THE LIMITING DISTRIBUTION OF BROWNIAN MOTION IN A BOUNDED REGION WITH INSTANTANEOUS RETURN[1]

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1. Summary. A point executes Brownian motion in a bounded, connected, and open three dimensional region D. When it reaches the boundary Γ, at point α, it is instantaneously returned to D according to probability measure $\mu(\alpha)$ (we write $\mu(\alpha, A)$ for the measure of set A), and the Brownian motion is resumed. This is a Markov process and, subject to certain regularity conditions on Γ and $\mu(\alpha)$, we derive the limiting distribution of the process. Processes of this sort have been considered by Feller [1]; he has obtained the transition probabilities of such processes. He is concerned more generally with Markov processes with continuous sample functions on a linear interval; the return may be instantaneous or after a random period of time.

Let $p^0(t, \xi, A)$ be the probability that the point is in set A of D at time t when it is initially at point ξ of D, with the additional restriction that no boundary contacts have been made. It is known that

$$p^0(t, \xi, A) = \int_A u(t, \xi, x)\, dx, \tag{1}$$

where dx is the volume element about x and u is the solution of the equation

$$\tfrac{1}{2}\Delta u = u_t\,,$$

subject to the conditions

$$u(t, \xi, \alpha) = 0, \qquad \alpha \,\varepsilon\, \Gamma, \qquad \lim_{t \to 0} \int_C u(t, \xi, x)\, dx = 1,$$

where C is any sphere of non-zero radius with center ξ which is entirely within D. We may write explicitly

$$u(t, \xi, x) = \sum_{k=1}^{\infty} v_k(\xi) v_k(x) e^{-\lambda_k t},$$

where λ_k is the kth eigenvalue and $v_k(x)$ the corresponding eigenfunction of the equation $\Delta u + 2\lambda u = 0$ subject to the boundary condition $u = 0$ on Γ. If $K(\xi, x)$ is the Green's function of $\Delta u = 0$ in D, then[2] ([2], and [3], page 273)

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$$K(\xi, x) = \frac{1}{2}\int_0^\infty u(t, \xi, x)\, dt. \tag{2}$$

Let $\phi(t, \xi, \alpha)\, dt\, d\alpha$ be the probability the point is absorbed at surface element $d\alpha$ of Γ between t and $t + dt$ when it is initially at point ξ of D. Then ϕ is half the interior normal derivative of u at point α of Γ([3], page 273). When the point is initially at ξ the probability of ultimate absorption in set S of Γ is given by

$$\pi(\xi, S) = \int_S \int_0^\infty \phi(t, \xi, \alpha)\, dt\, d\alpha. \tag{3}$$

We may define a discrete parameter Markov process with Γ as state space by taking as transition probability

$$\pi(\alpha, S) = \int_D \pi(\xi, S)\mu(\alpha, d\xi) \tag{4}$$

This Markov process has a limiting distribution π which satisfies the equation

$$\pi(S) = \int_\Gamma \pi(\alpha, S)\pi(d\alpha). \tag{5}$$

We define a measure of sets of D by

$$\lambda(A) = \int_\Gamma \mu(\alpha, A)\pi(d\alpha).$$

We may now write the density function for the limiting distribution. If $M(\xi)$ is the mean time of reaching the boundary when the point is initially at ξ,

$$M(\xi) = \int_\Gamma \int_0^\infty t\phi(t, \xi, \alpha)\, dt\, d\alpha.$$

then the density function of the limiting distribution is

$$\frac{2\int_D K(\xi, x)\lambda(d\xi)}{\int_D M(\xi)\lambda(d\xi)}. \tag{6}$$

If we are given a probability measure λ in D and the return is always according to λ, then it is clear that the limiting density of this process is also given by (6). If λ concentrates at a single point ξ we may drop the integrals in (6), and in particular we get

$$M(\xi) = 2\int_D K(\xi, x)\, dx.$$

We note that (6) is essentially the steady distribution of temperature in the following problem: D is a homogeneous heat conducting body whose boundary is kept at temperature 0 and in which there is a constant source of heat distributed according to λ.

Regarding the regularity conditions, we shall assume that Γ is made up of finitely many surfaces, each with a continuously turning tangent plane and that D has a Green's function ([4], page 262). We will assume there is a closed set B in D such that

$$\inf_{\alpha \varepsilon \Gamma} \mu(\alpha, B) = \gamma > 0.$$

2. Origin of the problem. This problem had its origins in the ecological research of Professor Thomas Park of the University of Chicago. He has been investigating problems of population stability and inter-species competition of flour beetles. It was discovered, on statistical investigations suggested in part by Jerzy Neyman, that the distribution of the beetles in the container of flour was not uniform, with the density increasing toward the boundaries of the container. The problem arose as to whether the nonuniformity might be simply a consequence of the random motion of the beetles or whether it ought to be attributed to some inhomogeneity such as a temperature gradient in the flour. To check the plausibility of the idea that the nonuniformity might arise from random motion alone, we have set up a model which may have some relevance to the actual situation. The region D represents the volume of flour. We assume the independence of the motions of the beetles so that we may confine ourselves to the random motion of a single point. This is a reasonable assumption if the density of the beetles is low. For the random motion we take Brownian motion; this is appropriate if we want path continuity and spatial homogeneity. Finally, we must introduce some mechanism of return from the boundary; we use the device of instantaneous return. If the return distribution is concentrated near the point of contact on the boundary then the device has some semblance of plausibility. More precisely we may suppose $\mu(\alpha, A(\alpha)) = 1$, where $A(\alpha)$ is that set of points of D whose distance from α is less than or equal to δ, a small positive number. Then if E is the subset of points of D whose distance from Γ is in excess of δ we have $\lambda(E) = 0$. If we are prepared to accept the density of distribution in E, as given by (6), as a theoretical model for what is observed then we are faced with a contradiction. For the density is a harmonic function in E, by virtue of $\lambda(E) = 0$. Because of the minimum-maximum properties of such functions we cannot have increasing density from the central parts of E outward to the boundary of E since that would entail a minimum at an interior point of E.

3. Derivation of the limiting distribution. We sketch a proof that we have defined a process by the instantaneous return mechanism. This is equivalent to proving that finitely many contacts occur in a finite time with probability 1. By the assumption on μ it will happen infinitely often with probability 1 that the point is returned to B. Let $T(x)$ be the time to reach Γ, starting at x. If δ is positive Prob $(T(x) > \delta)$ is a continuous function of x which achieves a positive minimum on B. Thus of the times the point is returned to B it will happen infinitely often with probability 1 that the time to reach Γ is in excess of δ. This implies that infinitely many contacts in a finite time has probability 0.

Let $p(t, x, A)$ be the transition probability of the process, i.e., the probability the point is in A at time t when it is initially at x. Then we prove the limiting distribution p exists and

$$p(t, x, A) = p(A) + f(t, x, A), \tag{7}$$

where

$$|f(t, x, A)| < ae^{-kt}. \tag{8}$$

Here a and k are positive and independent of x and A. To simplify the notation we will make the following convention: if $f(x)$ is a function on D then we define a corresponding function $f(\alpha)$ on Γ by taking the integral, over D, of f with respect to the measure $\mu(\alpha)$. With this convention we may replace x by α in (7) and (8). We note that both $f(t, x, A)$ and $f(t, \alpha, A)$ are integrable with respect to t from 0 to ∞.

Proceeding with the proof we use the fact that $u(t, \xi, x)$ is strictly positive for all ξ and x in D and for positive t. Then the minimum $v(x, \delta, t)$, achieved by $u(T, \xi, x)$ subject to $\xi \in B$ and $t - \delta \leqq T \leqq t$, is also strictly positive, and it follows directly that for $t - \delta \leqq T \leqq t$,

$$p^0(T, \alpha, A) \geqq \gamma \int_A v(x, \delta, t)\, dx.$$

It is clear that

$$h(\delta) = \inf_{x \in D} \int_\Gamma \int_0^\delta \phi(t, x, \alpha)\, dt\, d\alpha$$

satisfies $0 < h(\delta) < 1$ for all $\delta > 0$. Let $p^1(t, \xi, A)$ be the probability the point, initially at ξ, is in A at time t having made exactly one boundary contact. Then if $t > \delta$,

$$\begin{aligned} p^1(t, \xi, A) &= \int_\Gamma \int_0^t \phi(\tau, \xi, \alpha) p^0(t - \tau, \alpha, A)\, d\tau\, d\alpha \\ &\geqq \int_\Gamma \int_0^\delta \phi(\tau, \xi, \alpha) p^0(t - \tau, \alpha, A)\, d\tau\, d\alpha \\ &\geqq \int_\Gamma \int_0^\delta \phi(\tau, \xi, \alpha)\, d\tau\, d\alpha \cdot \gamma \int_A v(x, \delta, t)\, dx \\ &\geqq h(\delta)\gamma \int_A v(x, \delta, t)\, dx. \end{aligned}$$

We follow now the proof of a similar theorem given by Doob ([5], page 197). If $m(t, A)$ and $M(t, A)$ are respectively the infimum and supremum of $p(t, \xi, A)$ as ξ varies over D, then $M(t, A) \geqq m(t, A)$ and by the Chapman-Kolmogorov equation it can be seen that $M(t, A)$ is non-increasing and $m(t, A)$ is non-decreasing. For fixed t_0, ξ_0, x_0 define the set function

$$\psi(A) = p(t_0, \xi_0, A) - p(t_0, x_0, A).$$

There is a set A^+ on which ψ is maximum, such that $\psi(A) \geqq 0$ for any subset A of A^+, and such that $\psi(A) \leqq 0$ for any subset A of $A^- = D - A^+$. We have, assuming δ such that $0 < \delta < t_0$,

$$\begin{aligned} \psi(A^+) &= 1 - p(t_0, \xi_0, A^-) - p(t_0, x_0, A^+) \\ &\leqq 1 - p^1(t_0, \xi_0, A^-) - p^1(t_0, x_0, A^+) \\ &\leqq 1 - h(\delta)\gamma \int_D v(x, \delta, t_0)\, dx = c < 1. \end{aligned}$$

Following now a line of argument analogous to Doob's we have

$$M(t, A) - m(t, A) \leqq c^{(t/t_0)-1},$$

from which it follows that $M(t, A)$ and $m(t, A)$ have a common limit $p(A)$ and that

$$|p(t, x, A) - p(A)| \leqq M(t, A) - m(t, A) \leqq c^{(t/t_0)-1}.$$

Thus (7) and (8) are established, with $a = c^{-1}$ and $k = 1/t_0 \log c^{-1}$.

Before deriving (6) we have to establish the existence of the limiting distribution π of the boundary process. To this end we prove the lemma

$$(9) \qquad \zeta = \sup_{S \varepsilon \Gamma} (\max_{x \varepsilon B} \pi(x, S) - \min_{x \varepsilon B} \pi(x, S)) < 1.$$

We note that $\pi(x, S)$ is, for fixed S, a harmonic function of x([3], page 273). If (9) is not true there will be a sequence of sets S_k such that

$$(10) \qquad \max_{x \varepsilon B} \pi(x, S_k) \to 1, \qquad \min_{x \varepsilon B} \pi(x, S_k) \to 0.$$

Since $\pi(x, S_k)$ is a sequence of harmonic functions with $0 \leqq \pi(x, S_k) \leqq 1$, we may extract a subsequence which converge to a harmonic function $f(x)$ uniformly on any compact subset of D([4], page 249). Without change of notation we suppose this done. However (10) implies that $f(x)$ achieves the values 1 and 0 on B, which contradicts the fact that $f(x)$ is harmonic in D and $0 \leqq f(x) \leqq 1$.

To prove the existence of the limiting distribution of the boundary process we again follow the lines of Doob's proof. For fixed α and β we define the set function

$$\psi(S) = \pi(\alpha, S) - \pi(\beta, S).$$

Associated with ψ are the sets S^+ and S^-, and we have

$$\begin{aligned} \psi(S^+) &= 1 - (\pi(\alpha, S^-) + \pi(\beta, S^+)) \\ &\leqq 1 - \left(\int_B \pi(x, S^-)\mu(\alpha, dx) + \int_B \pi(x, S^+)\mu(\beta, dx)\right) \\ &\leqq 1 - \gamma\, (\min_{x \varepsilon B} \pi(x, S^-) + \min_{x \varepsilon B} \pi(x, S^+)) \\ &= 1 - \gamma(1 - \max_{x \varepsilon B} \pi(x, S^+) + \min_{x \varepsilon B} \pi(x, S^+)) \\ &\leqq 1 - \gamma(1 - \zeta). \end{aligned}$$

If now we introduce $m^{(n)}(S)$ and $M^{(n)}(S)$, the infimum and supremum of the n step transition probability $\pi^{(n)}(\alpha, S)$ as α ranges over Γ, then following Doob's proof

$$M^{(n)}(S) - m^{(n)}(S) \leqq (1 - \gamma(1 - \zeta))^n,$$

from which it follows that the limiting distribution exists and satisfies (5).

We are now in position to derive (6). We have

$$p(t, \xi, A) = p^0(t, \xi, A) + \int_\Gamma \int_0^t \phi(\tau, \xi, \alpha) p(t - \tau, \alpha, A) \, d\tau \, d\alpha.$$

Introducing (7) and integrating with respect to t we get, after some reductions,

$$\int_0^T p^0(t, \xi, A) \, dt = p(A) \left[T \left(1 - \int_0^T \int_\Gamma \phi(\tau, \xi, \alpha) \, d\alpha \, d\tau \right) + \int_0^T \int_\Gamma \tau \phi(\tau, \xi, \alpha) \, d\alpha \, d\tau \right] + I(T, \xi, A), \tag{11}$$

where

$$I(T, \xi, A) = \int_0^T f(t, \xi, A) \, dt - \int_0^T \int_\Gamma \int_0^t \phi(\tau, \xi, \alpha) f(t - \tau, \alpha, A) \, d\tau \, d\alpha \, dt. \tag{12}$$

The second term in the bracket on the right of (11) tends to $M(\xi)$ as $T \to \infty$, and we show that the first term tends to 0. This term can be written

$$T \text{ Prob } (x(t) \varepsilon D, 0 < t \leqq T \mid x(0) = \xi). \tag{13}$$

Let the coordinates of point x be x_1, x_2, x_3 and suppose D is contained between the planes $x_1 = a$ and $x_1 = -a$. Then (13) tends to 0 if the expression

$$T \text{ Prob } (-a < x_1(t) < a, 0 < t \leqq T \mid x(0) = \xi) \tag{14}$$

tends to 0. We may write (14) explicitly

$$T \sum_{n=1}^{\infty} \frac{4}{(2n + 1)\pi} \sin \frac{(2n + 1)\pi}{2a} (\xi + a) \exp \left(-\frac{(2n + 1)^2 \pi^2 T}{8a^2} \right),$$

which is less than

$$T \sum_{n=1}^{\infty} \frac{4}{(2n + 1)\pi} \exp \left(-\frac{(2n + 1)^2 \pi^2 T}{8a^2} \right), \tag{15}$$

and it is easily proved that (15) tends to 0. Letting $T \to \infty$ in (11) we get

$$\int_0^\infty p^0(t, \xi, A) \, dt = p(A) M(\xi) + I(\infty, \xi, A). \tag{16}$$

Referring to (12) and (3) we may write, on introducing the variables $\tau' = \tau$ and $t' = t - \tau$,

$$I(\infty, \xi, A) = \int_0^\infty f(t, \xi, A)\, dt - \int_\Gamma \left(\int_0^\infty \phi(\tau', \xi, \alpha)\, d\tau'\right)\left(\int_0^\infty f(t', \alpha, A)\, dt'\right) d\alpha$$
$$= \int_0^\infty f(t, \xi, A)\, dt - \int_\Gamma \left(\int_0^\infty f(t, \alpha, A)\, dt\right) \pi(\xi, d\alpha).$$

The integration of the right side with respect to measure λ is equivalent to consecutive integrations with respect to $\mu(\beta)$ and π. The first integration gives, using (4),

$$\int_0^\infty f(t, \beta, A)\, dt - \int_\Gamma \left(\int_0^\infty f(t, \alpha, A)\, dt\right) \pi(\beta, d\alpha);$$

and the second, using (5), gives the value 0. Thus integrating on both sides of (16) with respect to measure λ we get

$$\int_D \int_0^\infty p^0(t, \xi, A)\, dt\, \lambda(d\xi) = p(A) \int_D M(\xi)\lambda(d\xi).$$

This equation, together with (1) and (2), implies that (6) is the density function of the limiting distribution p.

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