Consequently, ρ_1 will be a good approximation to $\hat{\rho}$ whenever ρ_1 is bounded away from unity and $m - \nu$ is large.

On substituting back, one obtains for the maximum likelihood estimates of λ and μ

$$\hat{\lambda} = \frac{(n + m)\hat{\rho}}{\hat{\rho}T + \tau},$$

$$\hat{\mu} = \frac{n + m}{\hat{\rho}T + \tau}.$$

Whenever the approximation of ρ_1 for $\hat{\rho}$ is valid, the following simple approximations for $\hat{\lambda}$ and $\hat{\mu}$ result:

$$\hat{\lambda} \approx \frac{n + \nu}{T},$$

$$\hat{\mu} \approx \frac{m - \nu}{\tau}.$$

Note the difference between these formulas and the formulas n/T and m/τ which would result if the initial distribution was neglected as mentioned in Sec. 1.

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MOST POWERFUL RANK-TYPE TESTS

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For some non-parametric problems the use of the invariance principle reduces the class of suitable tests to those based on ranks of ordered observations. To obtain among these the test that is most powerful from some specified alternative distribution, it is necessary to have the marginal probability distribution of the rank statistic under the alternative. Hoeffding [1] gives a method that expresses the probabilities of such a distribution in terms of an expectation taken with respect to the hypothesis distribution. Applications have been made to the problem of location (Hoeffding [1]) and to the problem of randomness (Lehmann [2] and Terry [3]). We extend Hoeffding's method and, for the problem of location with symmetry, derive a locally most powerful rank-type test against normal alternatives.

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Suppose a point in a sample space $\mathfrak{X}$ can be given by coordinates r, t in such a way that $\mathfrak{X}_r \times \mathfrak{X}_t = \mathfrak{X}$, where $\mathfrak{X}_r$, $\mathfrak{X}_t$ are the sample spaces for r, t respectively. Also, suppose that there are two probability measures on the space $\mathfrak{X}$, a *hypothesis* measure

$$M_1 \qquad \Pr\{(r, t) \in R\} = \int_R f(r)g(t)\, d\mu(r)\, d\nu(t),$$

where $f(r)$, $g(t)$ are, in fact, the marginal densities of r, t with respect to the σ-finite measures μ, ν respectively, and an *alternative* measure

$$M_2 \qquad \Pr\{(r, t) \in R\} = \int_R p(r, t)\, d\mu(r)\, d\nu(t).$$

We assume that M_2 is absolutely continuous with respect to M_1, that is, $p(r, t)/f(r)g(t)$ is finite almost everywhere $\mu \times \nu$.

In the application we have in mind, r will be like a "rank statistic" and t will be like an "order statistic". For the problem of randomness, r would be the ranks $(r_1, \cdots, r_n)$ of the n observations $(x_1, \cdots, x_n)$, and t would be the order statistics $(x_{(1)}, \cdots, x_{(n)})$ where $x_{(1)} \cdots x_{(n)}$ are the numbers $x_1, \cdots, x_n$ arranged in order of magnitude. It is of interest to remark in passing that the term order statistic can be misleading; in calculating the "order statistic" one *loses* precisely the information on the "order" in which the different values occur in the sample.

THEOREM. *The marginal density function for r under M_2 is*

$$(1) \qquad f(r)E_{M_1}\left\{\frac{p(r, t)}{f(r)g(t)}\,\middle|\, r\right\},$$

which is the hypothesis density adjusted by the expectation of the density ratio taken with respect to the marginal distribution of t under the hypothesis.

PROOF. The proof is trivial. The marginal density function for r is

$$\int_{\mathfrak{X}_t} p(r, t)\, d\nu(t) = f(r) \int_{\mathfrak{X}_t} \frac{p(r, t)}{f(r)g(t)}\, g(t)\, d\nu(t)$$
$$= f(r)E_{M_1}\left\{\frac{p(r, t)}{f(r)g(t)}\,\middle|\, r\right\}.$$

This theorem can be of use whenever there is a hypothesis measure for which the marginal distribution of t is simple enough that the expectation can be evaluated or approximated easily.

EXAMPLE. Consider the problem of location. Let $x = (x_1, \cdots, x_n)$ be a sample of n from an absolutely continuous distribution, having density function $f(x)$ on the real line. For the problem

$$(2) \qquad \begin{aligned} &\text{Hypothesis: Median } \{f(x)\} = 0 \\ &\text{Alternative: Median } \{f(x)\} > 0, \end{aligned}$$

the sign test is uniformly most powerful. Changing the hypothesis, we obtain the problem of location and symmetry

$$\text{(3)} \qquad \begin{aligned} &\text{Hypothesis: } f(x) \text{ symmetric about } x = 0 \\ &\text{Alternative: } f(x) \text{ nonsymmetric about } x = 0. \end{aligned}$$

This problem has a larger class of tests of a given size and the sign test does not remain uniformly most powerful even with a one-sided alternative. Wilcoxon [5] has proposed a sign rank test.

We consider the formulation (3). Any topological transformation of the positive axis coupled with the same transformation of the negative axis obviously leaves the problem unchanged. It is reasonable then to examine invariant tests and in particular to find the maximal invariant function. Let $|x|_{(1)}, \cdots, |x|_{(n)}$ designate respectively the smallest, $\cdots$, the largest of the n values $|x_1|, \cdots, |x_n|$. Also, let $s_1, \cdots, s_n$ be the signs respectively of the x's producing $|x|_{(1)}, \cdots, |x|_{(n)}$. We take

$$r = r(\mathbf{x}) = (s_1, \cdots, s_n)$$
$$t = t(\mathbf{x}) = (|x|_{(1)}, \cdots, |x|_{(n)}).$$

(r, t) does not provide coordinates on the whole sample space R^n; however it does provide coordinates on the sample space of $x_{(1)}, \cdots, x_{(n)}$ which is a sufficient statistic for the problem (the region having any coordinates equal $x_{(i)} = x_{(j)}$ has measure zero and is disregarded). r is the maximal invariant function. t is a sufficient statistic for the measures of the hypothesis in (3). The sample space of r has 2^n points and they have equal probability under the hypothesis.

To find an invariant test that is most powerful for a specific alternative, we need the marginal distribution of r under the alternative, and we can obtain this from the theorem above. An alternative of interest is the normal distribution with mean μ and variance σ^2. A reasonable hypothesis distribution to use with this for the theorem would be the normal distribution with mean 0 and variance σ^2. We evaluate Pr $\{s_1, \cdots, s_n\}$. Obviously, this depends only on μ/σ. Acccordingly, we set $\mu/\sigma = \delta$ and work with normal distributions having unit variance.

$$\Pr\{s_1, \cdots, s_n\} = \frac{1}{2^n} E\left\{\frac{\text{alternative density}}{\text{hypothesis density}} \middle| t\right\}$$
$$= \frac{1}{2^n} \int_{0<x_1<\cdots<x_n} \cdots \int \frac{n!(2\pi)^{\frac{-n}{2}} \exp\left\{-\dfrac{\sum(s_i x_i - \delta)^2}{2}\right\}}{n!(2\pi)^{\frac{-n}{2}} \exp\left\{-\dfrac{\sum(s_i x_i)^2}{2}\right\}}$$
$$\cdot\, 2^n n!(2\pi)^{\frac{-n}{2}} \exp\left\{-\frac{\sum x_i^2}{2}\right\} \pi\, dx_i$$

$$= \frac{1}{2^n} e^{\frac{-n\delta^2}{2}} \iint\limits_{0<x_1<\cdots<x_n} \exp\{\delta \sum s_i x_i\}\, 2^n n!(2\pi)^{\frac{-n}{2}} \exp\left\{-\frac{\sum x_i^2}{2}\right\} \pi\, dx_i$$

$$= \frac{1}{2^n} e^{\frac{-n\delta^2}{2}} E\{e^{\delta \Sigma s_i |z|_{(i)}}\},$$

where $|z|_{(1)}, \cdots, |z|_{(n)}$ are the order statistics for a sample of absolute values from the standardized normal distribution. By applying the fundamental lemma of hypothesis testing, we find the most powerful test has test statistic

$$E\{e^{\delta \Sigma s_i |z|_{(i)}}\},$$

a function of $s_1, \cdots, s_n$. For δ small this can be approximated by

$$E\{1 + \delta \sum s_i |z|_{(i)}\} = 1 + \delta \sum s_i E\{|z|_{(i)}\}. \tag{4}$$

An equivalent statistic is

$$\sum s_i E\{|z|_{(i)}\}. \tag{5}$$

This is the Wilcoxon test statistic, $\sum s_i i$, with ranks replaced by expected order statistics for a sample of absolute values from the standardized normal. The limiting distribution of (5) under the hypothesis can be shown to be asymptotically normal by the use of the central limit theorem and a result of Hoeffding [6].

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