$$\text{var}\, u_p(0) \sim \left\{ \frac{(2q+1)(2q-1)\cdots 1}{2^q q!} \right\}^2 \sigma^2/n, \tag{13}$$

where q is $\frac{1}{2}p$ when p is even and $\frac{1}{2}(p-1)$ when p is odd. In the region of extrapolation, when $|x|$ is large (12.2) gives

$$\text{var}\, u_p(x) \doteqdot (2p+1)\{(2p)!/2^p p!^2\}^2 x^{2p} \sigma^2/n.$$

The deviations from these formulae when n is not large have been discussed and tabulated [4].

4. Comparison of the two methods. In the central part of the range the uniform spacing method gives a smaller variance than the minimax variance method. An asymptotic expansion of (13) using Stirling's factorial approximation shows that the ratio of the variances is roughly $2/\pi$. This ratio increases steadily with $|x|$, and at the ends of the range the variance for the uniform spacing method exceeds that for the minimax variance method by a factor $p+1$, while in the region of extrapolation this factor approaches $2+p^{-1}$. The crossover points for the two variance curves occur at ± 0.58 for the quadratic and ± 0.72 for the cubic. Thus over most of the region of interpolation the advantage lies with the uniform spacing method, but at the extremes of the region of interpolation and in the region of extrapolation the advantage lies decidedly with the minimax variance method.

Fig. 1 shows the shape of the two variance curves in the region of interpolation for the second and third degree polynomials. Since the curves are symmetrical about the origin of x, only half of each curve is drawn.

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CONDITIONS THAT A STOCHASTIC PROCESS BE ERGODIC[1]

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In his work on statistical inference on stochastic processes, Grenander has pointed out ([2], p. 257) that "the concept of metric transitivity seems to be

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important in the problem of estimation of a stationary stochastic process." In this note, we give necessary and sufficient conditions in terms of characteristic functions that a strictly stationary stochastic process $X(t)$ be metrically transitive or ergodic (see Doob [1], pp. 452–457 for definition of the terminology) More importantly, we state a mean ergodic theorem (or weak law of large numbers) for stochastic processes which are strictly stationary of order K, by which is meant that for every choice of K points $t_1, \cdots, t_K$, the random variables $X(t_1 + h), \cdots, X(t_K + h)$ have a joint probability distribution which does not depend on h.

THEOREM 1. Let the random variables $X(t)$ be defined for t in

$$T = \{0, \pm 1, +2, \cdots\}.$$

Let K be a positive integer. Let $t_1, \cdots, t_K$ be points in T. Assume that there is a characteristic function $\varphi(u_1, \cdots, u_K)$ such that, for all $u_1, \cdots, u_K$,

$$E[\exp i\{u_1X(t_1 + h) + \cdots + u_KX(t_K + h)\}] = \varphi(u_1, \cdots, u_K) \quad \text{for all } h \text{ in } T. \tag{1.1}$$

Assume that, for each τ in T, there is a characteristic function $\varphi(u_1, \cdots, u_K; \tau)$ such that

$$E[\exp i\{u_1(X(t_1 + h) - X(t_1 + h + \tau)) + \cdots + u_K(X(t_K + h) - X(t_K + h + \tau))\}] = \varphi(u_1, \cdots, u_K; \tau) \quad \text{for all } h \text{ in } T. \tag{1.2}$$

Let $r \geqq 1$. Then for every Borel function $g(x_1, \cdots, x_K)$ such that

$$E\,|\,g(X(t_1), \cdots, X(t_K))|^r < \infty,$$

the sample means

$$M_n(g) = \frac{1}{n+1}\sum_{h=0}^{n} g(X(t_1 + h), \cdots, X(t_K + h)) \tag{1.3}$$

converge as a limit in r-mean. A necessary and sufficient condition that the limit of the $M_n(g)$ be the ensemble mean $E(g) = Eg(X(t_1), \cdots, X(t_K))$ is that, for all real $u_1, \cdots, u_K$,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{\tau=0}^{n} \varphi(u_1, \cdots, u_K; \tau) = |\,\varphi(u_1, \cdots, u_K)\,|^2. \tag{1.4}$$

The meaning of these conditions is as follows: (1.1) states, in terms of characteristic functions, that the stochastic process is strictly stationary of order K; (1.2) states that the process of increments $Y(t) = X(t) - X(t + \tau)$ is strictly stationary of order K; (1.4) represents a very weak form of asymptotic independence.

From Theorem 1, together with the Birkhoff-Khintchine ergodic theorem (see Doob [1], pp. 464–473) we immediately obtain the following theorem.

THEOREM 2: A strictly stationary stochastic process $X(t)$ is metrically transitive if, and only if, for every positive integer K, for any choice of K points $t_1, \cdots, t_K$, and for any real numbers $u_1, \cdots, u_K$, (1.4) holds.

The conditions of Theorem 2 constitute a formulation in terms of characteristic functions of known conditions for metric transitivity (see Loève [4], p. 435).

As an indication of the power of these theorems, let us mention that with their aid one can readily establish the following statement made without proof in the book of Grenander and Rosenblatt ([3], p. 44): If $X(t)$ is a normal process, a necessary and sufficient condition for it to be ergodic (metrically transitive) is that its spectrum be continuous. If $X(t)$ is a linear process, then it is ergodic.

Theorem 1, and consequently Theorem 2, may be extended to the case of continuous parameter stochastic processes. They provide a new proof of the theorem of Maruyama (see [2], p. 257) that a continuous stationary normal process is metrically transitive if, and only if, its spectrum is continuous.

Theorem 1 is very closely related to the weak law of large numbers for wide-sense stationary processes (see Doob [1], p. 489), from which it differs in that it does not require existence of second moments for $X(t)$.

The proof of Theorem 1 is fairly immediate. From (1.1), (1.2), and (1.4), it follows (either by the weak law of large numbers for wide-sense stationary processes, or directly by a simple argument [6]) that the theorem holds for trigonometric polynomials $g(x_1, \cdots, x_K) = \exp i(u_1x_1 + \cdots + u_Kx_K)$. To extend the theorem to Borel functions $g(x_1, \cdots, x_K)$ such that $E\,|\,g\,|^r < \infty$, one uses the fact that to any $\epsilon > 0$ one may find a trigonometric polynomial $g_\epsilon(x_1, \cdots, x_K)$ such that

$$E\,|\,g(X(t_1), \cdots, X(t_K)) - g_\epsilon(X(t_1), \cdots, X(t_K))\,|^r < \epsilon.$$

In [5] one may find related theorems, including a discussion of convergence with probability one of certain sample means $M_n(g)$ of stochastic processes which are strictly stationary of order K.

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