ABSTRACTS OF PAPERS

(Additional abtsracts of papers presented at the Atlantic City Meeting of the Institute, September 10–13, 1957)

1. **Tests for Significance in Bivariate Harmonic Analysis,** HAROLD HOTELLING, University of North Carolina, AND DONALD F. MORRISON, National Institute of Mental Health.

Detection of a common period in two or more observed variates, such as the radial velocity and brightness of a star or a pair of economic variates, may be undertaken by means of any of several generalizations of univariate periodogram analysis. Three such generalizations are considered in this paper. Two are forms of the multivariate analysis of variance; of these one uses the Wilks determinantal statistic, the other the Hotelling T_0. The third statistic, originated by the junior author, has a distribution whose large-sample approximation is relatively easy to handle. (Received June 10, 1957.)

2. **Conditions that a Stochastic Process be Ergodic,** EMANUEL PARZEN, Stanford University.

In his thesis on statistical inference on stochastic processes, Grenander has pointed out that "the concept of metric transitivity seems to be important in the problem of estimation of a stationary stochastic process." In this note, we give necessary and sufficient conditions in terms of characteristic functions that a strictly stationary (discrete or continuous parameter) stochastic process $X(t)$ be metrically transitive or ergodic. More importantly, we state a mean ergodic theorem (or weak law of large numbers) for stochastic processes which are strictly stationary of order k, by which is meant that for every choice of k points $t_1, \cdots t_k$, the random variables $X(t_1 + h), \cdots, X(t_k + h)$ have a joint probability distribution which does not depend on h. With the aid of these theorems, one can readily establish the following theorems: If $X(t)$ is a normal stationary process, a necessary and sufficient condition for it to be ergodic is that its spectrum be continuous. If $X(t)$ is a linear stationary process, then it is ergodic. (Received June 13, 1957.)

3. **Testing Homogeneity of Means in the Presence of Heterogeneity of Variance,** JOHN GURLAND AND LLOYD ROSENBERG.

A finite series representation for the distribution of statistics with a structure similar to that of the t-statistic is utilized in obtaining under simple restrictions the exact size of a test when variance heterogeneity is present. Further modification of the technique was utilized to obtain the exact power of the tests. Exact probabilities have also been compared with approximations based on an approximate numerator and/or an approximate denominator in the ratios under consideration. The possibility of extending the techniques to the case of more than two samples is also considered. (Received June 21, 1957.)

4. **Generalization of Steinhaus' Results on Fair Division,** PETER NEWMAN, University College of the West Indies.

In *Econometrica*, 1948, H. Steinhaus posed the problem of fair division. A non-homogeneous object X is to be divided among n people, each of whom has a valuation function v_i, assumed to be an increasing, bounded, countably additive, non-atomic, positive measure,

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defined on some Boolean σ-algebra S of subsets of X. Steinhaus asserted that there exists a partition $\bigcup_{i=1}^{i=n} E_i$ such that, for each i, $v_i(E_i) \geqq v_i(X)/n$. This was proved by K. Urbanik (*Fund. Math.*, 1954) who further showed that (i) provided that, for at least one pair i, j and one $E \varepsilon S, v_i(E)/v_i(X) \neq v_j(E)/v_j(X)$, then there is at least one k such that $v_k(E_k) > v_k(X)/n$; (ii) provided further that the measures are equivalent, there exists a partition $\bigcup_{i=1}^{i=n} F_i$ such that, for all i, $v_i(F_i) > v_i(X)/n$. ("good division"). It is shown by an elementary and constructive proof that the condition that the v_i be countably additive measures can be replaced, for all three results, by the condition that they be sub-additive set functions. If they are further assumed to be strictly sub-additive, assumption (i) above can also be dropped. (Received May 27, 1957; revised June 26, 1957.)

5. Graphic Methods based upon Properties of Advancing Centroids, S. I. Askovitz, University of Pennsylvania.

The centroid is defined in elementary physics as the center of gravity or balance point of a composite mass. Centroids of sets of isolated points have found a number of applications in statistics. The fact that centroids can often be located quite readily by graphic methods has made them fairly useful. The simplest application is to the graphic determination of mean values. This can generally be carried out in a matter of a few seconds directly on the original graph, with a pencil and straightedge alone. An entire set of moving averages can likewise be drawn by the use of a single polygonal line, without any calculation. By considering combinations of unequally weighted points, methods have been developed for drawing the line of best fit according to the least squares criterion, again without computation. The change in the least squares lines when new points are added can be worked out easily. The mean value and standard deviation of frequency distributions can also be determined entirely graphically. Other applications, for example, to rank correlation procedures, are being completed. (Received June 28, 1957.)

6. On the Decomposition of Certain χ^2 Variables, Robert V. Hogg and Allen T. Craig, University of Iowa, (By Title).

Let $Q = Q_1 + \cdots + Q_{k-1} + Q_k$, $k \geqq 2$, where $Q_1, \cdots, Q_{k-1}, Q_k$ are real symmetric quadratic forms in central or noncentral, stochastically independent or dependent, normal variables. Let $Q, Q_1, \cdots, Q_{k-1}$ have central or non-central chi-square distributions with parameters r, θ and r_i, θ_i, $i = 1, \cdots, k-1$, respectively where r and r_i are the degrees of freedom and θ and θ_i are the non-centrality parameters. It is proved that if Q_k is a nonnegative quadratic form, then $Q_1, \cdots, Q_{k-1}, Q_k$ are mutually stochastically independent. It follows immediately from the mutual stochastic independence that Q_k has a chi-square distribution with parameters $r_k = r - \sum_1^{k-1} r_i$, $\theta_k = -\sum_1^{k-1} \theta_i$. (Received June 28, 1957.)

7. The Limiting Distribution of a Likelihood Ratio Test for the Serial Correlation Coefficient, John S. White, Minneapolis-Honeywell Regulator Company.

Let x_t be a discrete Gaussian process satisfying the auto-regressive equation $x_t - \alpha x_{t-1} = u_t$ $(t = 1, 2, 3, \cdots)$, where the u's are NID $(0, \sigma^2)$, $|\alpha| < 1$ is an unknown parameter and x_0 is a constant. It is shown that if λ is the likelihood ratio for testing the hypothesis $H:\alpha = \alpha_0$ against the alternative hypothesis $H':\alpha \neq \alpha_0$ then $-2 \log \lambda$ has a chi-squared distribution with 1 df. This result also holds in the so-called explosive case; i.e., $|\alpha| > 1$. (Received June 28, 1957.)

8. On the (Nonrandomized) Optimality of Symmetrical Designs, J. C. Kiefer, Cornell University, (By Title).

Many commonly employed symmetrical designs such as Balanced Incomplete Block Designs, Youden Squares (in particular, Latin Squares), etc., are shown to be optimum among the class of *non-randomized* designs in situations where they are employed to test (H_0) absence of treatment effects. Letting V_d = covariance matrix of best linear estimators of treatment effects when design d is used, the optimality criteria considered are: (1) minimization of det V_d ; (2) minimization of the largest eigenvalue of V_d ; (3) maximization of the minimum power (over all tests and designs) on a fixed contour; (4) the accomplishment of (3) locally (i.e., to first order terms near H_0). Of these, (1) and (2) were demonstrated by Wald and Ehrenfeld, respectively, for the Latin Square. The optimal nature of (2) involves the tacit assumption that the F-test should be used, and since it is not generally true that that test achieves (3) or (4) for a fixed design, criterion (2) has less intrinsic meaning. (3) is generally difficult to verify because of the question of which test to use for each d. In many settings where there is an appropriately symmetric design, (1) implies (2) and (4); these last three criteria are verified in the cited examples. (Received July 2, 1957.)

9. On the Non-optimality of Symmetrical Designs among Randomized Designs, J. C. Kiefer, Cornell University, (By Title).

The following is the simplest example of a general phenomenon. Suppose X_{ij} independent and normal with unit variance and mean $\mu_i (i, j = 1, 2)$. Consider the problem of selecting (before observation) exactly two of the X_{ij} and using them to test $H_0: \mu_1 = \mu_2 = 0$ with size α. The "symmetrical" design d selects (X_{11}, X_{21}) and uses the usual χ^2 test with 2 degrees of freedom. Let d' select (X_{i1}, X_{i2}) with probability 1/2 for each i and use the χ^2 test with 1 degree of freedom, whichever i is chosen. It is shown that the power function of d' is uniformly greater than that of d in a neighborhood of H_0. This is the simplest example of a general phenomenon which persists for any number of populations and observations, whether or not the variance is known. In cases like those where Balanced Incomplete Block Designs, Youden Squares, etc., are usually employed to test that all contrasts (of treatment effects) are 0, the same phenomenon persists as $\alpha \rightarrow 0$. The results are also true for other distributions. (Received July 7, 1957.)

10. On an Optimal Property of Variance-components Estimates, Werner Gautschi, Indiana University.

Recently Graybill and Wortham (*J. Amer. Stat. Assoc.*, Vol. 51 (1956), pp. 266–268) have stated the following result for balanced designs: Among all unbiased estimates for a variance-component, the standard estimate, as given by the method of analysis of variance, has uniformly smallest variance. The authors have sketched a proof which, however, is not quite complete in various aspects. This paper presents a general method of proving results of the above type with applications to various particular designs. The method consists in three steps:

(i) In order to find a sufficient and complete statistic T for the variance-components, an orthogonal transformation is applied which reduces the observation vector y to a "canonical" vector z with independent components. This involves finding the eigenvalues and eigenvectors of the covariance matrix $\sum_y$.

(ii) To avoid laborious transformations of quadratic forms, a lemma is given by means of which among all forms $Q = z'Bz$ which are unbiased estimates for a variance-component, the form Q^* with smallest variance is easily found.

(iii) Q^* depends only through T and thus according to Lehmann and Scheffé (*Sankhyā*, Vol. 10 (1950), pp. 305–340 and Vol. 15 (1955), pp. 219–236) has uniformly smallest variance among *all* unbiased estimates. No transformation backwards is needed, since Q^* is seen to have the same distribution as the standard estimate in y. (Received July 8, 1957.)

11. Quantization for Least Mean Squares Error, Stuart P. Lloyd, Bell Telephone Laboratories.

A quantizing scheme for a real random variable X consists of a partition $\{Q_1, Q_2, \cdots, Q_\nu\}$ of the range of X together with a set $\{q_1, q_2, \cdots, q_\nu\}$ of representative values. An observation falling in Q_α is reported as q_α, $\alpha = 1, 2, \cdots, \nu$. With the number ν of quanta preassigned, and the c.d.f. F of X given, one seeks the $\{Q_\alpha\}$ and $\{q_\alpha\}$ which minimize the mean squared quantization error $\sum \int_{Q_\alpha} (x - q_\alpha)^2 \, dF(x)$. Necessary conditions obtained are (1) q_α is the center of mass $[dF]$ of Q_α, $\alpha = 1, 2, \cdots, \nu$ and (2) modulo sets of measure zero $[dF]$, the $\{Q_\alpha\}$ are intervals whose endpoints bisect the segments between adjacent $\{q_\alpha\}$. Two trial-and-error methods for finding such $\{Q_\alpha\}$ and $\{q_\alpha\}$ are described. The nonsufficiency of conditions (1) and (2) is demonstrated. For suitably restricted F, asymptotic properties for large ν are given. (Received July 8, 1957.)

12. Tests of Multiple Independence and the Associated Confidence Bounds, S. N. Roy and R. E. Bargmann, University of North Carolina.

In this paper a test based on the union-intersection principle is proposed for over-all independence between p variates or p sets of variates with a multivariate normal distribution. Methods used in earlier papers have been applied to invert these tests for each situation and to obtain, with a joint confidence coefficient greater than or equal to a preassigned value, simultaneous confidence bounds on certain parametric functions measuring departures from independence of variate 1 or the set (1) with variates $2, 3, \cdots, p$ or the sets $(2), (3), \cdots, (p)$; variate 2 of the set (2) with variates $3, 4, \cdots, p$ or the sets $(3), (4), \cdots, (p)$; and so on. One of the objects of these confidence bounds is the detection of the "culprit variates" in the case of rejection of the "complex" hypothesis of multiple independence; for the "complex" hypothesis is, in this case, the intersection of several more "elementary" hypotheses of two-by-two independence. (Received July 8, 1957.)

13. Confidence Bounds on the "Ratio of Means" and "Ratio of Variances" for Correlated Variates, S. N. Roy and R. F. Potthoff, University of North Carolina.

In this paper confidence bounds are obtained (i) on the ratio of variances of a (possibly) correlated bivariate normal distribution, and then, by generalization, (ii) on a set of parametric functions of a (possibly) correlated $p + p$ variate normal distribution, which plays the same role for a $2p$-variate distribution as the ratio of variances does for the bivariate case, (iii) on the ratio of means of the distribution indicated in (i) and, by generalization, (iv) on a set of parametric functions of the distribution indicated in (ii), which plays the same role for this problem as the ratio of means does for the bivariate case. For (i) and (iii) the confidence coefficient is any preassigned $1 - \alpha$, and the distribution involved is the *central* t-distribution, while for (ii) and (iv) the confidence statement is a simultaneous one with a joint confidence coefficient greater than or equal to a pre-assigned $1 - \alpha$. For (ii) the distribution involved is that of the *central* largest canonical correlation coefficient (squared), and for (iv) the distribution involved is that of the *central* Hotelling's T^2. (Received July 8, 1957.)

14. On Aggregation and Consolidation in Finite Substochastic Systems, I., DAVID ROSENBLATT, American University, (By Title).

We call a system $x(I - A) = w$ a finite *substochastic system* if the $n \times n$ matrix A is substochastic (row sums $\leqq 1$) and w is nonnegative; if $w \neq \theta$, the null vector, without loss of generality we take w to be a stochastic vector. A solution x of a substochastic system is called *admissible* if x is finite, nonnegative, but not null. An *aggregation matrix* C is an $n \times r$ stochastic matrix with exactly one positive entry in each row, $1 \leqq r < n$. A *weight matrix* E is an $n \times n$ diagonal matrix containing nonnegative entries on the diagonal. A *consolidation* of a substochastic matrix is an $r \times r$ matrix $B(C; E; A) = (C'EC)^{-1}C'EAC$, where $C'EC$ is regular, $1 \leqq r < n$. A consolidation $B(C; E; A)$ is said to be *representative* for a given system $x(I - A) = w$ in respect to an admissible solution $\hat{x}$ if and only if

$$\hat{x}(I - A)C = \hat{x}C(I - B(C; E; A)).$$

A consolidation is said to be *canonical* for a given system $x(I - A) = w$ if and only if it is representative in respect to all admissible solutions of the system.

THEOREM 1: *Consider two finite-dimensional stationary Markov chains characterized by* $\{x_0 ; x_{k+1} = x_k A \mid k = 0, 1, \cdots\}$, $\{y_0 ; y_{k+1} = y_k B \mid k = 0, 1, \cdots\}$ *and an aggregation matrix* C *such that* $y_0 = x_0 C$. *A necessary and sufficient condition that* $y_k = y_k C$ *for all* k *and any* x_0 *is that* $AC = CB$. *B is necessarily a canonical consolidation of* $x(I - A) = \theta$, *invariant for all weight matrices* T *consistent with* $(C'EC)$ *regular.* (Received July 8, 1957.)

15. On Aggregation and Consolidation in Finite Substochastic Systems, II., DAVID ROSENBLATT, American University, (By Title).

Let C be an aggregation matrix. For any column of C containing more than one positive entry, the index set of the rows containing the positive entries will be called an *aggregation set* of C. For any finite stochastic matrix, we distinguish (exhaustively) between transient and ergodic indices and call any closed set of indices an *ergodic set* of indices. For convenience, call all (transient or ergodic) indices connected (in the directed graph of the stochastic matrix) to indices of a given ergodic set, the *associated indices* of that set. THEOREM 2: *Let* $x(I - A) = \theta$ *be a stochastic system involving two or more ergodic sets of indices. Let a canonical consolidation* $B(C; E; A)$ *exist for the system, E regular. The following then obtains: If there exists an aggregation set of C containing one or more associated indices of each of the ergodic sets in a collection* $\{H_1, \cdots, H_p ; p \geqq 2\}$ *and containing at least one index of an ergodic set* H_j, *then* $x_j C = x_h C$ $(j, h = 1, \cdots, p)$ *holds in the stationary stochastic vectors for these sets. Consequently, each index of every ergodic set in the collection* $\{H_1, \cdots, H_p\}$ *must be contained in some aggregation set of C. Moreover, all ergodic sets in the collection exhibit one or more indices in any given aggregation set, or none do.* (Received July 8, 1957.)

16. On Aggregation and Consolidation in Finite Substochastic Systems, III., DAVID ROSENBLATT, American University, (By Title).

THEOREM 3: *Let* $x(I - A) = \theta$ *be a stochastic system. Let a canonical consolidation exist for the system for E regular and given C. Let the associated points of each ergodic set be represented in at most one aggregation set of C. Then C determines an invariant canonical consolidation of the system if and only if each aggregation set contains at least one ergodic index.* Let V_d denote the dth aggregation set of an arbitrary aggregation matrix C; V_d containing m_d indices, $d = 1, \cdots, g$. Let $u = \sum_{d=1}^{g} m_d$. Let $(I - A)_C$ denote the distinguished $u \times n$ submatrix of $(I - A)$ with row indices belonging to aggregation sets of the $n \times (g + n - u)$ aggregation matrix C. Let f_j denote the diagonal elements of a weight matrix E, $j = 1, \cdots, n$. For convenience, we extend the preceding definitions to consideration of finite non-negative systems, $x(\alpha I - A) = w$, A and w nonnegative and α a positive scalar. THEOREM 4: *Let* $x(\alpha I - A) = w$ *be a nonnegative system with one or more admissible solu-*

tions. A consolidation $B(C; E; A)$ is a representative consolidation of the system in respect to an admissible solution $\hat{x}$ of the system if and only if the following holds: $\sum_{d=1}^{g} \sum_{j \varepsilon V_d} (\hat{x}_j - f_j \sum_{k \varepsilon V_d} \hat{x}_k / \sum_{k \varepsilon V_d} f_k) d_{jq} = 0$ *for all* $q = 1, \cdots, (g + n - u)$, *where* d_{jq} *is the typical element of* $(\alpha I - A)_C C$. This result permits treatment of aggregation and consolidation in a significant specialization of the von Neumann model of economic equilibrium. (Received July 8, 1957.)

17. On Aggregation and Consolidation in Finite Substochastic Systems, IV., David Rosenblatt, American University, (By Title).

Theorem 5: *Let A be a substochastic matrix of order $n \geqq 3$. Let a system $x(I - A) = w$ have admissible solutions not all of which are positive only in a single fixed component. Any consolidation $B(C; E; A)$ is canonical for the system if and only if A has the form*

$$A = \alpha I + \beta U,$$

where U is stochastic with all rows identical, and $\alpha + \beta \leqq 1$. The present inquiry is partly motivated by phenomenological models of substochastic variety in mathematical economics and econometrics, e.g., certain interindustrial (input-output) models, intersectoral trade or exchange models, models of macroeconomic stability. The underlying abstract conception of these models goes back to the stationary process of the *Tableau Economique* of François Quesnay (published 1758). In many of these models, the coefficients a_{ij} of A are macrostatistical observables representing empirical economic "flows." We are led to the following proposal in certain applied input-output contexts, in place of matrix inversion, consolidation, or both. Given $x_0(I - A) = w_0$, A substochastic and $(I - A)$ regular, where x_0, A, and w_0 are empirically given. Required to find x for $x(I - A) = w$ for given w. Assume $w \neq \mu w_0$, μ scalar. Let $\tilde{A}_{w0}$, $\tilde{A}_w$ denote the matrices for the corresponding stochastic systems. Compute the stationary stochastic vector $\tilde{z}(w)$ by use of the iterative scheme: $\tilde{z}_{k+1}(w) = \tilde{z}_0 \tilde{A}_w^{k+1}$ $(k = 0, 1, 2, \cdots)$; x is computable from $\tilde{z}(w)$. $\tilde{A}_w^{k+1}$ is always convergent in practice. The present inquiry employs a graphtheoretic approach to problems of aggregation and consolidation in linear systems. (Received July 8, 1957.)

18. Bayes Acceptance Sampling Procedures for Large Lots, Donald Guthrie, Jr. and M. V. Johns, Jr., Stanford University.

A lot of N items is accepted or rejected on the basis of a sample of fixed size n. The consequences of acceptance or rejection are appraised in terms of economic costs consisting of a cost of inspection, and a cost due to accepting or rejecting the lot. If the lot is accepted then the cost due to passing each uninspected item is proportional to a random variable associated with that item. This random variable is assumed to have a distribution which is a member of an exponential family over which an a priori probability distribution is defined. If the lot is rejected, the cost is proportional to the number of items in the uninspected remainder of the lot. Explicit asymptotic expressions are given characterizing the Bayes rejection procedures and sample sizes for large values of the lot size N. (Received July 10, 1957.)

19. On the Equality of the Variances of Several Univariate Normal Populations and some Multivariate Extensions, R. Gnanadesikan, University of North Carolina.

Suppose we have independent random samples of sizes $n_0, n_1, \cdots, n_k$, respectively, from $N(\mu, \sigma^2)$, $N(\mu_1, \sigma_1^2)$, $\cdots$, and $N(\mu_k, \sigma_k^2)$ where the means and the variances are unknown. Using the heuristic union-intersection principle, a test is derived for the null hypothesis $H_0:\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$, and some power properties of the test are studied. The associated simultaneous confidence bounds, with a joint confidence coefficient $(1 - \alpha)$, on $\sigma_1^2/\sigma^2, \cdots$ and σ_k^2/σ^2 are then obtained. Next, for the corresponding multivariate problem of testing the null hypothesis of equality of dispersion matrices, i.e., $H_0:\Sigma_1(p \times p) = \cdots = \Sigma_k(p \times p) = \Sigma(p \times p)$, a test is proposed and the associated simultaneous confidence bounds, with a joint confidence coefficient $\geqq (1 - \alpha)$, on the characteristic roots $c(\Sigma_1\Sigma^{-1}, \cdots, c(\Sigma_k\Sigma^{-1})$ are obtained. (Received July 11, 1957.)

20. Further Contributions to Confidence Bounds on Multivariate Variance Components, S. N. Roy and R. Gnanadesikan, University of North Carolina.

Under the general multivariate linear hypotheses model or Model I, for a restricted k-way classification (including the multivariate analogues of the usual complete and incomplete block connected designs), k matrices, $S_1, \cdots, S_k$, due to the hypotheses of equality of the row vectors of $\xi_i(m_i \times p)$ (for $i = 1, 2, \cdots, k$) and a matrix, S_0, due to error are obtained. Next, for the multivariate variance components model, the k sets $\xi_i(m_i \times p)$'s (for $i = 1, 2, \cdots, k$) are treated as random components from k independent p-variate normal populations $N[\mu_i, \Sigma_i]$ $(i = 1, 2, \cdots, k)$ while the error component is from $N[0, \Sigma]$ with $\Sigma_i = \sigma_i^2\Sigma$. Under this model, when the matrices $S_0, S_1, \cdots, S_k$, obtained under Model I, satisfy the conditions for being distributed mutually independently in central pseudo-Wishart forms with appropriate degrees of freedom, then a set of simultaneous confidence bounds, with a joint confidence coefficient $\geqq (1 - \alpha)$, are obtained on $\sigma_1^2, \cdots, \sigma_2^k$ and all the characteristic roots, $c(\Sigma)$, of the matrix Σ. Next, even when the matrices $S_1, \cdots, S_k$ are not mutually independent, if they are independent of S_0, and if they all have central pseudo-Wishart distributions with appropriate degrees of freedom, then an alternate set of separate confidence bounds for the individual σ_i^2's $(i = 1, 2, \cdots, k)$ are obtained with exact confidence coefficients. (Received July 11, 1957.)

21. A Table of the Expected Value of the Quasi-range, H. Leon Harter, Wright Air Development Center.

The rth quasi-range, w_r, of a sample of n is defined as the range of $(n - 2r)$ sample values, omitting the r largest and the r smallest. Symbolically, $w_r = x_{r-r} - x_{r+1}$, where $x_1, x_2, \cdots, x_n$ are the ordered sample values. The expected value of the rth quasi-range for samples of n from the standard normal distribution $N(0, 1)$ is given by the relation $E(w_r) = 2(r + 1)\binom{n}{r+1}\int_{-\infty}^{\infty} x[\frac{1}{2} - \Phi(x)]^r[\frac{1}{2} + \Phi(x)]^{n-r-1}\Phi(x)\,dx$, where

$$\Phi(x) = (1\sqrt{2\pi})e^{-x^2/2}$$

and $\Phi(x) = \int_0^x \Phi(x)\,dx$. Tables of $E(w_r)$, accurate to within one in the sixth decimal place, are given for $n = 2(1)100$, $r = 0(1)8$. These tables were computed by numerical integration (trapezoidal rule), using the Burroughs E101 computer. The use of sample quasi-ranges in estimating the population standard deviation is discussed. (Received July 12, 1957.)

22. A Generalization of the Discriminant Function Analysis (Preliminary Report), M. M. Rao, University of Minnesota, (By Title). Introduced by R. C. Bose.

Let $\mathbf{x}_r$, $\mathbf{x}_s$ be row vectors of p components each of the rth and sth individuals of two random samples drawn from p-variate normal populations specified by $N(E(\mathbf{x}), \Sigma)$ where

$E(x_{ir}) = \xi_{i1} + \alpha_{1i}t_{1r} + \cdots + \alpha_{ki}t_{kr}$ and $E(x_{js}) = \xi_{j2} + \beta_{1j}t_{1s} + \cdots + \beta_{kj}t_{ks}(i,\ j = 1, 2, \cdots p, r = 1, 2, \cdots, n_1 : s = 1, 2, \cdots n_2)$ so that all components have regressions of the same, kth, order. The t's are fixed variates; e.g., in biological data, such as growth, the regression is age trend. We shall also consider q samples instead of two. The problem is to derive a statistic to test the differences between the samples. In this paper tests for two and q sample cases are derived which on setting α's and β's zero, reduce to Fisher's test of discriminant function (1938) and Wilks' Λ criterion (see C. R. Rao's book, 1952, p. 262), respectively. Also a set of discriminant functions for q samples is obtained simultaneously. We note, however, that the possible $qC_2 = q_1$ say, discriminant functions cannot always be obtained (if $q > 2$). Only s of them will come out, where s = rank C, the hypothesis matrix; i.e., $C\xi = 0$, $C(q_1 \times r)$ and $\xi(r \times p)$ where $r = q(k + 1)$ is the matrix of the parameters; for example, if $q = 3$, $k = 2$, then $s = 2$. An illustration is also considered. (Received July 12, 1957.)

23. Distribution of a Serial Correlation Coefficient Near the Ends of the Range, M. M. Siddiqui, University of North Carolina, (By Title).

If $x_1, \cdots, x_n$ are observations on a stationary time series at equal intervals of time and it is known that $Ex_t = 0$ for $t = 1, \cdots, n$, we may define a serial correlation coefficient with lag unity by $r^* = (\Sigma_1^{n-1} x_i x_{i+1})/[(\Sigma_1^{n-1} x_i^2)(\Sigma_1^{n-1} x_{i+1}^2)]$. Assuming the observations to be distributed independently as $N(0, 1)$ variates a geometrical approach suggested by Hotelling (*American Journal of Mathematics*, 1939) is utilized to obtain the order of contact of the distribution curve at $r^* = \pm 1$. It is shown that if for a number r_0 in $[0, 1]$ and close to 1, $P(r^* \geqq r_0)$ is expanded in a series of powers of $(1 - r_0)$ the first non-zero coefficient, I_n, is that of the power $(n - 2)/2$. Bounds on the value of I_n are obtained to be 0.435 and 0.638. (Received July 15, 1957.)

24. Jacobi Polynomials and Distributions of Some Serial Correlation Coefficients, M. M. Siddiqui, University of North Carolina, (By Title).

If y is a variate with range $[c_1, c_2]$, where $-1 \leqq c_1 \leqq c_2 \leqq 1$, and the moment generating function of y, $\chi_y(t)$, can be written in the form

$$\chi_y(t) = e^{-t}\Sigma_1^{\infty} a_k t^k F(\beta + k + 1, \alpha + \beta + 2k + 1, 2t),$$

where $\alpha, \beta > -1$ and $F(a, b, x)$ is a confluent hypergeometric function, then, under certain conditions, the pdf of y is given by $p(y) = f(y; \alpha, \beta)[\Sigma_0^{\infty} b_k P_k^{(\alpha,\beta)}(y)]$; here $f(y; \alpha, \beta) = C^{-1}(1 - y)^{\alpha}(1 + y)^{\beta}$, $C = 2^{\alpha+\beta+1}B(\alpha + 1, \beta + 1)$, $P_k^{(\alpha,\beta)}(y)$ is the kth degree Jacobi polynomial associated with $f(y; \alpha, \beta)$ and b_k is determined by a_k. Slight modifications in the form of $\chi_y(t)$ and $p(y)$ are necessary if the range of y is between 0 and 1. Let $x_1, \cdots, x_n$ be independent $N(0, \sigma)$ variates. Defining serial correlation coefficients by

$$r_s = (\Sigma_1^{n-s} x_i x_{i+s})/(\Sigma_1^n x_i^2),$$

$s = 1, 2, \cdots$, the moments of r_s are used to express its moment generating function in the required form and hence an approximation to its distribution is obtained as a product of a beta distribution and a series of Jacobi polynomials. It is proved that the series is asymptotic. By an easy generalization of this method, an approximation to the bivariate distribution of r_1 and r_2 is also obtained. (Received July 15, 1957.)

25. Age-dependent Branching Stochastic Processes in Cascade Theory—II. Case of Transformation Probabilities a Function of Absorber Depth, W. Max Woods, Stanford University and A. T. Bharucha-Reid, University of Oregon, (By Title).

In this paper we consider a simple model for an electron-photon cascade in which the transformation probabilities are functions of the absorber depth. This model is developed within the framework of the Waugh generalisation of the Bellman-Harris process. In particular, we assume $q_0(t) = 1 - \beta \exp(-\alpha t)$, $q_2(t) = \gamma \exp(-\alpha t)$, $q_1(t) = 1 - (q_0(t) + q_2(t))$, $\alpha > 0, 0 \leqq \beta \leqq 1, 0 \leqq \gamma \leqq 1$, where $q_i(t)$ is the probability that i electrons will be formed when transformation takes place at thickness t. The first and second moments of the probability distribution of the number of electrons in the cascade are obtained and their properties discussed. An expression which gives the probability that the cascade will terminate is also obtained. (Received July 15, 1957.)

26. **An Extension of the Theory of Cumulative Frequence Functions,** BERNARD J. DERWORT, North American Aircraft Corporation AND WALDO A. VEZEAU, St. Louis University, (By Title).

This work provides an extension in three areas: (1) some of the known theory for functions of one variable is extended to functions of two variables, (2) new theory for functions of one variable is developed, namely, a moment-generating function and cumulative semi-invariants, (3) nine classes of new cumulative functions of one variable are developed. (Received July 16, 1957.)

27. **Some Renewal Processes Related to Types I and II Counter Models,** RONALD PYKE, Stanford University.

The Type I and Type II counter problems (cf. W. Feller, "On probability problems in the theory of counters", *Courant Anniversary Volume* (1948), pp. 105–115) with arbitrary input and deadtime are studied. Let $\{X_j\}$, $\{Y_j\}$ be independent sequences of independent indentically distributed non-negative random variables (i.e., independent renewal processes). Let $S_0 = 0$, $S_k = X_1 + X_2 + \cdots + X_k (k \geqq 1)$ and define recursively $n_0 = m_0 = 0$, $n_j = \min \{k > n_{j-1}: S_k > Y_{j-1} + S_{n_{j-1}}\}$ and

$$m_j = \min \{k > m_{j-1}: S_k > S_i + Y_i, i = m_{j-1}, \cdots, k - 1\}.$$

The probability distributions of the n- and m-processes thus defined are obtained. Define $Z_j = S_{n_j}$ and $V_j = S_{m_j} (j \geqq 1)$. The Z- and V-processes thus defined are renewal processes associated respectively with Type I and Type II counter models. The distribution and characteristic functions for the Z-process are obtained explicitly. An integral equation determining the characteristic function for the V-process is derived. Other quantities connected with these processes are also studied. In particular,

$$\text{Prob } \{Z_k + Y_k \leqq t < Z_{k+1} \text{ for some } k \geqq 1\}$$

and a similar quantity with Z replaced by V are derived explicitly as well as their limits as $t \to \infty$. Several examples are listed. A more general counter model proposed by Albert and Nelson (*Ann. Math. Stat.*, Vol. 24 (1953), pp. 9–22) is also studied and its solution is given explicitly in terms of the solution of a corresponding Type II problem. (Received July 17, 1957.)

28. **Contributions to the Theory of Random Mappings,** BERNARD HARRIS, Stanford University.

A random mapping space (X, τ, P) is a triplet, where X is a finite set of elements x of cardinality n, τ is a set of transformations T of X into X, and P is a probability measure

over τ. If $x \varepsilon X$ and $T \varepsilon \tau$, $T^k x$ is defined as the kth iteration of T performed on x, where k is an integer either positive or negative. If for some $k \geqq 0$, $T^k x = y$, then y is said to be a kth image of x in T. The set of successors of x in T, $S_T(x)$ is defined as the set of all images of x, i.e., $S_T(x) = \{x, Tx, \cdots, T^n x\}$, which need not all be distinct elements. If for some $k \leqq 0$, $T^k x = y$, then y is said to be a kth preimage of x in T. The set of all kth preimages of x in T is $P_T^{(k)}(x)$ and $P_T(x) = \bigcup_{k=-n}^{0} P_T^{(k)}(x)$ is the set of predecessors of x. If there exists a $m > 0$, such that $T^m x = x$, then x is a cyclical element in T and the set of elements $x, Tx, \cdots, T^{m-1}x$, is the cycle containing x, $C_T(x)$. If there exists a pair of integers k, l, $T^k x = T^l y$, then $x \sim y$ under T. The resulting equivalence classes are called the components of T. The author considers several choices of T, in each case choosing P as the uniform distribution over T, and computes the distributions of the number of elements in $S_T(x)$, the number of elements in the cycle in the component containing x, the number of cyclical elements, the number of elements in $P_T(x)$, and the number of components of T. (Received July 17, 1957.)

29. Tabulation of the Trivariate Normal Integral, Preliminary Report, George P. Steck, Sandia Corporation, (By Title).

Let $F(h, k, m) = \int_{-\infty}^{h} \int_{-\infty}^{k} \int_{-\infty}^{m} dT(x, y, z; \rho_{12}, \rho_{13}, \rho_{23})$, where $T(x, y, z; \rho_{12}, \rho_{13}, \rho_{23})$ is the trivariate normal density function with zero means, unit variances, and correlation coefficients $\rho_{12}, \rho_{13}, \rho_{23}$. $F(h, k, m)$ can be expressed as a sum of 3 univariate normal integrals plus the sum of 6 T-functions (the T-function has been tabulated by D. B. Owen, *Ann. Math. Stat.*, Vol. 27, No. 4) plus the sum of 6 integrals of the form

$$S(m, a, b) = (1/\sqrt{2\pi}) \int_{-\infty}^{m} e^{-x^2/2}\, T(ax, b)\, dx.$$

The function $S(m, a, b)$ has been tabulated by numerical integration (but not checked) for $m = 0(.1)5.0$; $a = 0(.1)[\sqrt{25 - m^2}/10m]$, $b = 0(.1)1.0$. The method used for expressing the trivariate normal integral as a function of three variables applies equally well in expressing the n-dimensional normal integral as a function of n variables. (Received July 18, 1957.)

30. Exact Probabilities and Asymptotic Relationships for Some Statistics from m-th Order Markov Chains, Leo A. Goodman, University of Chicago.

Exact formulas are given for the joint probability distribution of the set of observed m-tuple frequencies ($m \geqq 1$) in an observed sequence $\{X_1, X_2, \cdots, X_N\}$ from a $(m - 1)$-th order Markov chain with a denumerable number of states. Formulas are also presented for the conditional distribution of the set of m-tuple frequencies, given the set of n-tuple frequencies, in a sequence from a chain of order $\leqq n - 1$. If the chain is of order $\leqq n - 1$, and has a finite number s of states, the conditional probability (of the m-tuple frequencies, given the n-tuple frequencies), when regarded as statistic computed from the observed sequence, is asymptotically equivalent to the joint probability (regarded as a statistic) of a corresponding set of observed cell entries in a set of s^{n-1} independent contingency tables with fixed marginal totals (each table has s^{m-n} rows and s columns), where independence in each table is assumed. Several simplified tests, related to standard tests of independence in contingency tables, are given for the null hypothesis H_{n-1} that the chain is of order $n - 1$ against the alternate hypothesis H_{m-1}. Results of P. G. Hoel (*Biometrika*, Vol. 41 (1954), pp. 430–439), P. Whittle (*J. Roy. Stat. Soc.*, B, Vol. 17 (1955), pp. 235–242), and R. Dawson and I. J. Good (*Ann. Math. Stat.*, Vol. 28 (1957)) are generalized herein. (Received July 18, 1957.)

31. Asymptotic Distributions of Some Goodness of Fit Criteria for m-th Order Markov Chains, Leo A. Goodman, University of Chicago, (By Title).

I. J. Good's review, for *Mathematical Reviews*, of P. Billingsley's article in *Ann. Math. Stat.*, Vol. 27 (1956), pp. 1123–1129, proposed two conjectured generalizations to mth order Markov chains ($m \geqq 0$) of Billingsley's results for zeroth order stationary chains. It is proved herein that the first conjecture is correct, while the second isn't. Billingsley's results used the theory of finite dimensional vector spaces, while the generalization is proved herein by approximating the goodness of fit criteria by functions of statistics whose asymptotic distributions were derived in L. A. Goodman's "Exact Probabilities and Asymptotic Relationships for Some Statistics from m-th Order Markov Chains." The generalization follows: Let $\{X_1, X_2, \cdots, X_N\}$ be an observed sequence from a stochastic process where each random variable takes as values only the integers 1, 2, $\cdots$, s. Let f_u be the observed frequency of the m-tuple $u = (u_1, u_2, \cdots, u_m)$. Let H_n be the composite hypothesis that the process is a chain of order $n(m + 1 > n \geqq 0)$. Let H_n' be any simple hypothesis within H_n, and let $\hat{H}_n$ be the maximum-likelihood estimate of H_n'. Let

$$\hat{\psi}^2_{m,n} = \textstyle\sum_u [f_u - \hat{f}_{u,n}]^2/\hat{f}_{u,n},$$

where $f_{u,n}$ is the expected value of $\dot{f}_u$, given H_n, in a new sequence of length N. Then, when H_n is true, $\hat{\psi}^2_{m,n}$ has asymptotically ($N \to \infty$) a distribution $*_{\lambda=1}^{m-n-1} K_{g(\lambda)}(x/\lambda)$, where $*$ denotes convolution, $g(\lambda) = s^{m-1-\lambda}(s-1)^2$, and $K_{g(\lambda)}(x/\lambda)$ is the χ^2-distribution with $g(\lambda)$ degrees of freedom. (Received July 18, 1957.)

32. On the Bivariate Sign Test, Isadore Blumen, Cornell University.

A test for the hypothesis that the median of a bivariate distribution is (μ, v) is called a sign test if it is based on the direction of the n vectors from (μ, v) to (x_i, y_i), where $i = 1, \cdots, n$. The test proposed here is obtained by ranking the vectors according to the slope $(y_i - v)/(x_i - \mu)$. The statistic used is $v^2 = (v_1^2 + v_2^2)/n$, where $v_1 = \sum_{j=1}^n a_j \cos(\pi j/n)$ and $v_2 = \sum_{j=1}^n a_j \sin(\pi j/n)$, a_j is ± 1 according as the difference $y_i - v$ is positive or negative, and the index j corresponds to the vector with the jth largest slope. The large sample distribution of this statistic is obtained and it is compared with a number of other sign tests in relative efficiency. (Received July 18, 1957.)

33. The Telephone Trunking Problem (Preliminary Report), Herbert Scarf, The RAND Corporation, (Introduced by T. E. Harris).

Customers arrive at a service point, with independent, identically distributed, interarrival distributions. There are N servers, each of which serves according to the same negative exponential distribution. The assumption is made that a customer departs immediately if all of the servers are busy at the moment of his arrival, so that no queue is formed. This model is solved for an arbitrary interarrival distribution, in the sense that an explicit formula is obtained for the probability distribution of the number of busy servers. In addition, a relatively simple formula is given for the expected fraction of the customers turned away. (Received July 19, 1957.)

34. Asymptotic Independence of Tests of Parametric Forms of Cell Probabilities in the Analysis of Categorical Data, Earl L. Diamond, University of North Carolina, (By Title).

This is a generalization of some results given by Mitra in "Contributions to the Statistical Analysis of Categorical Data" (North Carolina Institute of Statistics mimeograph series No. 142). We start from a product of multinomial distributions of the form

$$\phi = \prod_i [n_{i0}! \prod_j p_{ij}^{n_{ij}} / \prod_j n_{ij}!]$$

with $\sum_j p_{ij} = 1$; $i = i_1 i_2 \cdots i_k$; $j = j_1 j_2 \cdots j_l$; $i_1 \varepsilon (r_1)_{i_2 \cdots i_k}$ (a subset of r_1 depending on the subscript set $i_2 \cdots i_k$); $i_2 \varepsilon (r_2)_{i_3 \cdots i_k}$; $\cdots$; $i_{k-1} \varepsilon (r_{k-1})_{i_k}$; $i_k = 1, 2, \cdots, r_k$ and

$$j_1 = 1, 2, \cdots, S_1 ; \cdots ; j_l = 1, 2, \cdots, S_l .$$

We next consider the hypotheses $H_0^{(1)}: p_{ij} = f_{ij}^{(1)}(\theta_1, \cdots, \theta_{t_1})$ subject to

$$g_m^{(1)}(\theta_1, \cdots, \theta_{t_1}) = 0 \ (m = 1, 2, \cdots, u_1 < t_1)$$

and $H_0^{(2)}: p_{ij} = f_{ij}^{(2)}(\theta_1', \cdots, \theta_{t_2}')$ subject to $g_m^{(2)}(\theta_1', \cdots, \theta_{t_2}') = 0 \ (m = 1, 2, \cdots, u_2 < t_2)$ where t_1, t_2 < [total number of cells − total number of multinomial distributions]. Each hypothesis is a composite one in which the θ's or the θ''s are the nuisance parameters and $f_{ij}^{(1)}$, $f_{ij}^{(2)}$, $g_m^{(1)}$, and $g_m^{(2)}$ are known functions. Necessary and sufficient conditions for the asymptotic independence of $H_0^{(1)}$ and $H_0^{(2)}$ are derived, these conditions being extensions of similar conditions for more special cases discussed in Mitra's paper. (Received July 22, 1957.)

35. Tests of Parametric Forms of Cell Probabilities and their Asymptotic Power in the Analysis of Categorical Data, EARL L. DIAMOND, University of North Carolina, (By Title).

This is a generalization of some material in a previous paper by the author ("Extension of some results given by Mitra on 'Statistical Analysis of Categorical Data' ") presented at the March, 1957, IMS meetings in Washington, D.C. We start from a product of multinomial distributions of the form $\phi = \prod [n_{i0}! \prod_j p_{ij}^{n_{ij}} / \prod_j n_{ij}!]$ with $\sum_j p_{ij} = 1$; $i = i_1 i_2 \cdots i_k$; $j = j_1 j_2 \cdots j_l$; $i_1 \varepsilon (r_1)_{i_2 \cdots i_k}$ (a subset of r_1 depending on the subscript set $i_2 \cdots i_k$); $i_2 \varepsilon (r_2)_{i_3 \cdots i_k}$; $\cdots$; $i_{k-1} \varepsilon (r_{k-1})_{ik}$; $i_k = 1, 2, \cdots, r_k$ and $j_1 = 1, 2, \cdots, S_1$; $\cdots$; $j_l = 1, 2, \cdots, S_l$. We next consider the hypothesis $H_0: p_{ij} = f_{ij}(\theta_1, \cdots, \theta_t)$ subject to $g_m(\theta_1, \cdots, \theta_t) = 0 (m = 1, 2, \cdots, u < t)$ against the alternative

$$H_n: p_{ij} = f_{ij}(\theta_1, \cdots, \theta_t) + n^{-1/2}\delta_{ij}$$

subject to $g_m(\theta_1, \cdots, \theta_t) = 0$, where t < (total number of cells − total number of multinomial distributions). The hypothesis is a composite one in which the θ's are the nuisance parameters and f_{ij} and g_m are known functions. Tests are given for hypotheses analogous to the hypotheses of "no partial correlation," "no multiple correlation," "no canonical correlation," and "complete independence" in multivariate analysis, and analogous to the hypotheses of "no block effect," "no treatment effect," and "no block or treatment effect" in analysis of variance. The asymptotic power of each test is derived. (Received July 22, 1957.)

36. Tests of Functional Forms of Cell Probabilities and their Asymptotic Power in the Analysis of Categorical Data, EARL L. DIAMOND, University of North Carolina.

This is an extension of some results given by Mitra in "Contributions to the Statistical Analysis of Categorical Data" (North Carolina Institute of Statistics mimeograph series

No. 142) and amplified by Ogawa in "On the Mathematical Principles Underlying the Theory of the χ^2 Test" (North Carolina Institute of Statistics mimeograph series No. 162). We start from a product of multinomial distributions of the form

$$\phi = \Pi_i \, [n_{i0}! \, \Pi_j \, p_{ij}^{n_{ij}} / \Pi_j \, n_{ij}!]$$

with $\sum_j p_{ij} = 1$; $i = i_1 i_2 \cdots i_k$; $j = j_1 j_2 \cdots j_l$; $i_1 \varepsilon (r_1)_{i_2 \cdots i_k}$ (a subset of r_1 depending on the subscript set $i_2 \cdots i_k$); $i_2 \varepsilon (r_2)_{i_3 \cdots i_k}$; $\cdots$ $i_{k-1} \varepsilon (r_{k-1})_{i_k}$; $i_k = 1, 2, \cdots, r_k$ and $j_1 = 1, 2, \cdots, S_1$; $\cdots$; $j_l = 1, 2, \cdots, S_l$. We next consider the hypothesis

$$H_0 : f_m(p_{ij}\text{'s}) = 0 \ (m = 1, 2, \cdots, t)$$

with $t <$ (total number of cells $-$ total number of multivariate distributions), against the alternative $H_n : f_m(p_{ij}\text{'s}) = n^{-1/2}\delta_m$. A test is given for hypotheses of this form and the asymptotic power of this test is derived. As an example, the case in which

$$f_m(p_{ij}\text{'s}) (\text{for } m = 1, 2, \cdots, t)$$

are linear is developed in detail. (Received July 22, 1957.)

37. Asymptotic Normality and Efficiency of Certain Nonparametric Test Statistics, Herman Chernoff and I. Richard Savage, Stanford University.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be observations from the continuous cumulative distribution functions $F(x)$ and $G(x)$ respectively. If $z_{iN} = 1$ when the ith smallest of $N = m + n$ observations is from F and $z_{iN} = 0$ otherwise, then many nonparametric test statistics are of the form $T = \sum_{i=1}^{N} E_{iN} z_{iN}$. Theorems of Wald and Wolfowitz, Noether, Hoeffding, Lehmann, Madow, and Dwass have given sufficient conditions for the asymptotic normality of T. In this paper we extend these results to cover more situations with $F \neq G$. In particular it is shown for all alternative hypotheses that the Fisher-Yates, Terry-Hoeffding c_1-statistic is asymptotically normal and the test for translation based on it is at least as efficient as the t-test. (Received July 22, 1957.)

38. Effects and the Classical Analysis of Variance Mixed Model, Mary D. Lum, Wright Air Development Center.

Consider the two-factor mixed model $x_{ijr} = M + A_i + b_j + (Ab)_{ij} + e_{ijr}$,

$$(i = 1, \cdots, I; j = 1, \cdots, J; r = 1, \cdots, R),$$

where M, A_i are constants; b_j , $(Ab)_{ij}$, e_{ijr} are independently normally distributed with zero means and constant variances σ_b^2 , σ_{Ab}^2 , σ_e^2 , respectively. Besides the effects b_j , $(Ab)_{ij}$ one can alternatively consider the effects β_j , γ_{ij}: $\beta_j = b_j + (Ab)_{\cdot j}$, $\gamma_{ij} = (Ab)_{ij} - (Ab)_{\cdot j}$, where $(Ab)_{\cdot j} = \sum_{i=1}^{I} (Ab)_{ij}/I$. The effects γ_{ij} are subject to linear constraints $\sum_{i=1}^{I} \gamma_{ij} = 0$; nevertheless the mean squares are distributed as Chi-squares and F-tests are valid. The experimenter is usually more interested in the effects β_j , γ_{ij} rather than b_j , $(Ab)_{ij}$; though it may also be desirable to investigate the latter. An F-test of the hypothesis $\beta_j \equiv 0$ involves the mean square for error as the appropriate error term, whereas that of the hypothesis $b_j \equiv 0$ involves the mean square for interaction. It is thus shown that the suitability of any mean square as an error term simply depends on the particular effect in which one is interested. The same argument can be extended to the general n-factor mixed model. (Received July 22, 1957.)

39. Equally Spaced Levels for Multi-level Continuous Sampling Plans (Preliminary Report), Donald Guthrie, Jr. and M. V. Johns, Jr., Stanford University, (By Title).

The tightened multi-level continuous sampling scheme proposed by Derman, Littauer, and Solomon (*Ann. Math. Stat.*, Vol. 28, No. 2, June, 1957) calls for sampling at levels $1, f, f^2, \cdots, f^k$, with k possibly infinite. An alternative scheme proposed in this paper calls for sampling at levels $1, f, f/2, \cdots, f/k$. If $k = \infty$, then for plans with levels $1, f, f^2, \cdots$, all levels of inspection correspond to null recurrent states in the Markov chain describing the process for p less than the AOQL. In the scheme discussed here all levels of inspection are ergodic for all values of p. In this respect the proposed plan gives better protection against a sudden deterioration of quality. For $k = 2, 3, 4, 5$, the plans are compared on the basis of expected first passage time to 100% inspection if p goes from p_1 (good quality) to p_2 (bad quality) for fixed AOQL, and average fractions inspected equal at p_1 and p_2. In addition, for both types of plans, contours of constant AOQL are given for $k = 2, 3, 4, 5, \infty$. (Received July 22, 1957.)

40. Extension of the Mann-Whitney "U" Test to Samples Censored at the Same Fixed Point, Max Halperin, National Institutes of Health.

Suppose we have random samples of size m and n from populations with continuous cumulatives, G and F respectively. Denote an observation from G by y and from F by x. Let both samples be censored to the right at the same fixed point, $x = T$, $y = T$. A statistic, U_c, is defined, which is the sum of (1) The usual U statistic, as defined to test against the alternative $F(z) > G(z)$, all z, computed for the uncensored elements of both samples, and (2) The product of the number of uncensored y's and the number of censored x's. It is shown that, when $F = G$, $-\infty < x \leqq T$, and for the total number of censored elements in both samples fixed at the total number observed, say r, the distribution of U_c is independent of the specific nature of F, and U_c, properly standardized, is shown under appropriate conditions on m, n, r, to have an asymptotically normal distribution. A test of the null hypothesis, $F = G$, $-\infty < x \leqq T$, based on U_c is proposed and shown to be consistent against the alternative, $F(x)/F(T) > G(x)/G(T)$, $F(T) > G(T)$, $-\infty < x < T$. This alternative implies $F(x) > G(x)$, $-\infty < x \leqq T$. (Received July 22, 1957.)

41. On some Distribution-free Bias Properties of the Latent Roots of Real Symmetric Random Matrices, H. Robert van der Vaart, University of North Carolina, (By Title).

If the probability distribution of the $k \times k$ real symmetric random matrix F is continuous and satisfies $\mathcal{E}(F) = \Phi$, then $\mathcal{E}(l_1) < \lambda_1$, $\mathcal{E}(l_k) > \lambda_k$, $\mathcal{E}(l_k - l_1) > \lambda_k - \lambda_1$, and

$$\mathcal{E}(\textstyle\sum_{h=1}^{k} l_h) = \sum_{h=1}^{k} \lambda_h .$$

Under the same conditions Med $(l_1) \leqq \lambda_1$, Med $(l_k) \geqq \lambda_k$, whereas for many common distributions of F these inequalities regarding the medians are strict. In these statements *random matrix* means *matrix of random elements*; the *probability distribution of a matrix* stands for the *joint distribution of its elements*; the *expectation of a matrix* stands for the *matrix of the expectations of its elements*; $l_1 \leqq l_2 \leqq \cdots \leqq l_k$ are the latent roots of F (hence random variables) arranged according to increasing magnitude, while $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k$ are the latent roots of Φ arranged in a similar way. Denote by Λ the diagonal matrix consisting of the elements $\lambda_1, \lambda_2, \cdots, \lambda_k$ and by L the diagonal matrix consisting of the elements $l_1, l_2, \cdots, l_k$. Then the proof is based on the existence of orthogonal matrices Y, U and V such that $\Phi = Y\Lambda Y'$, $F = ULU'$ and $YV = U$. Here Y is a matrix of parameters and U and V are matrices of random variables. (Received July 22, 1957.)

42. On the Distribution of the Latent Roots of Real Symmetric Random Matrices with Multinormally Distributed Elements, H. ROBERT VAN DER VAART, University of North Carolina.

Let the $\frac{1}{2}k(k+1)$ random elements

$$f_{ij}(ij = 11, \cdots, 1k, 22, \cdots, 2k, \cdots, k-1\ k-1, k-1\ k, kk)$$

of a $k \times k$ real symmetric random matrix F be multinormally distributed with $\mathcal{E}(f_{ij}) = \varphi_{ij}$ and $\mathcal{E}(f_{ij}, f_{pq}) = \sigma_{ij,pq}$. Then in order that the joint distribution of the latent roots $l_1 \leqq l_2 \leqq \cdots \leqq l_k$ of the matrix F depends only on the various $\sigma_{ij,pq}$-values and on the latent roots $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k$ of the matrix $\Phi = \| \varphi_{ij} \|$, and not on the elements of any orthogonal matrix Y with $\Phi = Y\Lambda Y'$ (the elements of Y would be, in a sense, nuisance parameters), there is an interesting necessary and sufficient condition on the $\sigma_{ij,pq}$-values. Furthermore, in case $k = 2$ a number of results on the joint distribution of l_1 and l_2 (evidently depending on the matrix of $\sigma_{ij,pq}$-values) are presented. They regard both the amounts of bias of the l_i as estimators of the λ_i and their variances and covariances. (Received July 22, 1957.)

43. Bias in Certain Current Procedures of Response Surface Estimation, H. ROBERT VAN DER VAART, University of North Carolina (By Title).

Be ξ and β real $k \times 1$ matrices, Φ a $k \times k$ real symmetric matrix, η a real scalar variable of which η_0 is a certain value; ξ, β, Φ and η are non-random quantities. The equation of any quadratic response surface has form $\eta - \eta_0 = \beta'\xi + \xi'\Phi\xi$. Be the $k \times k$ real symmetric random matrix F continuously distributed with $\mathcal{E}(F) = \Phi$, then F is currently used as an estimator of Φ, the type of response surface being estimated from the latent roots of F. The latter estimation method is biased: one will estimate unduly often that the surface is saddle-shaped when in fact it has a minimum or maximum. Another corollary from the distribution theory of latent roots is that generally the variances of the "canonical quadratic effects" (i.e., the latent roots of F) are different from the variances of f_{11}, f_{22}, etc., even with second order rotatable designs. These designs do have the gratifying property that if the elements of F are multinormally distributed the joint distribution of the latent roots of F does not depend on the nuisance parameters represented by the elements of any orthogonal matrix Y with $\Phi = Y\Lambda Y'$. Here Λ is the diagonal matrix consisting of the latent roots of Φ. (Received July 22, 1957.)

44. On the Numerical Computation of Certain Multivariate Normal Integrals, H. ROBERT VAN DER VAART, University of North Carolina, (By Title).

Be y and $n \times 1$ matrix, C a real symmetric positive definite $n \times n$ matrix with $c_{ii} = 1 (i = 1, \cdots, n)$. Consider (1): $| C |^{-\frac{1}{2}} \cdot \int \exp(-\frac{1}{2}y'C^{-1}y)\, dy_1 \cdots dy_n$, where integration is over $y_i > 0 (i = 1, \cdots, n)$. If C is a Jacobi matrix (i.e., $c_{ij} = 0$ for $| i - j | > 1$) the integral (1) reducse to a sum of integrals (2): $\int | E |^{-\frac{1}{2}} de_{12}\, de_{34} \cdots de_{m-1m}$, with $m \leqq [\frac{1}{2}n]$, m even, where E represents any principal minor matrix of C with row indices

$$i_1 < i_2 < \cdots < i_{m-1} < i_m,$$

and integration is over $0 < e_{12} < c_{i_1 i_2}, \cdots, 0 < e_{m-1m} < c_{i_{m-1} i_m}$. If $m = 4$, then (2) is at most a double integral, which is for most sets of c_{ij}-values numerically computable to a for statistical purposes very satisfactory degree of accuracy by a nine-point integration method (equivalent to Gauss' three-point method for single integrals applied twice), or even—with less accuracy—by a five-point method which is not equivalent to any univariate method applied twice. The simplicity (greatly enhanced by certain symmetry properties of $| E |$ with respect to $c_{ij} = 0$) and accuracy of these methods make numerical computation of certain multinormal integrals very easy. Generalization to $m > 4$ is immediate. If integration in (1) is over $y_i > \omega_{i\bullet} \neq 0 (i = 1, \cdots, n)$ or if C is no longer a Jacobi matrix, essentially the same methods apply, though without the short-cuts made possible by the above-mentioned symmetries now being destroyed. (Received July 22, 1957.)

45. Birth and Death Random Walk Process in s Dimensions, J. Neyman and Elizabeth L. Scott, University of California.

Consider two finite sequences $\{a_i\}$ and $\{b_j\}$ of Borel sets in s dimensional Euclidean space R_s. The a_i are disjoint "cells". The b_j are disjoint "regions". Consider dimensionless particles ("organisms", male or female) located in R_s. For $0 \leqq T_1 \leqq T_2$, symbol $\theta(T_1, T_2)$ denotes conditional probability that an organism aged T_1 will live to be T_2. At times $t = 0, 1, 2, \cdots$ each surviving female gives birth to a litter composed of a random number ν of organisms of next generation. Φ is the probability that an organism born is female. All variables ν are identically distributed, mutually independent and independent of all other random variables of the system. Given that an organism survives up to time T_2 and given that at time $T_1 < T_2$ it is located at $X(T_1) = x_1 \varepsilon R_s$, the function

$$f(x_1, x_2, T_1, T_2)$$

represents the conditional density at $x_2 \varepsilon R_s$ of $X(T_2)$, the position of this organism at time T_2. All organisms random walk independently. For $t \geqq 0$ and for Borel set $c \subset R_s$, the symbols $\alpha(c, t)$, $\beta(c, t)$ represent the numbers of males and females of the ancestral generation born at $t = 0$ in the given sequence $\{a_i\}$ of cells, and $\gamma(c, t)$, $\delta(c, t)$ those of male and female descendents of ancestors born in $\{a_i\}$, all alive at time t and at that time located in c. The results obtained concern the joint distribution of n quadruples $\alpha(b_j, t), \beta(b_j, t), \gamma(b_j, t), \delta(b_j, t)$ corresponding to an arbitrary set of n regions $\{b_j\}$. This distribution is expressed in terms of the distribution of $\{\alpha(a_i, 0), \beta(a_i, 0)\}$ and in terms of unspecified θ, Φ and f. Applications include astronomy, ecology, and radioactive phenomena. (Received July 24, 1957.)

46. On Convergence of Distribution Functions and of Moments of Order Statistics, Mandakini Rohatgi, University of California, (By Title, introduced by J. Neyman).

Let $Y_{1n} < Y_{2n} < \cdots < Y_{sn}$ be the s smallest order statistics out of a sample of size n from a distribution $F(x)$. It is assumed that there exit constants $a_n > 0$ and b_n such that $Y_{1n}^* = (Y_{1n} - b_n)/a_n$ has a limiting distribution that is nondegenerate as n tends to infinity. Then $Y_{1n}^* < \cdots < Y_{sn}^*$, where $Y_{in}^* = (Y_{in} - b_n)/a_n$, have a joint limiting distribution. Furthermore the conditional distribution function of $Y_{2n}^* < \cdots < Y_{sn}^*$, given Y_{1n}^*, has a limit as n tends to infinity, which is a distribution function, and the conditional distribution function of $Y_{1n}^* < \cdots < Y_{(s-1)n}^*$, given Y_{sn}^*, has a limit which is also a distribution function. If n is a random variable N whose distribution depends on a parameter γ and if N/γ tends to unity in probability, as γ tends to infinity, then there exit constants α_γ and β_γ such that $(Y_{iN} - \beta_\gamma)/\alpha_\gamma$ has a limiting distribution as γ tends to infinity and α_γ and β_γ can be taken to be a_γ and b_γ respectively. In the special case when $F(x)$ is a Normal distribution function, the first moment of Y_{1n}^* converges to the first moment of the limiting distribution, the second moment of Y_{1n}^* is bounded for large n and the third moment of Y_{1n}^* diverges. (Received July 24, 1957.)

47. Best Unbiased Tests of Composite Hypotheses with s Constraints, Mandakini Rohatgi, University of California.

Let X be a random vector with probability density depending on $s + k$ parameters $\xi_1, \cdots, \xi_s, \theta_1, \cdots, \theta_k = (\xi, \boldsymbol{\theta})$. Hypotheses considered are $H_1: \xi = \xi_0$, $\boldsymbol{\theta}$ unknown, and $H_2; \xi_1 = \xi_2 = \cdots = \xi_s = \xi$, ξ unspecified, $\boldsymbol{\theta}$ unknown. Locally most powerful unbiased tests with constant power on ellipsoids are derived for testing H_1 and H_2, under a set of assumptions similar to those made in the Neyman-Pearson theory of testing hypotheses. As an example, a test for equality of variances in s Normal populations is given. In the case when the sample sizes are equal, the locally most powerful test with spherical power surfaces is $\sum_{i=1}^{i=s} S_i^4 \geqq \lambda$, where λ is a function of $\sum_{i=1}^{i=s} S_i^2$. (Received July 24, 1957.)

48. On the Asymptotic Distribution of the Likelihood-ratio in some Mixed Variates Populations, J. Ogawa, M. D. Moustafa and S. N. Roy, University of North Carolina, (By Title).

Let the likelihood function of the population under consideration be $P(X \mid H_0)$ and $P(X \mid H)$ under the null-hypothesis H and the alternative hypothesis H respectively, then it is well known that under certain conditions the random variable

$$-2 \log \lambda = -2 \log [\max P(X \mid H_0)/\max P(X \mid H)]$$

has the χ^2-distribution with suitable degrees of freedom in the limit as n, the sample size, tends to infinity, provided the null-hypothesis H is true. S. S. Wilks (1939) got this result based upon J. L. Doob's work (1934). Later (1943) A. Wald obtained the same result starting from somewhat stronger assumptions. However, as far as the authors are concerned, they have never seen any complete proof along the Wilks' line published so far. In this note, the authors are concerned with the asymptotic distributions of $-2 \log \lambda$ for testing various kinds of null-hypothesis in certain mixed variates populations. For that purpose the authors will present a complete proof of the above mentioned proposition, and then the validity of Doob's assumptions was verified in each case which was of the authors' main concern. (Received July 24, 1957.)

49. On Test of a Certain Hypothesis based upon Selected Sample Quantiles, J. Ogawa, University of North Carolina.

The author reported on the estimation of the location and scale parameters based upon the selected sample quantiles, and determined the optimum spacings for the normal and exponential distributions. The author will present here the theory of testing a certain hypothesis and show that the optimum spacings for estimation purpose turn out to be the spacings which give the greatest powers for testing purpose. (Received July 24, 1957.)

50. Run Tests and Likelihood Ratio Tests for Markov Chains, Leo A. Goodman, University of Chicago, (By Title).

This article first discusses some runs tests as tests of randomness in a single sequence of alternatives, where the number of kinds of alternatives is s and where the sequence is long. Simple derivations of some long sequence run tests are given by making use of some results concerning the asymptotic distribution of the observed transition numbers in a sequence from a Markov chain. Since the asymptotic distribution of the transition proportions is related to some standard asymptotic results for multinomial trials, a close relationship is observed between certain asymptotic results in the distribution theory of runs and in the standard distribution theory for multinomial trials. Large sequence runs tests are presented for certain generalizations of the null hypothesis of randomness and for certain alternate hypothesis concerning Markov chains, and the asymptotic distributions are obtained under both null and alternate hypothesis; thus, generalizing some standard results in the distribution theory of runs. Runs tests for the special case $s \equiv 2$ are studied in detail. Some simplified likelihood ratio statistics for testing generalization of the null hypothesis of randomness and hypotheses concerning the order of a Markov chain are studied in detail and compared with other statistics that have been suggested in the literature. In the errata to his paper in *Biometrika*, Vol. 42 (1955), pp. 531–533, I. J. Good refers the reader to the present work; some of the statistics mentioned in his paper have been studied further herein and a number of inaccuracies in his paper have been corrected. (Received July 25, 1957.)

51. **Further Results in Testing of Hypotheses on a Multivariate Population, some of the Variates being Continuous and the Rest Categorical,** M. D. Moustafa, University of North Carolina, (By Title).

Consider a multi-way table such that certain ways refer to continuous variates and the other ways are categorical. For certain problems all the categorical ways refer to variates; for certain other problems all of them refer to ways of classification; and for some problems some of the ways refer to variates and the rest to ways of classification. The author assumes that the conditional distribution of the continuous variates, given the categorical variates, is a multinormal distribution; and in case some of the categorical ways are ways of classification, the conditional distribution of the continuous variates will be a set of independent multinormal distributions. For such multi-way tables, tests for hypotheses like, say that of conditional independence or joint independence or total independence, etc., are formulated. Considering large sample tests, the statistic used is the $-2 \log \lambda$ statistic, which, in each of these situations, is shown, in another paper, to have asymptotically the χ^2-distribution; but to adopt it to this study, the fact that some of the variates are categorical should be noticed. The author suggests a statistic which is algebraically simpler, more convenient and is asymptotically equivalent, in probability, to the $-2 \log \lambda$ statistic when the latter is calculated directly from the likelihood ratio. (Received July 25, 1957.)

52. **Random Walks in the Plane with General Absorbing Barriers,** M. V. Johns, Jr., Stanford University.

Let Y_j, $j = 1, 2, \cdots$, be independent zero-one random variables with

$$\text{Prob}\,\{Y_j = 1\} = p.$$

The points (S_n, n), $n = 1, 2, \cdots$, where $S_n = \sum_{j=1}^{n} Y_n$, describe the path of a random walk. An absorbing barrier may be characterized by a non-decreasing sequence of positive integers $a_1, a_2 \cdots$, where absorption takes place at the nth stage if

$$S_j \leqq a_j, \; j = 1, 2, \cdots n - 1,$$

and $S_n > a_n$. Without loss of generality it may be assumed that $a_{j+1} \leqq a_j + 1$, for all j. The probability of attaining the point (k, n) without prior absorption is shown by an elementary argument to be

$$\left[\sum_{j=0}^{k} \gamma_j \binom{n}{k-j}\right] p^k(1-p)^{n-k},$$

where the γ's are determined recursively by

$$\gamma_m = -\sum_{j=0}^{m-1} \gamma_j \binom{\alpha_m - 1}{m - j}, m = 1, 2, \cdots,$$

where $\gamma_0 = 1$ and where the α's are determined as follows: $\alpha_j = j, j = 1, 2, \cdots j_0$, where j_0 is the smallest j such that $a_j = j$; $\alpha_j = \min \{i: a_i = j\}, j > j_0$. A similar expression is obtained for the case of a random walk between two absorbing barriers. Application of these results yields an explicit solution of the classical "gambler's ruin" problem where the number of plays is finite and the stakes risked at each play are fixed but not necessarily equal. (Received July 26, 1957.)

53. Mathematical Developments in the Theory of Human Lethal Dose, (Preliminary Report), Clifford J. Maloney, Fort Detrick.

A number of two parameter families of curves (probit, logit, sinit) and one one parameter family (exponential response curve) have been proposed to express the relation between biological response and intensity of a deleterious agent. Two measures of response in the case of disease agents are (1) sickness and (2) death. It was observed (Maloney, *Proceedings of the Second Army Conference on Design of Experiments*, 1956) that, provided morbidity and mortality are related to dose level by the same form of curve, and since all dead individuals had necessarily been ill, that the morbidity curve must always lie above the mortality curve when the two are plotted on the same graph, and hence can be used to infer one parameter of the mortality curve from those of the morbidity curve. The problem is so important that a search for minimum conditions for the validity of the conclusion is appropriate. The present paper considers the consequence of some relaxation of the assumption that the curve relating mortality to dose is a member of the same two parameter family that relates morbidity to dose. (Received July 26, 1957.)

54. A Matrix Definition of the Correlation between Two Sets of Variables, Andre G. Laurent, Michigan State University, (By Title).

Let X_1, X_2 be $p \times 1$ and $q \times 1$ random vectors, $p \geqq q$, and $X = (X_1, X_2)'$ be $N(0, \sigma)$ with covariance matrix $\sigma = (\sigma_{ij}), i = 1, 2\ j = 1, 2$, (all matrices non singular); let

$$\sigma_{ii} = \sigma_i \sigma_i'$$

where σ_i is triangular. Implicit in and consistent with Hotelling's definition of canonical correlations is the intuitively "natural" generalisation of the correlation between X_1 and X_2 as a matrix, namely $P = \sigma_2^{-1} \sigma_{21} \sigma_1'^{-1}$ which yields the covariance matrix of X_2, given X_1, as $\sigma_2(I - PP')\sigma_2'$. The squares of the canonical correlations are the roots of

$$| I\rho^2 - PP' | = 0.$$

$\sigma = \text{Diag. } (\sigma_i)\ \mathcal{R}\ \text{Diag. } (\sigma_i)'$, where $\mathcal{R}$ is the generalised correlation matrix

$$\begin{pmatrix} I & P' \\ P & I \end{pmatrix}, \mathfrak{D} = \begin{pmatrix} I & D_\rho \\ D_\rho' & I \end{pmatrix},$$

where $D_\rho = \text{Diag. } (\rho_k), k = 1, \cdots, p$, is the canonical correlation matrix and $\mathcal{R}$ and $\mathfrak{D}$ are "equivalent" in that sense that there exists Λ such that

$$\Lambda \mathcal{R} \Lambda' = \mathfrak{D}, \quad \Lambda = \text{Diag. } (\Lambda_1, \Lambda_2),$$

Λ_i orthogonal. If X is $N(0, I)$ and R is the "correlation" based on sample values, i.e., $R = S_2^{-1}S_{21}S_I'^{-1}$, the distribution of R is $a_n \mid I - RR' \mid^{(n-p-q-2)/2} dR$, with

$$a_n = \pi^{-pq/2}\Gamma\left(\frac{n-1}{2}\right)\cdots\Gamma\left(\frac{n-q}{2}\right)\Big/\Gamma\left(\frac{n-p-1}{2}\right)\cdots\Gamma\left(\frac{n-p-q}{2}\right).$$

A scalar correlation coefficient between X_1 and X_2 can be obtained by means of a proper scalar function of RR'. In case $X = (X_1, X_2, X_3)'$, similar generalisations of the multiple and the partial correlations, (starting from conditional distributions) yield the identity $(I - P_{3(21)}P'_{3(21)}) = (I - P_{32.1}P'_{32.1})(I - P_{31}P'_{31})$. (Received July 29, 1957.)

55. On Ranking Parameters of Location and Scale in Continuous Populations, K. C. Seal, Calcutta University, (By Title).

The general problem of selecting from a given set of continuous populations a subset, which should contain the most desirable population, such as the population having the largest or smallest parameter of location or scale with certain specified risk, is studied in this paper. The problem of ranking either the location or scale parameters when all other parameters are assumed to be known is at first considered. The closely similar problem when both location and scale parameters are assumed to be unknown but when one of these two parameters, either location or scale, can be eliminated by the method of studentization, is then discussed. It is also shown that the analogous problem of ranking parameters belonging to multivariate populations is readily solvable from the proposed solution to the above problems. When the same experiment is to be continued to more than one stage the modifications required for the solution to such allied multistage problems are also indicated in broad outline. The decision procedure suggested for these problems is shown to possess many desirable properties which include *properties of unbiasedness, gradation and monotonicity*. The suggested decision rule also minimizes the expected size of the finally retained subset in most situations and, in fact, may be taken to be the optimum from an infinite class of decision rules. (Received July 29, 1957.)

56. Statistical Estimate and Control of the Costs Caused by Accidents in a Factory, Hans Bühlmann, University of California, (Introduced by E. L. Scott).

Consider the random variable Z equal total costs of all accidents in a time interval of fixed length divided by sum of all salaries paid in the same time. Z depends on two factors, the number of accidents W and the amount of damage X caused by each accident. Approximating the frequency of W by a normal distribution and assuming the distribution of damage to be of Γ-type, the distribution of Z is obtained. Records of the number of accidents and the total costs in the past intervals provide maximum likelihood estimates of the parameters and the expected value of Z. In practice, we want to estimate the values of the parameters and to test the hypothesis that they are unchanged. The results obtained include sufficient statistics for each of the parameters with their distributions. (Received August 2, 1957.)