THE COMPARISON OF THE SENSITIVITIES OF SIMILAR EXPERIMENTS: THEORY[1]

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0. Summary. The comparison of the sensitivities of experiments using different scales of measurement or different experimental techniques can be effected through a comparison of noncentral variance ratios. The distribution of the ratio of two noncentral variance ratios is obtained and its properties are discussed. Based on this distribution, tests of hypotheses on the parameters on noncentrality of two noncentral variance-ratio distributions are developed.

It is shown that the distribution of the ratio of two noncentral variance ratios may be approximated adequately by the distribution of the ratio of two central variance ratios with appropriately adjusted degrees of freedom. A table for use in applications of the latter distribution is given for one-sided tests at the 5% level of significance.

Through the association of the distribution of the multiple correlation coefficient in regression models with that of the noncentral variance ratio, it was also possible to develop test procedures on multiple correlation coefficients.

Much of the discussion in this paper is on comparisons of similar experiments in the sense that variance ratios with the same degrees of freedom are compared. However, it is shown how these results may be generalized for comparisons of dissimilar experiments.

1. Introduction. The problem of comparing different scales of measurement for experimental results was discussed by Cochran [1] in considerable detail in 1943. He assumed that analysis of variance techniques were applicable and confined his attention to the case in which all scales measure the same experiment. It was noted that a comparison of the sensitivities of two scales should depend both on the experimental errors associated with them and on the magnitudes of the treatment effects in the scales. In the concluding section of his paper Cochran indicated how a result of Pitman [7] may be used to compare the sensitivities of two scales in two-treatment experiments and went on to state that in general the comparison should depend on a test of significance of a hypothesis on the parameters of two noncentral variance-ratio distributions. It is this problem that is considered in this paper. The results will be useful not only in comparing scales of measurement per se but in comparing different experimental techniques in a broader sense in similar experiments and, under certain conditions, in comparing two population multiple correlation coefficients.

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Let $\dot{F}$ be a noncentral variance ratio with $2a$ and $2b$ degrees of freedom (it is convenient to use the degrees of freedom in this form; a and b may be integers or half-integers and no generality is lost). Then we define

$$(1) \qquad \dot{u} = 2a\dot{F}/2b$$

and $\dot{u}$ has the density function,

$$(2) \qquad \dot{f}(\dot{u}; a, b, \lambda) = f(\dot{u}; a, b)e^{-\lambda}{}_1F_1[a + b, a, \lambda\dot{u}/(1 + \dot{u})].$$

Here λ is the parameter of noncentrality associated with $\dot{F}$ and $\dot{u}$.

$$(3) \qquad f(u; a, b) = \dot{f}(u; a, b, 0) = [B(a, b)]^{-1}u^{a-1}(1 + u)^{-(a+b)},$$

where B denotes the beta function, and

$${}_1F_1(\alpha, \beta, x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)}\frac{x^2}{2!} + \cdots$$

is the confluent hypergeometric function.[3] If u is related to a central variance-ratio F through (1), $f(u; a, b)$ is the density function of u. References [3], [13], and [11] are noted for discussions bearing on derivations of (2).

In Model I of the analysis of variance with so-called fixed parameters in the additive model, if τ_i is the effect of the ith of t treatments, $\sum_{i=1}^{t} \tau_i = 0$, and, if σ is the population experimental error,

$$(4) \qquad \lambda = k\sum_{i=1}^{t} \tau_i^2/2\sigma^2,$$

where k is the number of observations in each treatment mean. We see at once that λ is a parameter incorporating both the experimental error associated with the scale and the magnitudes of treatment effects in the scale. We take $\dot{u}_1$ and $\dot{u}_2$ as defined in (1) to be the appropriate statistics for two similar experiments on which to base comparisons of the sensitivities of the two scales or experimental techniques. We shall consider the distribution of

$$(5) \qquad \dot{w} = \dot{u}_1/\dot{u}_2$$

under the assumption that the two similar experiments are independent.

The term "similar experiments" has been used in the sense that F-ratios for treatment comparisons resulting from them have the same degrees of freedom. When, in addition, $k_1 = k_2$, k_i, the number of observations in each treatment mean of experiment i, we shall call the two experiments identical. Major emphasis will be placed on the distribution of $\dot{w}$ when $\dot{u}_1$ and $\dot{u}_2$ arise from identical experiments. Then the null hypothesis of equal sensitivities is equivalent to the hypothesis that the two parameters of noncentrality, λ_1 and λ_2, associated with $\dot{u}_1$ and $\dot{u}_2$ are equal. When the experiments are similar but not identical, the hypothesis should be on the equality of λ_1/k_1 and λ_2/k_2.

[3] We shall use the "dot" notation to indicate variates and distributions associated with noncentral valiance ratios. Functional forms will be abbreviated, e.g., $\dot{f}(\dot{u}; a, b, \lambda)$ to $\dot{f}(\dot{u})$, whenever it may be done without confusion.

While values of the distribution function of $\dot{w}$ will be obtained for small a and b, adequate tabulation of the distribution would be difficult with ordinary computing facilities and would result in a three-parameter classification even for a specified level of significance. Similar difficulties entered in the tabulation of the distribution function of $\dot{u}$ although tables and charts are available ([11], [4], [6], and [12]) and Patnaik [5] proposed an approximation in that case. Patnaik essentially approximated to the density $\dot{f}(\dot{u};\ a,\ b,\ \lambda)$ using the density $(1/K)f\left(\frac{\dot{u}}{K};\ a',\ b\right)$; the two densities for $\dot{u}$ were given equal first and second moments through choice of K and a'. In our notation,[4] it is required that

$$a' = (a + \lambda)^2/(a + 2\lambda) \quad \text{and} \quad K = (a + \lambda)/a. \tag{6}$$

In this paper we consider approximating to both the distributions of $\dot{u}_1$ and $\dot{u}_2$ following Patnaik's method and then obtain the distribution of $\dot{w}$ on the assumption that $\dot{w}$ is the ratio of two independent variates with distributions of the form (3). The distribution of the ratio of central variates,

$$w = u_1/u_2, \tag{7}$$

is then of interest. It will be shown that the distribution function of w with $2a'$ and $2b$ degrees of freedom is a good approximation to the distribution function of $\dot{w}$ with $2a$ and $2b$ degrees of freedom given $\lambda_1 = \lambda_2 = \lambda$ on the basis of a comparison of available percentage points of the two distributions.

Values of w_0 such that $P(w \geqq w_0) = .05$ have been tabulated for ranges of values of a' and b. The computation of such tables will be discussed.

2. The distribution of the ratio of similar noncentral variance ratios. The marginal distribution of $\dot{w}$ in (5) is obtainable from the joint distribution of $\dot{u}_1$ and $\dot{u}_2$ written as the product of two expressions like (2) on the assumption of independence of $\dot{u}_1$ and $\dot{u}_2$ and given similar experiments in that both variates depend on $2a$ and $2b$ degrees of freedom. With the specification of f and ${}_1F_1$ in (2), this joint density function is[5]

$$e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} [B(a + r, b)B(a + s, b)]^{-1} \dot{u}_1^{a+r-1} \dot{u}_2^{a+s-1} \cdot (1 + \dot{u}_1)^{-(a+b+r)}(1 + \dot{u}_2)^{-(a+b+s)}, \qquad 0 \leqq \dot{u}_1, \dot{u}_2 \leqq \infty.$$

When we define new variates $\dot{w}$ and x through the relations

$$\dot{u}_1 = \dot{w}(x - 1)/(\dot{w} - x) \quad \text{and} \quad \dot{u}_2 = (x - 1)/(\dot{w} - x),$$

$\dot{w}$ has the definition (5) and we may write down the joint distribution of $\dot{w}$ and x. From that joint distribution it is at once evident that the marginal distribution of $\dot{w}$ is

[4] Patnaik's notation differs from that used here. His degrees of freedom, ν_1, ν_2, and ν, correspond to $2a$, $2b$, and $2a'$, respectively; his λ is twice our λ. K in our notation corresponds to Patnaik's k.

[5] r and s take values $0, \dots, \infty$ unless otherwise specified.

$$\dot{g}(\dot{w}; a, b, \lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} [B(a + r, b)B(a + s, b)]^{-1} \cdot \dot{w}^{a+r-1} H(\dot{w}; a, b, r, s), \qquad 0 \leqq \dot{w} \leqq \infty, \tag{8}$$

where

$$H(\dot{w}) = (\dot{w} - 1)^{-(2a+2b+r+s-1)} \int_1^{\dot{w}} (x - 1)^{2a+r+s-1}(\dot{w} - x)^{2b-1} x^{-(a+b+r)}\, dx. \tag{9}$$

The form (8) is adequate for all values of $\dot{w}$ including $\dot{w} = 1$ in view of the following form (10) for $H(\dot{w})$.

The integral (9) appearing in (8) may be written in terms of hypergeometric functions following Erdélyi ([2], p. 115). Then with the transformations, $y = (x - 1)/(\dot{w} - x)$ when $0 \leqq \dot{w} \leqq 1$, and $y = \dot{w}(x - 1)/(\dot{w} - x)$ when $1 \leqq \dot{w} \leqq \infty$,

$$\begin{aligned} H(\dot{w}) &= \int_0^{\infty} y^{2a+r+s-1}(1 + \dot{w}y)^{-(a+b+r)}(1 + y)^{-(a+b+s)}\, dy \\ &= B(2a + r + s, 2b)\,{}_2F_1(a + b + r, 2a + r + s, \\ &\qquad 2a + 2b + r + s, 1 - \dot{w}), \qquad 0 \leqq \dot{w} \leqq 1, \end{aligned} \tag{10}$$

and

$$\begin{aligned} H(\dot{w}) &= \dot{w}^{-(2a+r+s)} \int_0^{\infty} y^{2a+r+s-1}\left(1 + \frac{y}{\dot{w}}\right)^{-(a+b+s)} (1 + y)^{-(a+b+r)}\, dy \\ &= \dot{w}^{-(2a+r+s)} B(2a + r + s, 2b)\,{}_2F_1[a + b + s, \\ &\qquad 2a + r + s, 2a + 2b + r + s, (\dot{w} - 1)/\dot{w})], \qquad 1 \leqq \dot{w} \leqq \infty. \end{aligned} \tag{11}$$

$H(\dot{w})$ is expressed in the two forms for convenience; $H(1)$ may be obtained from either form and there is no discontinuity for $H(\dot{w})$ at $\dot{w} = 1$. The integrals in (10) and (11) hold for $0 \leqq \dot{w} \leqq \infty$; the division is made only to obtain expressions in terms of convergent hypergeometric series. Final forms for $\dot{g}(\dot{w})$ could now be obtained by substitution for $H(\dot{w})$ in (8).

From the specification of $H(\dot{w})$ in (9) it is clear that we could obtain a finite series expansion using binomial expansions in the integrand for $2a$, $2b$, r, and s are integers. These finite sums can of course be related to the functions ${}_2F_1$ in (10) and (11).

3. Properties of the distribution, $\dot{g}(\dot{w})$. (i) *Bounds and values of $\dot{g}(\dot{w})$.* When $\dot{w} \geqq 1$, with the replacement of $(1 + \dot{w}y)$ by $(1 + y)$ in the denominator of the integrand of (10), it is apparent that $H(\dot{w}) \leqq B(2a + r + s, 2b)$ and then, from (8)

$$\dot{g}(\dot{w}) \leqq e^{-\lambda_1-\lambda_2}\dot{w}^{a-1} \sum_r \sum_s \frac{(\dot{w}\lambda_1)^r \lambda_2^s}{r!s!} B(2a + r + s, 2b) \cdot [B(a + r, b)B(a + s, b)]^{-1}, \qquad 1 \leqq \dot{w} \leqq \infty. \tag{12}$$

Similarly, when $0 \leqq \dot{w} \leqq 1$, with the replacement of $[1 + (y/\dot{w})]$ by $(1 + y)$ in the denominator of the integrand of (11), we have

$$H(\dot{w}) \leqq \dot{w}^{-(2a+r+s)} B(2a + r + s, 2b)$$

and again from (8)

$$\dot{g}(\dot{w}) \leqq e^{-\lambda_1 - \lambda_2} \dot{w}^{-(a+1)} \sum_r \sum_s \frac{\lambda_1^r (\lambda_2/\dot{w})^s}{r!s!} B(2a + r + s, 2b) \cdot [B(a + r, b)B(a + s, b)]^{-1}, \qquad 0 \leqq \dot{w} \leqq 1. \tag{13}$$

But $B(2a + r + s, 2b)[B(a + r, b)B(a + s, b)]^{-1} = B(a + b + r, a + b + s) \cdot [B(b, b)B(a + r, a + s)]^{-1} < 1/B(b, b)$ and

$$\dot{g}(\dot{w}) < e^{\lambda_1(\dot{w}-1)} \dot{w}^{a-1}/B(b, b), \qquad 1 \leqq \dot{w} \leqq \infty, \tag{14}$$

and

$$\dot{g}(\dot{w}) < e^{\lambda_2(1-\dot{w})/\dot{w}}/\dot{w}^{a+1}B(b, b), \qquad 0 \leqq \dot{w} \leqq 1. \tag{15}$$

Now convergence of the series for $\dot{g}(\dot{w})$ is established for $0 < \dot{w} < \infty$ for all terms of that series are positive.

Limits for $\dot{g}(\dot{w})$ as $\dot{w} \to 0$ and as $\dot{w} \to \infty$ may be obtained. Returning to (8) and (9), we note that

$$\begin{aligned} \dot{w}^{a+r-1}H(\dot{w}) &= \frac{\dot{w}^{a+r-1}(-1)^{2b}}{(\dot{w} - 1)^{2a+2b+r+s-1}} \int_{\dot{w}}^1 (x - 1)^{2a+r+s-1}(x - \dot{w})^{2b-1} x^{-(a+b+r)}\, dx \\ &< \frac{\dot{w}^{\xi}}{(1 - \dot{w})^{2a+2b+r+s-1}} \int_{\dot{w}}^1 (1 - x)^{2a+r+s-1} x^{b-2-\xi}\, dx, \\ &\qquad \dot{w} < 1, \xi > 0, a > 1 + \xi, \\ &< \frac{\dot{w}^{\xi}}{(1 - \dot{w})^{2a+2b+r+s-1}} B(2a + r + s, b - 1 - \xi), \\ &\qquad \dot{w} < 1, \xi > 0, a, b > 1 + \xi. \end{aligned} \tag{16}$$

Hence $\lim_{\dot{w}\to 0} \dot{w}^{a+r-1}H(\dot{w}) = 0$ if $a, b > 1$, for ξ is at our disposal. This implies that $\lim_{\dot{w}\to 0} \dot{g}(\dot{w}) = 0$ if $a, b > 1$, which will be the case in all practical situations. A similar argument shows that $\lim_{\dot{w}\to\infty} \dot{g}(\dot{w}) = 0$ if $a, b > 1$.

In summary,

$$\begin{aligned} &\dot{g}(0) = 0, \qquad a, b > 1, \\ &\dot{g}(1) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} B(2a + r + s, 2b)[B(a + r, b)B(a + s, b)]^{-1} \\ &\qquad < 1/B(b, b), \\ &\dot{g}(\infty) = 0, \qquad a, b > 1, \end{aligned} \tag{17}$$

and the series for $\dot{g}(\dot{w})$ converges for $0 \leqq \dot{w} \leqq \infty$. When $\lambda_1 = \lambda_2$, as will later be required, $P(\dot{u}_1 \leqq \dot{u}_2) = P(\dot{u}_2 \leqq \dot{u}_1) = \frac{1}{2}$ and we see that $\dot{g}(\dot{w})$ then has a median at $\dot{w} = 1$. In general $\dot{g}(\dot{w})$ is a unimodal distribution with mode between 0 and 1 and approaches 0 as $\dot{w} \to 0$ and as $\dot{w} \to \infty$.

(ii) *Moments of* $\dot{g}(\dot{w})$. The kth moment of $\dot{w}$, $E(\dot{w}^k) = \dot{\mu}_k'$ can be obtained most easily from the joint distribution of $\dot{u}_1$ and $\dot{u}_2$ and the definition (5) for $\dot{w}$. Then it is a simple problem in integration to show that

$$\begin{aligned}\dot{\mu}_k' &= e^{-\lambda_1-\lambda_2} \frac{\Gamma(b+k)\Gamma(b-k)}{\{\Gamma(b)\}^2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} \frac{B(a+k+r, a-k+s)}{B(a+r, a+s)} \\ &= e^{-\lambda_1-\lambda_2} \frac{\Gamma(b+k)\Gamma(b-k)}{\{\Gamma(b)\}^2} \frac{\Gamma(a+k)\Gamma(a-k)}{\{\Gamma(a)\}^2} {}_1F_1(a+k, a, \lambda_1) \\ &\qquad \cdot {}_1F_1(a-k, a, \lambda_2)\end{aligned} \tag{18}$$

for $a, b > k$. When $k = 0$, it is clear that $\dot{\mu}_0' = 1$.

When $k = 1$, we sum (18) first with respect to r and obtain

$$\dot{\mu}_1' = e^{-\lambda_2} \frac{b}{b-1} (a + \lambda_1) \left[\frac{1}{a-1} + \frac{\lambda_2}{a} + \frac{\lambda_2^2}{2(a+1)} + \cdots\right].$$

This may be rewritten as the integral (obtained by multiplying each term in the series by λ_2^{a-1} and noting that it is an integral of $\lambda_2^{a-2}e^{\lambda_2}$ followed by some reduction),

$$\dot{\mu}_1' = \frac{b}{b-1} (a + \lambda_1) \int_0^1 y^{a-2} e^{\lambda_2(y-1)}\, dy. \tag{19}$$

Expansion of the exponential in the integrand of (19) and subsequent integration yields

$$\dot{\mu}_1' = \frac{b(a+\lambda_1)}{(a-1)(b-1)} \left[1 - \frac{\lambda_2}{a} + \frac{\lambda_2^2}{a(a+1)} - \cdots\right], \qquad a, b > 1. \tag{20}$$

In the special case where $a = 2$, it is apparent from (19) that

$$\dot{\mu}_1' = b(2 + \lambda_1)(1 - e^{-\lambda_2})/(b-1)\lambda_2 .$$

When $k = 2$, reduction similar to that when $k = 1$ yields

$$\begin{aligned}\dot{\mu}_2' &= a(a+1)b(b+1)[(a-1)(a-2)(b-1)(b-2)]^{-1} \\ &\cdot [1 + 2\lambda_1/a + \lambda_1^2/a(a+1)][1 - 2\lambda_2/a + 3\lambda_2^2/a(a+1) - \cdots], \\ &\qquad a, b > 2.\end{aligned} \tag{21}$$

When a and/or b is $\leqq 1$, $\dot{\mu}_k'$, $k \geqq 1$, does not exist; when a and/or $b \leqq 2$, $\dot{\mu}_k'$, $k \geqq 2$, does not exist.

4. The distribution of the ratio of noncentral variance ratios (general). While the distribution of $\dot{w}$ obtained in Sec. 2 will usually be the one required in practical work, it is not much more difficult to obtain more general results.

Consider independent variates $\dot{F}_1$ and $\dot{F}_2$, $\dot{u}_1 = a_1\dot{F}_1/b_1$, $\dot{u}_2 = a_2\dot{F}_2/b_2$, and $\dot{w} = c\dot{u}_1/\dot{u}_2$ where $\dot{F}_1$ and $\dot{F}_2$ are noncentral variance ratios with degrees of freedom, $2a_1$ and $2b_1$, and $2a_2$ and $2b_2$ with $c = a_2b_1/a_1b_2$ and parameters of noncentrality, λ_1 and λ_2, respectively. $\dot{u}_1$ and $\dot{u}_2$ will then have distributions of the form (2). An argument similar to that of Sec. 2 yields for the density function of $\dot{w}$

$$\dot{g}(\dot{w}; a_1, b_1, a_2, b_2, \lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} [B(a_1 + r, b_1) \cdot B(a_2 + s, b_2)]^{-1} c^{-(a_1+r)} \dot{w}^{a_1+r-1} H(\dot{w}; a_1, b_1, a_2, b_2, r, s) \tag{22}$$

where

$$H(\dot{w}) = \int_0^\infty y^{a_1+a_2+r+s-1}(1 + \dot{w}y/c)^{-(a_1+b_1+r)}(1 + y)^{-(a_2+b_2+s)}\, dy \tag{23}$$

similar to (10). The integral (23) can be used to express $H(\dot{w})$ in terms of a hypergeometric series when $0 \leqq \dot{w} \leqq c$ and it can be transformed to a form similar to (11) and thence in terms of a hypergeometric series when $c \leqq \dot{w} \leqq \infty$.

This general case is not of interest in comparing the sensitivities of experiments except as it reduces to the case of similar experiments and possibly when $a_1 = a_2$, $b_1 \neq b_2$.

5. The distribution of the ratio of central variance ratios and its properties. We have already indicated in Sec. 1 that we shall approximate to the distribution of the ratio of two independent noncentral variance ratios using the distribution of the ratio of two independent central variance ratios. The necessary distribution and its properties may be considered by the specialization $\lambda_1 = \lambda_2 = 0$ in results given above. We now summarize those results for this special case for they will be required in following sections.[6]

General results (22) and (23) now become

$$g(w; a_1, b_1, a_2, b_2) = c^{-a_1}[B(a_1, b_1)B(a_2, b_2)]^{-1} w^{a_1-1} H(w; a_1, b_1, a_2, b_2), \quad 0 \leqq w \leqq \infty, \tag{24}$$

where

$$H(w) = \int_0^\infty y^{a_1+a_2-1}(1 + wy/c)^{-(a_1+b_1)}(1 + y)^{-(a_2+b_2)}\, dy. \tag{25}$$

We consider in more detail the case for which F_1 and F_2 arise from similar experiments. Then, with $a_1 = a_2 = a$, $b_1 = b_2 = b$, we have

$$g(w; a, b) = w^{a-1}H(w; a, b)/[B(a, b)]^2, \qquad 0 \leqq w \leqq \infty, \tag{26}$$

with

$$H(w) = \int_0^\infty y^{2a-1}[(1 + wy)(1 + y)]^{-(a+b)}\, dy. \tag{27}$$

[6] We retain the notation already adopted but drop the "dot" when discussing central variates and their distributions.

$H(w)$ in (27) may be rewritten from (10), (11), and (9) respectively as

$$
\begin{aligned}
H(w) &= B(2a, 2b)\,{}_2F_1(a + b, 2a, 2a + 2b, 1 - w), \qquad 0 \leqq w \leqq 1 \\
&= w^{-2a}B(2a, 2b)\,{}_2F_1[a + b, 2a, 2a + 2b, (w - 1)/w], \qquad 1 \leqq w \leqq \infty \qquad (28)\\
&= (w - 1)^{-(2a+2b-1)} \int_1^w (x - 1)^{2a-1}(w - x)^{2b-1}x^{-(a+b)}\, dx, \qquad 0 \leqq w \leqq \infty.
\end{aligned}
$$

We now list certain special cases of $g(w)$ that were used to some extent in checking tables prepared by more general methods in Sec. 8. These results are most readily obtained from the third form of (28).

Parameters	$g(w)$
(i) $a = \frac{1}{2}, b = \frac{1}{2}$.	$\ln w/\pi^2\sqrt{w}\,(w - 1)$
(ii) $a = \frac{1}{2}, b = 1$.	$1/2\sqrt{w}(1 + \sqrt{w})^2$
(iii) $a = \frac{1}{2}, b = \frac{3}{2}$.	$4(w^2 - 1 - 2w \ln w)/\pi^2\sqrt{w}(w - 1)^3$
(iv) $b = a + \frac{1}{2}$.	$[2B(2a, 2a)]^{-1}w^{a-1}(1 + \sqrt{w})^{-4a}$

(v) $a = \frac{1}{2}$, $(a + b)$ an integer. $\frac{1}{2}e^{-2x}\left[\dfrac{(-1)^{b-\frac{1}{2}}2x}{\pi B(\frac{1}{2}, b)\sinh^{2b} x} + \dfrac{\cosh x}{\pi}\sum_{i=1}^{b-\frac{1}{2}}(-1)^{i+1}\right.$

$$\left. \cdot \frac{B(b + \frac{1}{2}, b + \frac{1}{2} - i)}{B(b, b + 1 - i)} \sinh^{-2i} x\right], \qquad x = \tfrac{1}{2}\ln w.$$

Results similar to (iv) may be given for $b = a - \frac{1}{2}$, $b = a \pm 3/2$, $b = a \pm 5/2$ and $b = a \pm 7/2$ without much trouble.

Results on the form of $\dot{g}(\dot{w})$ carry over to $g(w)$. Thus the limit of $g(w)$ is zero as $w \to 0$ and as $w \to \infty$. In addition the median value of w is unity as it was for $\dot{w}$ with $\lambda_1 = \lambda_2$. Moments of w about zero follow from the more general case also. Now

$$\mu_k' = B(a + k, b - k)B(a - k, b + k)/[B(a, b)]^2. \qquad (29)$$

In particular,

$$\mu_1' = ab/(a - 1)(b - 1), \qquad a, b > 1, \qquad (30)$$

and

$$\mu_2' = ab(a + 1)(b + 1)/(a - 1)(b - 1)(a - 2)(b - 2), \qquad a, b > 2. \qquad (31)$$

The variance of w is

$$\sigma^2 = ab(2a^2b + 2ab^2 - a^2 - b^2 - 4ab + 1)/(a - 1)^2(b - 1)^2(a - 2)(b - 2). \qquad (32)$$

Consider $\dot{w}$ for similar experiments with $2a$ and $2b$ degrees of freedom and assume that $\lambda_1 = \lambda_2 = \lambda$. We then compare the moments of $\dot{w}$ with the moments of w for similar experiments with $2a'$ and $2b$ degrees of freedom with $a' = (a + \lambda)^2/(a + 2\lambda)$ as given in (6). From (30) it now follows that

$$\mu_1' = (a+\lambda)b/(a-1)(b-1)\left[1 + \frac{\lambda(a+\lambda-1)}{(a-1)(a+\lambda)}\right]$$

$$\doteq (a+\lambda)b/(a-1)(b-1)(1+\lambda/a) \qquad \text{for large } a.$$

Also, from (20),

$$\dot{\mu}_1' = (a+\lambda)b\left[1 - \frac{\lambda}{a} + \frac{\lambda^2}{a(a+1)} - \cdots\right] \Big/ (a-1)(b-1)$$

$$\doteq (a+\lambda)b/(a-1)(b-1)(1+\lambda/a) \qquad \text{for large } a.$$

Hence μ_1' and $\dot{\mu}_1'$ are approximately equal under the stated conditions for large a. Similarly it may be shown that, for large a,

$$\mu_2' \doteq \dot{\mu}_2' \doteq b(b+1)(a^2+2a\lambda+\lambda^2+a+2\lambda)/(a-1)(b-1) \cdot (a-2)(b-2)(1+\lambda/a)^2.$$

6. The probability integral of $\dot{g}(\dot{w})$. We first consider $\dot{w}$ based on similar experiments, $\dot{g}(\dot{w})$ in (8), and turn our attention to means of evaluating $\dot{G}(\dot{w}_0) = P(\dot{w} \leqq \dot{w}_0) = \int_0^{\dot{w}_0} \dot{g}(\dot{w})\, d\dot{w}$. With interchange of order of integration and the definition of $H(\dot{w})$ in (10), it follows that

$$\dot{G}(\dot{w}_0; a, b, \lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} [B(a+r,b)B(a+s,b)]^{-1} \cdot \int_0^\infty y^{a+s-1}(1+y)^{-(a+b+s)} \int_0^{\dot{w}_0} (y\dot{w})^{a+r-1}(1+y\dot{w})^{-(a+b+r)} y\, d\dot{w}\, dy. \tag{33}$$

Transformation from $\dot{w}$ through setting $x = \dot{w}y/(1+\dot{w}y)$ and integration with respect to x of the resulting integrand, $x^{a+r-1}(1-x)^{b-1}$, following expansion in binomial series allows us to write

$$\dot{G}(\dot{w}_0) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} [B(a+r,b)B(a+s,b)]^{-1} \cdot \int_0^\infty y^{a+s-1}(1+y)^{-(a+b+s)}(\dot{w}_0 y)^{a+r}(1+\dot{w}_0 y)^{-(a+r)} \left[\frac{1}{a+r} - \frac{(b-1)\dot{w}_0 y}{(a+r+1)(1+\dot{w}_0 y)} + \frac{(b-1)(b-2)(\dot{w}_0 y)^2}{2!(a+r+2)(1+\dot{w}_0 y)^2} - \cdots\right] dy. \tag{34}$$

When $b \geqq 1$ is an integer, the series in square brackets will be finite. Furthermore, it has been shown [8] that

$$\dot{G}(\dot{w}_0; a, n, \lambda_1, \lambda_2) \leqq \dot{G}(\dot{w}_0; a, n+\tfrac{1}{2}, \lambda_1, \lambda_2) \leqq \dot{G}(\dot{w}_0; a, n+1, \lambda_1, \lambda_2)$$

for n an integer. We shall then only consider the evaluation of (33) or (34) when b is an integer and obtain values of $\dot{G}(\dot{w}_0)$ by interpolation when b is not an integer. This can be attempted in several ways.

(i) *Direct evaluation of* $\dot{G}(\dot{w}_0)$ *from* (34). Evaluation of $\dot{G}(\dot{w}_0)$ from (34) will depend on the evaluation of integrals of the form

$$h(\dot{w}_0\,;\,m,n,p) = \int_0^\infty y^{m-1}(1+y)^{-n}(1+\dot{w}_0 y)^{-p}\,dy \tag{35}$$

like $H(w)$ formerly defined and where for fixed r and s in (34), $m = 2a + r + s + j$, $n = a + b + s$, $p = a + r + j$, $j = 0, \cdots, (b-1)$, b an integer. As for $H(w)$, $h(\dot{w}_0)$ may be written in terms of hypergeometric series,

$$h(\dot{w}_0) = B(m, n+p-m)\,{}_2F_1(p, m, n+p, 1-\dot{w}_0), \qquad 0 \leqq \dot{w}_0 \leqq 1,$$

and

$$h(\dot{w}_0) = \dot{w}_0^{-m}B(m, n+p-m)\,{}_2F_1[n, m, n+p, (\dot{w}_0-1)/\dot{w}_0], \qquad 1 \leqq \dot{w}_0 \leqq \infty,$$

when $n + p - m > 0$ as will be the case here. If tables of the hypergeometric function were available, we could evaluate $\dot{G}(\dot{w}_0)$ through evaluations of $h(\dot{w}_0)$ and using a finite number of terms of (34) which may be shown to converge through a method similar to that used in Sec. 3(i).

As an alternative to the use of hypergeometric series, $h(\dot{w}_0)$ may be transformed to a form like (9) and evaluated through the use of binomial expansions in the integrand.

Use of (34) to evaluate $\dot{G}(\dot{w}_0)$ will in most instances require evaluation of a large number of terms to attain even 2- or 3-decimal accuracy.

(ii) *Integration for* $\dot{G}(\dot{w}_0)$ *after summation in* (34). When b is small (say $b \leqq 6$), the following method based on interchange of the order of summation and integration in (34) gives good accuracy in evaluating $\dot{G}(\dot{w}_0)$. Observe that

$$\sum_s \left(\frac{\lambda_2 y}{1+y}\right)^s \frac{\Gamma(a+b+s)}{s!\Gamma(a+s)} = \frac{\Gamma(a+b)}{\Gamma(a)}\,{}_1F_1\left(a+b, a, \frac{\lambda_2 y}{1+y}\right)$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)} \exp\left(\frac{\lambda_2 y}{1+y}\right) {}_1F_1\left(-b, a, -\frac{\lambda_2 y}{1+y}\right),$$

the latter form based on the use of Kummer's relation ([2], p. 253). This second confluent hypergeometric series is finite. We can also write

$$\sum_r \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)^r \frac{\Gamma(a+b+r)}{r!\Gamma(a+r)}$$
$$\cdot \left[\frac{1}{a+r} - \frac{(b-1)\dot{w}_0 y}{(a+r+1)(1+\dot{w}_0 y)} + \cdots + \frac{(-1)^{b-1}}{(a+b+r-1)}\left(\frac{w_0 y}{1+\dot{w}_0 y}\right)^{b-1}\right]$$
$$= \sum_r \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)^r \frac{1}{r!}$$
$$\cdot [P_0 + rP_1 + r(r-1)P_2 + \cdots + r(r-1)\cdots(r-b+2)P_{b-1}]$$
$$= \exp\left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)\left[P_0 + \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)P_1 + \cdots + \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)^{b-1} P_{b-1}\right],$$

where P_i, $i = 0, \cdots, (b-1)$ are polynomials in $\dot{w}_0 y/(1 + \dot{w}_0 y)$ independent of r and determined by appropriate grouping of terms in the first form above. With the substitution of the results of this paragraph in (34),

$$\dot{G}(\dot{w}_0) = \frac{\Gamma(a+b)}{\Gamma(a)[\Gamma(b)]^2} \dot{w}_0^a \int_0^\infty y^{2a-1}(1+y)^{-(a+b)}(1+\dot{w}_0 y)^{-a}$$

$$(36) \qquad \cdot \exp\left(-\frac{\lambda_1}{1+\dot{w}_0 y} - \frac{\lambda_2}{1+y}\right) {}_1F_1\left(-b, a, -\frac{\lambda_2 y}{1+\dot{w}_0 y}\right)$$

$$\cdot \left[P_0 + \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right) P_1 + \cdots + \left(\frac{\lambda_1 \dot{w}_0 y}{1+\dot{w}_0 y}\right)^{b-1} P_{b-1}\right] dy.$$

Evaluation of $\dot{G}(\dot{w}_0)$ now depends on evaluating a finite number of integrals of the form

$$(37) \qquad I(\dot{w}_0; \lambda, p, q) = \int_0^\infty \exp\left[-\lambda\left\{\frac{1}{1+\dot{w}_0 y} + \frac{1}{1+y}\right\}\right] \cdot (1+y)^{-p}(1+\dot{w}_0 y)^{-q}\, dy,$$

where p and q are integers or half-integers and when $\lambda_1 = \lambda_2 = \lambda$, the case of most immediate interest. Reduction of (36) to integrals like (37) follows when powers of y and powers of $\dot{w}_0 y$ in the numerator of the integrand of (36) are written respectively as powers of $[(1 + y) - 1]$ and $[(1 + \dot{w}_0 y) - 1]$ and expanded in finite binomial expansions. Recursion formulas reduce evaluations of (37) to forms depending on five basic integrals, $I(\dot{w}_0; \lambda, 0, 2)$, $I(\dot{w}_0; \lambda, 1, 1)$, $I(\dot{w}_0; \lambda, 3/2, \frac{1}{2})$, $I(\dot{w}_0; \lambda, \frac{1}{2}, 3/2)$ and $I(\dot{w}_0; \lambda, 5/2, \frac{1}{2})$. Some of the recursion formulas are

$$I(\dot{w}_0; \lambda, p, q) = (\dot{w}_0 - 1)^{-1}[\dot{w}_0 I(\dot{w}_0; \lambda, p-1, q) - I(\dot{w}_0; \lambda, p, q-1)], \qquad p, q \geqq 1,$$

$$I(\dot{w}_0; -\lambda, 0, q) = \lambda^{-1}[(q-2)I(\dot{w}_0; \lambda, 0, q-1) - \dot{w}_0^{-1}e^{-2\lambda}] - \dot{w}_0^{-1}I(\dot{w}_0; \lambda, 2, q-2), \qquad q > 2,$$

$$I(\dot{w}_0; \lambda, p, 0) = \lambda^{-1}[(p-2)I(\dot{w}_0; \lambda, p-1, 0) - e^{-2\lambda}] - \dot{w}_0 I(\dot{w}_0; \lambda, p-2, 2), \qquad p > 2,$$

$$I(\dot{w}_0; \lambda, 2, 0) = \lambda^{-1}[1 - e^{-2\lambda}] - \dot{w}_0 I(\dot{w}_0; \lambda, 0, 2).$$

The basic integrals may be evaluated by expanding their integrands in Taylor series about $\dot{w}_0 = 1$. The coefficients in the resultant expansions depend on integrals of the form

$$\int_0^\infty \frac{y^r}{(1+y)^{q+r}} e^{-\frac{2\lambda}{1+y}}\, dy = \int_0^1 e^{-2\lambda u}(1-u)^r u^{q+2}\, du,$$

which may be evaluated directly. When $\dot{w}_0 \geqq 2$, it is advantageous to obtain the series in an expansion in terms of $1/\dot{w}_0$ about $1/\dot{w}_0 = 1$. This is not a different problem since it is easy to show that $I(1/\dot{w}_0 ; \lambda, p, q) = \dot{w}_0 I(\dot{w}_0 ; \lambda, q, p)$. Schumann [8] has shown that if t_i, $i = 0, \cdots, r$, are terms in the Taylor expansion of $I(1/\dot{w}_0 ; \lambda, p, q)$ and if R_{r+1} is the remainder after these $(r + 1)$ terms,

$$1 > \dot{w}_0^r R_{r+1}/t_{r+1}(\dot{w}_0 - 1)^{r+1} > t_{r-1}/[\dot{w}_0 t_{r-1} - (\dot{w}_0 - 1)t_r].$$

This relationship was used to determine the accuracy of evaluations of the basic integrals. For brevity we have omitted the demonstration of the stated inequalities.

This method (ii) was used to obtain the values of $\dot{G}(\dot{w}_0)$ given in Table II and for the comparisons in Table IV. It was found that the Taylor series for the basic integrals converge slowly and as many as 18 terms were required to obtain 6 decimal accuracy in some of the basic integrals. Since the five basic integrals are used in recursion formulas, errors tend to be magnified; however the combination of integrals $I(\dot{w}_0 ; \lambda, p, q)$ required to evaluate $\dot{G}(\dot{w}_0)$ enter in such a way that errors in the $I(\dot{w}_0 ; \lambda, p, q)$ are to a large extent compensating for each other and reasonable accuracy can be obtained.

The computation required to construct even a limited table of values of $\dot{G}(\dot{w}_0)$ is very extensive and an approximate method was found.

(iii) *Approximation to* $\dot{G}(\dot{w}_0)$ *using* $G(\dot{w}_0)$. We have indicated that we shall approximate to $\dot{g}(\dot{w})$ with $\lambda_1 = \lambda_2 = \lambda$ and $2a$ and $2b$ degrees of freedom by using $g(w)$ with $2a'$ and $2b$ degrees of freedom. We shall extend this idea to yield an approximation to $\dot{G}(\dot{w}_0)$ using $G(\dot{w}_0)$ where

$$G(\dot{w}_0) = \int_0^{\dot{w}_0} g(w)\, dw. \tag{38}$$

Evaluations of $G(w_0)$ will be considered in the next section. The results in Table IV indicate that the approximation will be adequate for most practical situations.

Extension. The discussion of $\dot{G}(\dot{w}_0)$ has so far been limited to the case for similar experiments. The extension to $\dot{g}(\dot{w}_0)$ in (22) is straightforward. Difficulties in computation are however almost prohibitive.

For the general case, the form of $\dot{G}(\dot{w}_0)$ like (34) is

$$\dot{G}(\dot{w}_0 ; a_1, b_1, a_2, b_2, \lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!}$$
$$\cdot [B(a_1 + r, b_1)B(a_2 + s, b_2)]^{-1} \int_0^\infty y^{a_2+s-1}(1 + y)^{-(a_2+b_2+s)}(y\dot{w}_0/c)^{a_1+r} \tag{39}$$
$$\cdot (1 + y\dot{w}_0/c)^{-(a_1+r)} \left[\frac{1}{a_1 + r} - \frac{(b_1 - 1)y\dot{w}_0/c}{(a_1 + r + 1)(1 + y\dot{w}_0/c)} + \cdots\right] dy.$$

This distribution function has the same form as (34) and methods of evaluation discussed for (34) could therefore be applied.

7. The probability integral of $g(w)$. Results on the probability integral of $\dot{g}(\dot{w}_0)$ with $\lambda_1 = \lambda_2 = \lambda$ carry over to the probability integral of $g(w)$ when $\lambda = 0$. For similar experiments the form comparable to (34) is

$$G(w_0; a, b) = [B(a, b)]^{-2} \int_0^\infty y^{a-1}(1 + y)^{-(a+b)}(w_0 y)^a (1 + w_0 y)^{-a} \tag{40}$$
$$\cdot \left[\frac{1}{a} - \frac{(b-1)w_0 y}{(a+1)(1+w_0 y)} + \frac{(b-1)(b-2)(w_0 y)^2}{2!(a+2)(1+w_0 y)^2} - \cdots\right] dy.$$

Again the series in square brackets is finite when b is an integer.

$G(w_0)$ can be obtained in a form like (36) when b is an integer and depends on integrals

$$J(w_0; 0, p, q) = I(w_0; p, q) = \int_0^\infty (1 + y)^{-p}(1 + w_0 y)^{-q}\, dy. \tag{41}$$

For example, when $a = 2$, b an integer,

$$G(w_0) = 1 - b(b+1)[(b+1)I(w_0; b+1, b) - (b+1)I(w_0; b+2, b)$$
$$- bI(w_0; b+1, b+1) + bI(w_0; b+2, b+1)].$$

Similar expressions may be found for other values of a, given b an integer. Interpolation for b not an integer is again possible or direct evaluation may be used.

TABLE I

Values of w_0 for similar experiments such that $1 - G(w_0) = .05$

a \ b		1	2	3	4	5	6	7	8	9	10	15	∞
	d.f.	2	4	6	8	10	12	14	16	18	20	30	∞
1	2	66.12	32.76	26.76	24.37	23.10	22.31	21.77	21.39	21.10	20.87	20.21	19.00
3/2	3	40.81	18.35	14.40	12.81	11.97	11.45	11.09	10.85	10.65	10.50	10.07	9.28
2	4		13.91	10.62	9.32	8.62	8.19	7.90	7.69	7.54	7.41	7.06	6.39
5/2	5		11.82	8.87	7.70	7.11	6.68	6.42	6.23	6.09	5.96	5.64	5.05
3	6			7.86	6.77	6.18	5.78	5.54	5.37	5.25	5.13	4.82	4.28
7/2	7			7.22	6.17	5.61	5.26	5.03	4.85	4.72	4.61	4.30	3.79
4	8				5.75	5.21	4.88	4.65	4.48	4.35	4.24	3.94	3.44
9/2	9				5.45	4.92	4.59	4.36	4.19	4.07	3.97	3.68	3.18
5	10					4.70	4.37	4.14	3.98	3.86	3.76	3.48	2.98
11/2	11					4.52	4.19	3.98	3.82	3.70	3.60	3.32	2.82
6	12						4.05	3.84	3.68	3.56	3.46	3.18	2.69
13/2	13						3.94	3.72	3.57	3.45	3.35	3.07	2.58
7	14							3.63	3.47	3.36	3.26	2.98	2.49
15/2	15							3.54	3.39	3.27	3.18	2.90	2.40
8	16								3.32	3.20	3.11	2.83	2.34
17/2	17								3.26	3.14	3.05	2.77	2.28
9	18									3.09	3.00	2.72	2.22
19/2	19									3.04	2.95	2.67	2.17
10	20										2.91	2.63	2.12
21/2	21										2.87	2.59	2.08
15	30											2.36	1.85

TABLE II

Values of $\dot{w}_0$ such that $1-\dot{G}(\dot{w}_0) = .05$

d.f.		λ								
2a	2b	4	6	8	12	16	24	40	60	100
2	2	—	—	24.1	22.3	21.4	20.6	20.0	19.7	19.4
2	4	11.6	10.1	9.3	8.5	8.1	7.8	7.6	7.2	7.2
4	4	10.4	9.4	8.9	—	—	—	—	—	—
2	6	8.7	7.4	6.6	—	—	—	—	—	—

TABLE III

Values of w_0 such that $1-G(w_0\,; a', b) = .05$

d.f.		λ								
2a	2b	4	6	8	12	16	24	40	60	100
2	2	—	—	23.4	21.9	21.1	20.4	—	—	19.2
2	4	11.0	9.6	8.8	8.0	7.6	7.2	—	—	6.6
4	4	9.8	8.9	8.4	—	—	—	—	—	—
2	6	8.3	7.0	6.3	—	—	—	—	—	—

The recursion formulas previously given may be applied when λ is set equal to zero. The computation of $G(w_0)$ is considerably easier than that for $\dot{G}(\dot{w}_0)$.

In certain special cases alternative methods of evaluating $G(w_0)$ are available based on the special cases indicated in Sec. 5. For example, when $b = a + \frac{1}{2}$, $G(w_0) = I_x(2a, 2a)$ where I_x is the incomplete beta function with $x = w_0^{\frac{1}{2}}/(1 + w_0^{\frac{1}{2}})$. Other special cases with $b = a - \frac{1}{2}$, $b = a \pm \frac{3}{2}$, etc., were used as a check on some of the computing.

The general form like (40) based on (24) is

$$\begin{aligned} G(w_0\,; a_1, b_1, a_2, b_2) &= [B(a_1, b_1)B(a_2, b_2)]^{-1} \int_0^{\infty} y^{a_2-1}(1 + y)^{-(a_2+b_2)} \\ &\cdot (yw_0/c)^{a_1}(1 + yw_0/c)^{-a_1}\left[\frac{1}{a_1} - \frac{(b_1 - 1)yw_0/c}{(a_1 + 1)(1 + yw_0/c)} + \cdots\right] dy. \end{aligned} \tag{42}$$

8. Tables of $\dot{G}(\dot{w}_0)$ and $G(w_0)$.

Table I. Values of w_0 such that $1 - G(w_0) = .05$ are given in Table I for similar experiments for ranges of values of a and of b wide enough to meet most practical situations. The table is essentially restricted to values of the parameters, $a \leqq b$ for $g(w; a, b) = g(w; b, a)$ and indeed $a \leqq b$ again covers most situations commonly met.

The formulas given in Secs. 6 and 7 were used in constructing Table I, and "trial and error" methods were used to arrive at the appropriate values of w_0. Tables of the incomplete beta function were used to check certain entries in view of the special cases mentioned in Sec. 7.

TABLE IV

Bounds on $1-\dot{G}(w_0\,;a,b,\lambda)$ *for certain values of* w_0' *in Table* III

a	b	λ	w_0	$\dot{G}(w_0, a, b, \lambda)$
1	1	8	23.4	.945-.950
1	2	4	11.0	.935-.940
1	2	6	9.6	.940-.945
1	2	8	8.8	.940-.945
2	2	4	9.8	.945-.950
2	2	6	8.9	.945-.950
2	2	8	8.4	.945-.950
1	3	4	8.3	.940-.945
1	3	6	7.0	.940-.945
1	3	8	6.3	.940-.945

Table II. Some of the values of $\dot{w}_0$ such that $1 - \dot{G}(\dot{w}_0) = .05$ are given in Table II. For this table, $\lambda_1 = \lambda_2 = \lambda$, and $\dot{F}_1$ and $\dot{F}_2$ defining $\dot{w}$ were both taken to have $2a$ and $2b$ degrees of freedom. Formula (36) was used after the basic integrals $I(\dot{w}_0\,; \lambda, 0, 2)$ and $I(\dot{w}_0\,; \lambda, 1, 1)$ were evaluated in the manner described in Sec. 6(ii). Values of λ, a, b in Table II are too limited to make the table of real practical use and its main purpose is for comparison with Table III.

Table III. We approximate to $\dot{G}(\dot{w}_0\,; a, b, \lambda)$ using $G(w_0\,; a', b)$ where a' is defined in (6). This approximation has been used to obtain values of w_0 listed in Table III such that $1 - G(w_0\,; a', b) = .05$, the values being obtained by interpolation in Table I. Table III then contains values of w_0 appropriate for comparison with values of $\dot{w}_0$ in Table II. We see immediately that values of w_0 and $\dot{w}_0$ agree quite well. We can more easily assess the importance of the small differences observed by examining Table IV.

Table IV. In Sec. 6(ii), bounds on the error in computing $\dot{G}(\dot{w}_0\,; a, b, \lambda)$ were stated. In order to further compare the approximation to values of $\dot{w}_0$ in Table II by values of w_0 in Table III, we have considered the values of w_0 in the first three columns in Table III and evaluated $1 - \dot{G}(w_0\,; a, b, \lambda)$ as indicated for Table II and in Sec. 6(ii). Then bounds on $1 - \dot{G}(w_0)$ are given in Table IV. Each value of w_0 is such that $1 - G(w_0\,; a', b) = .05$. From Table IV it is clear that the values w_0 are sufficiently close to the appropriate values of $\dot{w}_0$ to be satisfactory for most purposes.

Some general comments based on Tables II, III, and IV are in order.

(i) Values of $\dot{G}(\dot{w}_0\,; a, b, \lambda)$ and $G(w_0\,; a', b)$ are fairly stable even for considerable variation in values of λ. This implies that it will have little effect in applications if we enter tables at a value of λ somewhat different from its true (and usually unknown) value.

(ii) Values given in Table III are close enough to the corresponding values in Table II, even for small degrees of freedom, to make their use meaningful. Since the construction of percentage points of $g(w)$ is much easier than the construction of such values for $\dot{g}(\dot{w})$, it was decided that Table I be constructed and that its use will be satisfactory.

9. The use of Table I.

(i) *Tests of hypotheses on sensitivity.* In the comparison of the sensitivities of identical experiments, we shall be interested in tests of the hypothesis,

$$H_0: \quad \lambda_1 = \lambda_2 = \lambda, \tag{43}$$

against one-sided and two-sided alternatives,

$$\begin{aligned} H_a\colon &(1)\ \lambda_1 > \lambda_2 \\ &(2)\ \lambda_1 \neq \lambda_2 . \end{aligned} \tag{44}$$

The test statistic is $\dot{w}$ and we limit consideration to the case of identical experiments, $\dot{w} = \dot{F}_1/\dot{F}_2$, $\dot{F}_1$ and $\dot{F}_2$ independent noncentral variance ratios with parameters λ_1 and λ_2, respectively, and $2a$ and $2b$ degrees of freedom. The test procedure will be to reject H_0 with significance level α for H_a : (1) when $\dot{w} > \dot{w}_0(\alpha)$ and to reject H_0 with significance level 2α for H_a:(2) when $\dot{w} > \dot{w}_0(\alpha)$ or $1/\dot{w} > \dot{w}_0(\alpha)$. The other one-sided alternative, H_a:$\lambda_1 < \lambda_2$, is included under H_a:(1) by interchange of definitions of $\dot{F}_1$ and $\dot{F}_2$.

It is apparent from $\dot{g}(\dot{w})$ that λ enters as a nuisance parameter in the calculation of $\dot{w}_0(\alpha)$. Fortunately $\dot{w}_0(\alpha)$ is not greatly affected by changes in λ and consequently it should be satisfactory to estimate λ by taking the average of estimates of λ_1 and λ_2. Such estimates may be found through estimating λ in (4) through equating well known expectations of mean squares to observed mean squares in the analysis of variance.

We use Table I to obtain an approximation to $\dot{w}_0(.05)$. It is necessary only to compute a' in (6) using a and the estimate of λ and then to interpolate in Table I for w_0, the required approximation to $\dot{w}_0(.05)$. Since we are at present limited to the use of Table I, we must take $\alpha = .05$.

(ii) *Tests of hypotheses on multiple correlation coefficients.* Consider R, the multiple correlation coefficient of the dependent variable on p independent variables, in usual multiple regression with assumed nonstochastic independent variables. Let R be based on N observation vectors. Then it is well known that $R^2/(1 - R^2)$ has the distribution $\dot{f}[R^2/(1 - R^2); p/2, (N - p - 1)/2, \lambda]$ defined in (2) and now

$$\lambda = (a + b)\rho^2/(1 - \rho^2), \tag{45}$$

where ρ is the population multiple correlation coefficient estimated by R.

To compare two population multiple correlation coefficients in identical regression experiments, we test

$$H_0: \quad \rho_1^2 = \rho_2^2 = \rho^2 \tag{46}$$

against

$$\begin{aligned} H_a\colon &(1)\ \rho_1^2 > \rho_2^2 \\ &(2)\ \rho_1^2 \neq \rho_2^2 . \end{aligned} \tag{47}$$

This test is identical with that of hypotheses (43) and (44) upon proper association of the parameters. We have redefined λ in (45) and now require

$$
\begin{aligned}
a &= p/2 \\
b &= (N - p - 1)/2 \qquad (48) \\
\dot{w} &= R_1^2(1 - R_2^2)/R_2^2(1 - R_1^2).
\end{aligned}
$$

λ, and hence ρ_1^2 and ρ_2^2, must again be estimated and we suggest the method proposed by Snedecor ([10], p. 348). This method is given by

$$\rho^2(\text{estimated}) = 1 - (1 - R^2)(a + b)/b \qquad (49)$$

in our notation.

It is of interest to compare values of ρ^2 in identical multiple regressions, for R^2 is commonly used as a measure of the fraction of the variation in the dependent variable explained by regression on the independent variables.

10. Concluding remarks. The main effort in this paper has been devoted to considerations on the distributions of w and $\dot{w}$ for similar experiments. We have indicated, however, the necessary generalizations for consideration of the distributions of w and $\dot{w}$ based on central or noncentral variance ratios with $2a_1$ and $2b_1$ and $2a_2$ and $2b_2$ degrees of freedom. For example, $G(w)$ in this general situation is given in (42).

The main applications of this work are in the comparison of the sensitivities of identical experiments and in the comparison of the squares of multiple correlation coefficients from identical regressions [in the sense that $(a + b)$, a and b in (45) and (48) are the same for both experiments]. Schumann [8] has suggested that, when a_1 and a_2 and b_1 and b_2 differ only slightly and are moderately large, an adequate approximation to $w_0(.05)$ may still be obtained from Table I.

Consider the more general situation with $a_1 = a_2 = a$, b_1 not assumed equal to b_2, and k_1 not assumed equal to k_2, k_1, and k_2 being values of k in (4) for the two independent experiments. Then, for a test on sensitivities, we would take the null hypothesis to be

$$H_0\colon \quad \lambda_1/k_1 = \lambda_2/k_2 = \Lambda. \qquad (50)$$

Our test statistic would be $\dot{w} = \dot{F}_1/\dot{F}_2$ where $\dot{F}_i$, $i = 1, 2$, is a noncentral variance ratio with $2a$ and $2b_i$ degrees of freedom and parameter of noncentrality λ_i. If $\dot{F}_i$, $i = 1, 2$, is distributed approximately like K_iF_i, written $\dot{F}_i \sim K_i F_i$, where F_i is taken to have the central variance-ratio distribution with $2a_i' = 2(a + \lambda_i)^2/(a + 2\lambda_i)$ and $2b_i$ degrees of freedom, we take

$$
\begin{aligned}
\dot{w} \sim K_1F_1/K_2F_2 &= (a + k_1\Lambda)F_1/(a + k_2\Lambda)F_2 \qquad (51) \\
&= cu_1/u_2,
\end{aligned}
$$

where

$$c = b_1(a + k_2\Lambda)(a + 2k_1\Lambda)/b_2(a + k_1\Lambda)(a + 2k_2\Lambda) \qquad (52)$$

in view of (50), (6), and the definition for u parallel with (1). For the test on sensitivities, we suggest the following procedure based on the apparent stability of values of w_0 in Table I, but the suitability of this procedure is not now subject to check. Estimate c by first obtaining estimates of λ_1 and λ_2 from the separate experiments and by using the average of the estimates of λ_1/k_1 and λ_2/k_2 to estimate Λ. Use the distribution of $w = u_1/u_2$ for similar experiments taking $a^* = (a_1' + a_2')/2$ for a and $b^* = (b_1 + b_2)/2$ for b and read the critical value w_0 from Table I. The critical value of $\dot{w}_0$ is then taken to be cw_0 as an approximation. A similar procedure may be used with the correlation coefficients. If $(a_1 + b_1) \neq (a_2 + b_2)$, the hypothesis that $\rho_1^2 = \rho_2^2$ is not equivalent to the hypothesis that $\lambda_1 = \lambda_2$ in view of the form for λ in (45). The steps required for an approximation are parallel to those discussed in this paragraph.

Extensions of Table I for values of α other than .05 are desired. Means of extending Table I for values, $\alpha = .025$, .01, and .005 are being investigated and it is hoped that these extensions can eventually be obtained. Extensions of Table I to the general case where $a_1 \neq a_2$ and $b_1 \neq b_2$ are not being considered. These tables would only occasionally be required in applications.

We have suggested situations wherein this work will be useful. Examples in taste testing, field experimentation, and regression have been worked out in detail by Schumann and Bradley [9].

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