ASYMPTOTIC BEHAVIOR OF TESTS ON THE MEAN OF A LOGARITHMICO-NORMAL DISTRIBUTION WITH KNOWN VARIANCE[1]

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1. Summary. Three tests were considered by Severo and Olds [1] for testing an hypothesis on a mean of a logarithmico-normal distribution with known variance. The purpose of this note is to discuss the asymptotic behavior of these tests for large sample size.

2. Asymptotic properties of the tests for large n. We adopt the terminology and notation of [1] in order to discuss the asymptotic properties of the T_1, T_2, and T_3 tests for large sample size n. The particular cases considered in [1] indicate that the power of each test increases as n increases. The question arises as to whether or not the approach is to some particular power function which has well-known properties.

The T_1 test. When $\mu_x = {}_0\mu_x$ then $\beta_{T_1} = \Phi(z_\alpha)$ for all n. When $\mu_x > {}_0\mu_x$, then the expression

$$\ln \frac{\mu_x^2}{\sqrt{1+\mu_x^2}} - \ln \frac{{}_0\mu_x^2}{\sqrt{1+{}_0\mu_x^2}} = \ln \frac{\mu_x^2}{\sqrt{1+\mu_x^2}} \frac{\sqrt{1+{}_0\mu_x^2}}{{}_0\mu_x^2}$$

is always greater than zero. Therefore

$$\lim_{n\to\infty} \beta_{T_1} = \begin{cases} \Phi(z_\alpha) = 1-\alpha, & \mu_x = {}_0\mu_x \\ \Phi(-\infty) = 0, & \mu_x > {}_0\mu_x \end{cases}$$

which is simply the ideal operating characteristic of a statistical test. Thus, increasing the sample size does not alter the functional form of the operating characteristic of the T_1 test.

The T_2 test. The T_2 test involves the mean $\bar{x}$, of n logarithmico-normal variates each having the same mean μ_x, and the same variance 1. By an application of the Lindeberg-Levy form of the Central Limit Theorem [2], $\bar{x}$ is asymptotically $N(\mu_x, 1/n)$. Hence, for large n, it follows that the operating characteristic of the T_2 test at any $\mu_x > {}_0\mu_x$ may be approximated by

$$\beta_{T_2} \doteq \Phi\left\{\frac{\left(z_\alpha \frac{1}{\sqrt{n}} + {}_0\mu_x\right) - \mu_x}{\frac{1}{\sqrt{n}}}\right\}$$

$$= \Phi\{z_\alpha - (\mu_x - {}_0\mu_x)\sqrt{n}\} = \Phi\{z_\alpha - \delta\sqrt{n}\}.$$

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Thus, for large sample sizes the T_2 test behaves like the most powerful one-sided test for testing a simple hypothesis on the mean of a normal distribution with known variance.

The T_3 test. The discussion of the asymptotic behavior of the T_3 test for large n employs the notation:

$$\xi_i = \frac{\ln x_i - \frac{b}{a}}{\sigma_y}, \tag{1}$$

$$E(\xi_i) = \frac{\mu_y - \frac{b}{a}}{\sigma_y} = m, \tag{2}$$

where $E(z)$ stands for the expected value of z. The noncentral χ^2 variate involved in the T_3 test may then be written as

$$\chi'^2 = \sum_{i=1}^{n} \xi_i^2, \tag{3}$$

with parameters $\lambda = nm^2$ and n.

The large sample behavior of the T_3 statistic is summarized in the following theorem which follows as a direct application of the Lindeberg-Levy form of the Central Limit Theorem.

THEOREM. *The noncentral χ^2 variate given by* (3) *is asymptotically* $N[n(1 + m^2), 2n(1 + 2m^2)]$ *as n approaches infinity.*

Thus, as n gets large, the χ'^2 distribution may be approximated by a normal distribution with mean $n(1 + m^2)$ and variance $2n(1 + 2m^2)$ where m is given by (2). This suggests that for large n the cut-off point for the T_3 test criterion may be approximated by

$$\chi'^2_{0,\alpha} \doteq z_\alpha\sqrt{2n(1 + 2m_0^2)} + n(1 + m_0^2),$$

where m_0 denotes the value of m evaluated at $\mu_x = {}_0\mu_x$.

Similarly, for large n, the theorem enables the operating characteristic of the T_3 test at any $\mu_x > {}_0\mu_x$ to be approximated by

$$\begin{aligned}\beta_{T_3} &\doteq P\{\chi'^2 \geqq z_\alpha\sqrt{2(1 + 2m_0^2)n} + (1 + m_0^2)n\} \\ &= P\left\{\frac{\chi'^2 - (1 + m^2)n}{\sqrt{2(1 + 2m^2)n}} \geqq \frac{z_\alpha\sqrt{2(1 + 2m_0^2)} - (m^2 - m_0^2)\sqrt{n}}{\sqrt{2(1 + 2m^2)}}\right\} \\ &= \Phi\left\{\frac{z_\alpha\sqrt{2(1 + 2m_0^2)} - (m^2 - m_0^2)\sqrt{n}}{\sqrt{2(1 + 2m^2)}}\right\}.\end{aligned} \tag{4}$$

Hence, when n is large, the functional form of the T_3 test and of its operating characteristic is replaced by the normal function. The rate of convergence of (4) is slow and for that reason approximations to the noncentral χ^2 suggested by Patnaik [3] or Abdel-Aty [4] are recommended in practice.

REFERENCES

[1] N. C. Severo and E. G. Olds, "A comparison of tests on the mean of a logarithmico-normal distribution with known variance," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 670–686.

[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1951.

[3] P. B. Patnaik, "The non-central χ^2 and F-distribution and their applications," *Biometrika*, Vol. 36 (1949), pp. 202–232.

[4] S. H. Abdel-Aty, "Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution," *Biometrika*, Vol. 41 (1954), pp. 538–540.

A t-TEST FOR THE SERIAL CORRELATION COEFFICIENT

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Summary. Let r be the sample serial correlation coefficient computed from a sample of size N drawn from a serially correlated process with parameter ρ. It is shown that the statistic

$$t = \frac{(r - \rho)\sqrt{N + 1}}{\sqrt{1 - r^2}}$$

is approximately distributed as Student's t with $N + 1$ degrees of freedom.

Introduction. Let (x_t) be a discrete process satisfying the stochastic difference equation

$$x_t = \rho x_{t-1} + u_t \qquad (t = 1, 2, \cdots)$$

where the u's are NID (0, 1) and ρ is an unknown parameter. If, considering a sample of size N, we assume that $x_{N+1} = x_1$, then the distribution of the x's is uniquely determined by that of the u's and the x's are said to be circularly correlated. The parameter ρ is called the (circular) serial correlation coefficient and may be estimated by

$$r = \frac{\sum_{t=1}^{N} x_t\, x_{t+1}}{\sum_{t=1}^{N} x_t^2}, \qquad (x_{N+1} = x_1).$$

Leipnik [1] obtained the following as an approximate (say $N > 20$) distribution for r

$$f(x) = \frac{(1 - x^2)^{(N-1)/2}}{B\left(\frac{1}{2}, \frac{N+1}{2}\right)(1 + \rho^2 - 2\rho x)^{N/2}}$$

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