POWER FUNCTIONS OF THE GAMMA DISTRIBUTION

By Gerald D. Berndt

Advisory Committee on Weather Control

Power functions are given for testing hypotheses on an increase in the mean μ of a gamma variable.

Let x be a random variable from a gamma population and let the frequency distribution of x be given by

$$f_0 = f(x; \beta, \gamma) = (\beta^\gamma \Gamma(\gamma))^{-1} x^{\gamma-1} \exp(-x/\beta), \qquad x > 0,$$
$$= 0, \qquad x \leqq 0,$$

where $\beta > 0$ and $\gamma > 0$. If x then undergoes a scale change of the form $x \to \delta x$ with $\delta > 1$, it is easily verified that the frequency distribution of δx is given by

$$f_1 = f(\delta x; \delta\beta, \gamma) = ((\delta\beta)^\gamma \Gamma(\gamma))^{-1} x^{\gamma-1} \exp(-x/\delta\beta), \qquad x > 0,$$
$$= 0, \qquad x \leqq 0.$$

Now in testing the null hypothesis $H_0 : \mu = \beta\gamma$ against the alternative hypothesis $H_1 : \mu = \delta\beta\gamma$, $\delta > 1$ and specified, the probability of detecting the hypothesized change in the mean, or the power of the test, is given by

$$\pi_\delta = \int_{x(\alpha)}^{\infty} f_1 \, dx,$$

where $x(\alpha)$ is such that

$$\alpha = \int_{x(\alpha)}^{\infty} f_0 \, dx$$

and α is the significance level of the test.

Curves of power functions of testing H_0 against H_1 are given for

$$\gamma = \tfrac{1}{2}, 1(1)5, 7, 10(5)50,$$
$$1.0 \leqq \delta \leqq 4.0,$$

and

$$\alpha = 0.01, 0.05, \text{and } 0.10.$$

For sufficiently large γ, the distribution of x converges to the normal distribution, and for many purposes the power of the test may then be evaluated by simply using the tables of the normal distribution function with standardized variates

$$t_\alpha = (x(\alpha) - \beta\gamma)/\beta\sqrt{\gamma},$$

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which is exceeded with probability α under H_0, and

$$t_\pi = (t_\alpha \beta \sqrt{\gamma} + \beta\gamma(1 - \delta))/\delta\beta\sqrt{\gamma},$$

which is exceeded with probability π under H_1. The upper bound of the error in using the normal approximation to the gamma distribution for $\gamma \geqq 50$ is calculated, by trial, to be

$$\sup_x | G(x) - N(x) | < 0.019,$$

where $G(x)$ is the distribution function of x as a gamma variable and $N(x)$ is the distribution function of x as a normal variable.

Consider the following example of the use of the accompanying power curves. In illustrating the use of the power curves we first take note of a well known property of the gamma distribution. That is, if $x_i(i = 1, \cdots, n)$ are independent random variables from gamma populations with parameters β and γ_i, then the sample mean is also a gamma variable with parameters β/n and

$$\gamma = \textstyle\sum_{i=1}^{n} \gamma_i .$$

See, for example, [1].

Suppose, for illustration, that a sample of size $n = 10$ is drawn, and that the $x_i(i = 1, \cdots, 10)$ are known to be independently and identically distributed gamma variables with $\gamma_i = 2.0$ and the same β for each i. It is desired to test $H_0\colon \mu = \mu_0$ against $H_1\colon \mu = 1.5\mu_0 = \mu_1$ with probability $\alpha = 0.05$ of accepting H_1 when in fact H_0 is true. What is the probability of detecting μ_1? Here $\delta = 1.5$ and $\gamma = 20.0$. In Fig. 2 we find $\delta = 1.5$ on the abscissa and move vertically to the point of intersection with the curve $\gamma = 20.0$. The power, $\pi_{1.5} = 0.598$, is then the ordinate value at this point of intersection.

How large a sample should be drawn in order that there is at least a probability of $\pi_{1.5} = 0.75$ of detecting the specified increase $\delta = 1.5$ in μ_0? Interpolating for the value of γ at $\delta = 1.5$ and $\pi_{1.5} = 0.75$, we find $\gamma = 32$. Hence the sample size should be at least $n = 32/\gamma_i = 16$ in this case.

The calculations on which the power curves are based were made using 3-point Lagrangian interpolation in Pearson's tables of the incomplete gamma function [2]. All calculations have been verified by actual integration of the gamma functions using high-speed computing machinery. This verification was carried out under the supervision of Dr. Max A. Woodbury at New York University.

I wish to acknowledge the work of Dr. Woodbury and his staff in making these calculations, and also to thank Elaine Berndt who performed all of the interpolations we have required and who drafted the accompanying figures.

REFERENCES

[1] C. R. Rao, *Advanced Statistical Methods in Biometric Research*, John Wiley and Sons, New York, 1952.

[2] *Tables of the Incomplete Gamma Function*, Karl Pearson, editor, Cambridge University Press, re-issued in 1951.

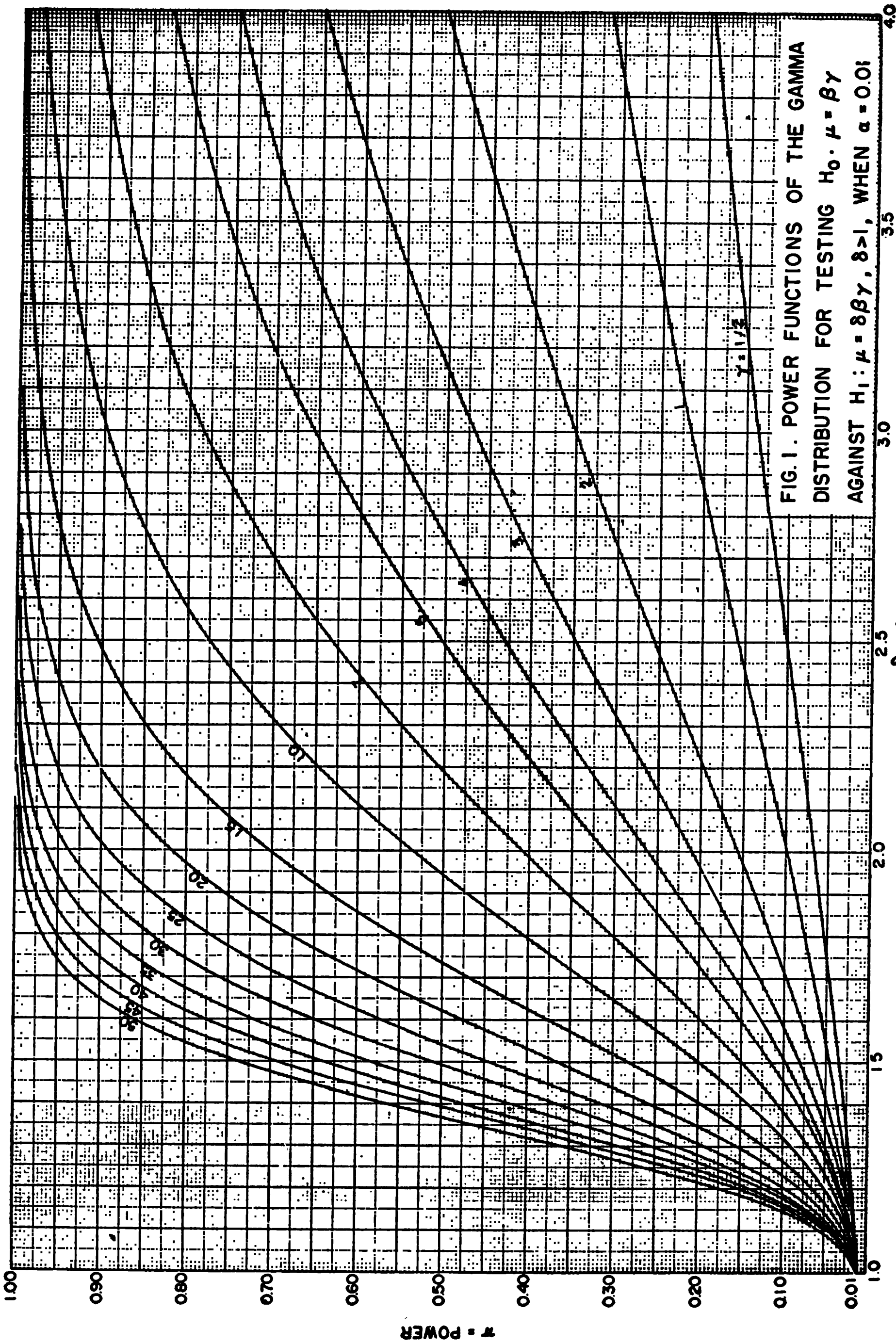

FIG. 1. POWER FUNCTIONS OF THE GAMMA DISTRIBUTION FOR TESTING $H_0 : \mu = \beta\gamma$ AGAINST $H_1 : \mu = \delta\beta\gamma$, $\delta > 1$, WHEN $\alpha = 0.01$

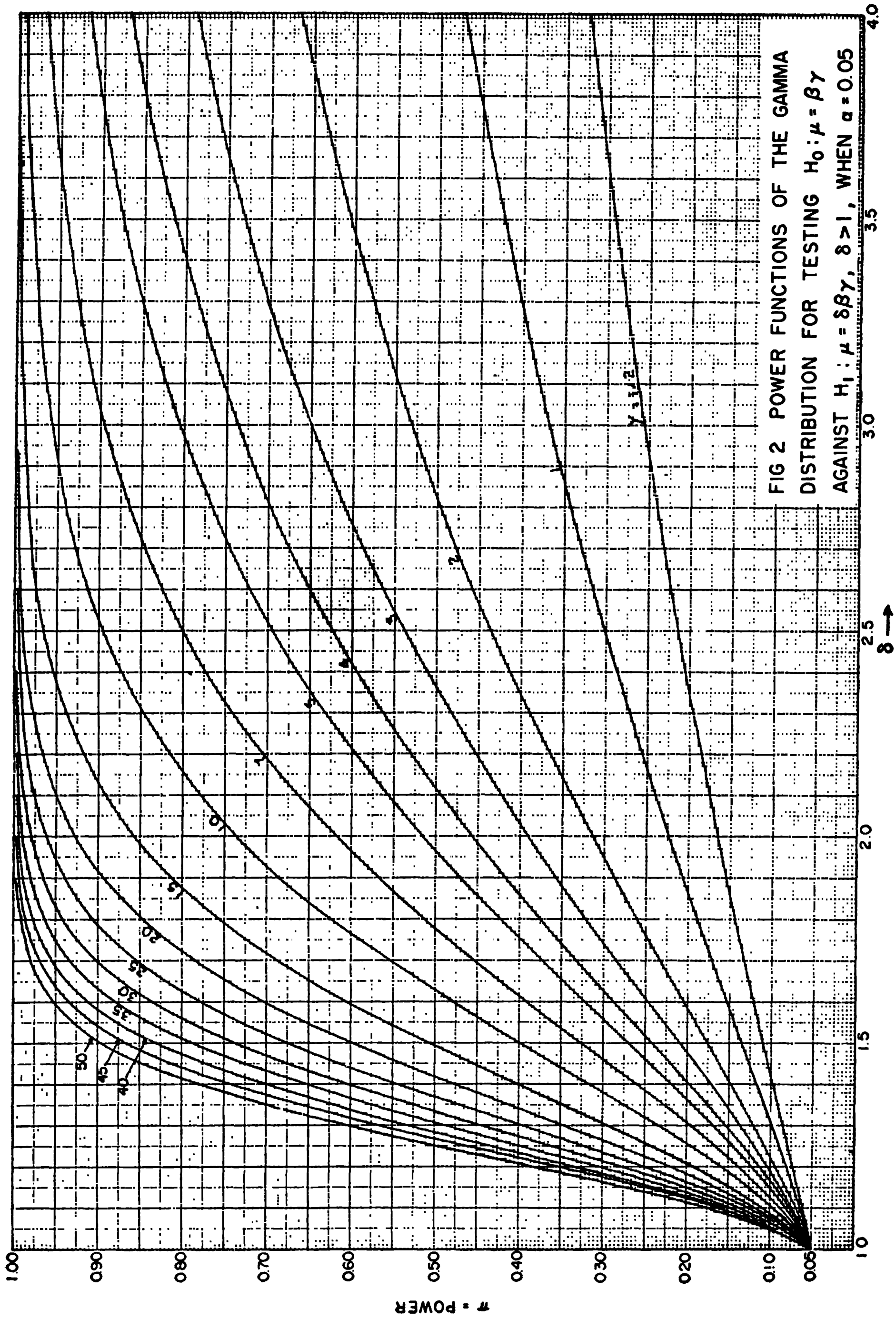

FIG 2 POWER FUNCTIONS OF THE GAMMA DISTRIBUTION FOR TESTING $H_0: \mu = \beta\gamma$ AGAINST $H_1: \mu = \delta\beta\gamma,\ \delta > 1$, WHEN $\alpha = 0.05$

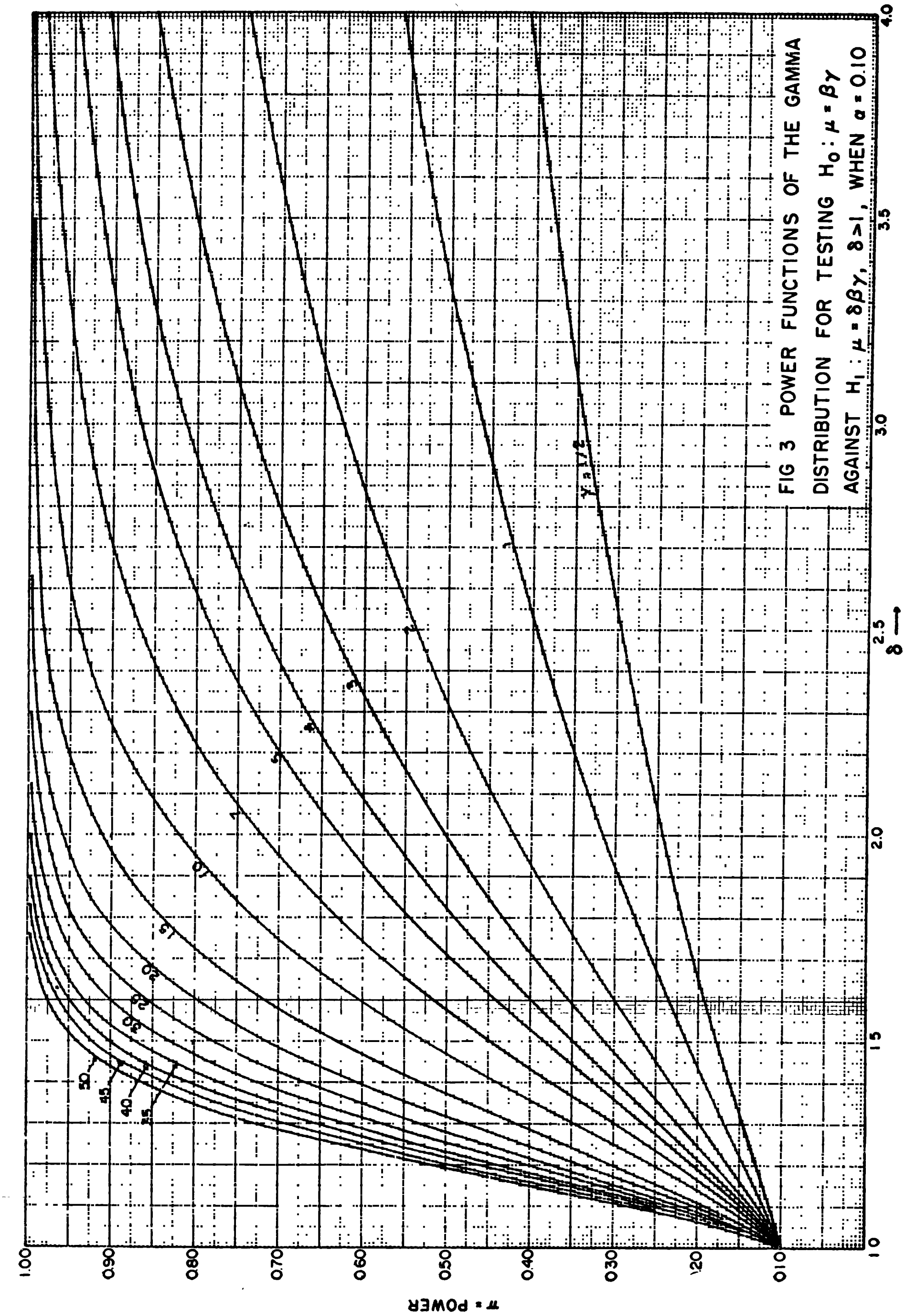

FIG 3 POWER FUNCTIONS OF THE GAMMA DISTRIBUTION FOR TESTING $H_0: \mu = \beta\gamma$ AGAINST $H_1: \mu = \delta\beta\gamma$, $\delta > 1$, WHEN $\alpha = 0.10$