\lim_{n\to\infty} \mu_n^{(r)} = \mu^{(r)}$, $(r \geqq 2)$ and this together with Lemma 2 completes the proof of Theorem 6.

REFERENCES

[1] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, translation by K. L. Chung, Addison-Wesley, 1954.

[2] J. M. Shapiro, "A condition for existence of moments of infinitely divisible distributions," *Canadian J. Math.*, Vol. 8 (1956), pp. 69–71.