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A NOTE ON BALANCED DESIGNS

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0. Summary. It is proved that a necessary and sufficient condition for a general design to be balanced is that the matrix of the adjusted normal equations for the estimates of treatment effects has $v - 1$ equal latent roots other than zero.

1. Estimates and their properties. We consider a design whose incidence matrix is $N_{v \times b} = [n_{ij}]$ in which the ith treatment is replicated r_i times and the blocks are of sizes $k_1, \cdots, k_b$. With the usual assumptions, the adjusted normal equations for the treatment effects are

$$Q = C\hat{\tau}, \tag{1.1}$$

where

$$Q = T - N \operatorname{diag}\left(\frac{1}{k_1}, \cdots, \frac{1}{k_b}\right) B \tag{1.2}$$

and

$$C = \operatorname{diag}(r_1, \cdots, r_v) - N \operatorname{diag}\left(\frac{1}{k_1}, \cdots, \frac{1}{k_b}\right) N' \tag{1.3}$$

with the condition

$$E_{1v}\hat{\tau} = 0 \tag{1.4}$$

(where E_{pq} denotes a $p \times q$ matrix with all its elements as unity).

It is well known that if rank $C = v - t$, a set of $t - 1$ independent treatment contrasts are not estimable. But if rank $C = v - 1$ every contrast is estimable and in this case the design is said to be connected.

If the design is connected there are $v - 1$ non-zero latent roots, say, λ_1, $\lambda_2, \cdots, \lambda_{v-1}$. As the rows of C add to zero, $(v^{-1/2}, \cdots, v^{-1/2})$ is the latent vector corresponding to the root zero.

Let

$$L = \left[\frac{L_1}{v^{-\frac{1}{2}}E_{1v}}\right] = \left[\frac{(l_{ij})}{v^{-\frac{1}{2}}E_{1v}}\right] \tag{1.5}$$

be an orthogonal matrix transforming C into diagonal form.

Received February 28, 1957; revised July 15, 1957.

Since L is orthogonal,

$$I = L'L = L_1'L_1 + \frac{1}{v} E_{vv}. \tag{1.6}$$

Pre-multiplying (1.1) by L, we get

$$LQ = LC\hat{\tau} = \left(\frac{\operatorname{diag}(\lambda_1, \cdots, \lambda_{v-1})}{0}\right) L_1 \hat{\tau}. \tag{1.7}$$

Hence we get

$$L_1 \hat{\tau} = \operatorname{diag}\left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{v-1}}\right) L_1 Q.$$

Premultiplying by L_1' and using (1.6) and (1.4), we obtain

$$\hat{\tau} = DQ, \tag{1.8}$$

Where

$$D = [d_{ij}] = L_1' \operatorname{diag}\left(\frac{1}{\lambda_1}, \cdots \cdots, \frac{1}{\lambda_{v-1}}\right) L_1. \tag{1.9}$$

From the solution (1.8), it follows that

$$\begin{aligned} V(\hat{\tau}) &= D\sigma^2, \\ V(\hat{\tau}_i - \hat{\tau}_j) &= (d_{ii} + d_{jj} - 2\, d_{ij})\sigma^2 \\ &= \sum_{\nu=1}^{v-1} \frac{(l_{\nu i} - l_{\nu j})^2}{\lambda \nu} \sigma^2. \end{aligned} \tag{1.10}$$

$$\begin{aligned} \text{Average variance} &= \frac{1}{v(v-1)} \sum_{i} \sum_{\substack{j=1 \\ i \neq j}}^{v} V(\hat{\tau}_i - \hat{\tau}_j) \\ &= \frac{2\sigma^2}{v-1} \sum_{\nu=1}^{v-1} \frac{1}{\lambda \nu} \end{aligned} \tag{1.11}$$

in view of the orthogonality conditions, a result which was obtained by O. Kempthorne in an alternative way [1].

Definition. A design is said to be balanced if every elementary contrast, $\tau_i - \tau_j$ is estimated with the same variance.

2. Theorem. *A necessary and sufficient condition for a design to be balanced is that C has $v - 1$ equal latent roots other than zero.*

To prove that the condition is necessary it is enough to show that $\lambda_1 = \cdots = \lambda_{v-1}$, for the C matrix of a balanced design is of rank $v - 1$.

From (1.10) and (1.11), we get

$$\sum_{\nu=1}^{v-1} \frac{(l_{\nu i} - l_{\nu j})^2}{\lambda_\nu} = \frac{2}{v-1} \sum_{j=1}^{v-1} \frac{1}{\lambda_\nu} = \frac{1}{v-1} \sum_{k=1}^{v-1} \frac{1}{\lambda_k} \sum_{\nu=1}^{v-1} (l_{\nu i} - l_{\nu j})^2.$$

Hence

$$\sum_{\nu=1}^{v-1} (l_{\nu i} - l_{\nu j})^2 \left(\frac{1}{\lambda_\nu} - \frac{1}{v-1} \sum_{k=1}^{v-1} \frac{1}{\lambda_k}\right) = 0. \tag{2.1}$$

Consider

$$V(\hat{\tau}_j - \hat{\tau}_k) = V(\hat{\tau}_i - \hat{\tau}_j) + V(\hat{\tau}_i - \hat{\tau}_k) - 2 \operatorname{Cov}(\hat{\tau}_i - \hat{\tau}_j, \hat{\tau}_i - \hat{\tau}_k).$$

Hence,

$$\operatorname{Cov}(\hat{\tau}_i - \hat{\tau}_j, \hat{\tau}_i - \hat{\tau}_k) = \frac{1}{v-1} \sum_{\nu=1}^{v-1} \frac{1}{\lambda_\nu},$$

i.e.,

$$\sum_{\nu=1}^{v-1} \frac{(l_{\nu i} - l_{\nu j})(l_{\nu i} - l_{\nu k})}{\lambda_\nu} = \frac{1}{v-1} \sum_{\nu=1}^{v-1} \frac{1}{\lambda_\nu} \sum_{\nu'=1}^{v-1} (l_{\nu' i} - l_{\nu' j})(l_{\nu' i} - l_{\nu' k}).$$

Hence,

$$\sum_{\nu=1}^{v-1} (l_{\nu j} - l_{\nu j})(l_{\nu j} - l_{\nu k}) \left(\frac{1}{\lambda_\nu} - \frac{1}{v-1} \sum_{\nu'=1}^{v-1} \frac{1}{\lambda_{\nu'}}\right) = 0 \quad \text{for } i \neq j \neq k, \tag{2.2}$$

From (2.1) and (2.2) taking $i = 1$, we get

$$d^{(j)\prime} \operatorname{diag}\left(\frac{1}{\lambda_1} - \frac{1}{v-1} \sum \frac{1}{\lambda_\nu}, \cdots, \frac{1}{\lambda_{v-1}} - \frac{1}{v-1} \sum \frac{1}{\lambda_\nu}\right) d^{(k)} = 0 \tag{2.3}$$

$$\text{for } j, k = 2, 3, \cdots v,$$

where $d^{(j)}$ is the column vector

$$\{l_{11} - l_{1j}, l_{21} - l_{2j}, \cdots, l_{v-11} - l_{v-1j}\}.$$

If

$$M = [d^{(2)}, d^{(3)}, \cdots, d^{(v)}],$$

then

$$M'M = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdots & & \cdots & \\ 1 & 1 & \cdots & 2 \end{bmatrix},$$

and det. $M'M = v \neq 0$. Hence $M'M$ and hence M are non-singular.

Therefore $d^{(2)}, \cdots, d^{(v)}$ are $v - 1$ linearly independent $(v - 1)$-vectors. Any $(v - 1)$ vector, say ξ, can be uniquely expressed in terms of these vectors, say

$$\xi = C_2 d^{(2)} + C_3 d^{(3)} + \cdots + C_v d^{(v)}.$$

From (2.3) it follows that

$$(2.4)\quad \begin{aligned} &\xi' \operatorname{diag}\left(\frac{1}{\lambda_1} - \frac{1}{v-1}\sum\frac{1}{\lambda_\nu}, \cdots, \frac{1}{\lambda_{v-1}} - \frac{1}{v-1}\sum\frac{1}{\lambda_\nu}\right)\xi = 0 \\ &= \sum_{i,j=2}^{v} C_i C_j d^{(j')} \operatorname{diag}\left(\frac{1}{\lambda_1} - \frac{1}{v-1}\sum\frac{1}{\lambda_\nu}, \cdots, \frac{1}{\lambda_{v-1}} - \frac{1}{v-1}\sum\frac{1}{\lambda_\nu}\right) d^{(j)} = 0. \end{aligned}$$

By taking successively ξ to be the $(v - 1)$ vectors $(1, 0, \cdots, 0), \cdots$ and $(0, 0, \cdots, 1)$, we get

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_2} = \cdots = \frac{1}{\lambda_{v-1}}.$$

Hence

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1}.$$

The condition is sufficient, for, if rank $C = v - 1$ and $\lambda_1 = \cdots = \lambda_{v-1} = \lambda$ (say), it follows immediately that every elementary contrast is estimable, and the solutions become

$$\hat{\tau} = \frac{1}{\lambda} L_1' L_1 Q = \frac{1}{\lambda}\left(I - \frac{1}{v} E_{vv}\right) Q = \frac{Q}{\lambda},$$

which shows that $V(\hat{\tau}_i - \hat{\tau}_j) = (2/\lambda)\sigma^2$, which is independent of both i and j and hence, the design is balanced. Q.E.D.

Corollaries. (i) *If the design is balanced, then*

$$(2.5)\qquad C = \lambda I - \frac{\lambda}{v} E_{vv}$$

and the solutions are

$$(2.6)\qquad \hat{\tau}_j = \frac{Q_j}{\lambda}.$$

(ii) *In a balanced design with equal block sizes, k, the replicate numbers must be equal.*

Proof. $C = \operatorname{diag}(r_1, \cdots, r_v) - 1/kNN'$ if block size is constant. Hence by Eq. (2.5), if the design is also balanced, we have

$$r_i - \frac{r_i}{k} = \lambda - \frac{\lambda}{v}.$$

Hence, r_i is the same constant for all i. Q.E.D.

(iii) *If all the treatments are replicated the same number of times and the blocks are of the same size then the only balanced design is* BIBD, *if such a design exists.*

Proof. If r is the number of replications and k is the block size, then

$$(2.7)\quad \begin{aligned} C &= rI - \frac{1}{k} NN' \\ &= \lambda I - \frac{\lambda}{v} E_{vv} \text{ by Corollary (i).} \end{aligned}$$

Hence comparing off-diagonal elements, we get

$$\lambda_{ij} = \frac{k\lambda}{v},$$

where λ_{ij} is the number of times the pair of treatments i, j occur together in the blocks. Since λ_{ij}'s are all equal the design is Balanced Incomplete Block Design (BIBD) [2]. This result was proved in an alternative form by W. A. Thompson [3].

3. Concluding remarks. But these do not exclude the possibilities of the existence of balanced designs with different block sizes and the same number of replications. As an example consider the design whose incidence matrix is

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$$r = (6, 6, 6, 6); \qquad k = (3, 3, 3, 3, 2, 2, 2, 2, 2, 2).$$

Here it can be verified that every elementary contrast is estimated with a variance equal to $3\sigma^2/7$, but the design is not a Balanced Incomplete Block Design.

It can also be seen that the example given above is obtained by adjoining two BIBD's with the same number of treatments. Such designs can be constructed from two BIBD's with the same number of treatments. Investigations on these lines are being carried out.

Acknowledgement. The author wishes to express his indebtedness to Professor M. C. Chakrabarti for suggesting this problem and for his help and guidance in preparing this note.

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THE SPACING OF OBSERVATIONS IN POLYNOMIAL REGRESSION

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1. Introduction and summary. De la Garza ([1], [2]) has considered the estimation of a polynomial of degree p from n observations in a given range of the

Received April 8, 1957.