SEMIMARTINGALES OF MARKOV CHAINS

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1. Introduction. We shall deal throughout this paper with absorbing Markov chains with a finite number of states. An absorbing Markov chain is one that has a set of "boundary" states which once reached cannot be left, and such that from any state the process reaches the boundary with probability 1. The chain is given by the transition matrix P, with entries p_{ij}.

More precisely, a state i is a "boundary" state if $p_{ii} = 1$. The remaining states will be called "interior" states. We must require that it is possible to reach the boundary from every interior state, not necessarily in one step. We assume, that there are r absorbing states and s interior states. The set of boundary states will be called B, the set of interior states I.

An upper semimartingale is a function on the states of the chain, such that the expected value of the function after one step from any state is greater than or equal to the value of the function at the state. A lower semimartingale is defined similarly, with the inequalities reversed. A martingale is a function on the states that is both an upper and a lower semimartingale.

A function on the states can be conveniently represented by a column vector. Such a vector z is an *upper semimartingale* if $Pz \geqq z$, a *lower semimartingale* if $Pz \leqq z$, and a *martingale* if $Pz = z$.

We assume that a set of nonnegative *boundary values* is assigned to the elements of B, v_j being assigned to state j. We denote by U the set of all nonnegative upper semimartingales and by U^* the set of all nonnegative lower semimartingales having the right boundary values. Thus U is the set of all vectors such that

$$\text{(a) } Pz \geqq z, \quad \text{(b) } z \geqq 0, \quad \text{(c) } \{z\}_j = v_j \quad \text{for} \quad j \,\varepsilon\, B.$$

The set U^* consists of the vectors satisfying conditions (b) and (c), and condition (a) with the inequality sign reversed.

Throughout the paper $\{z\}_j$ will denote the jth componnet of the vector z. Inequality signs between vectors will assert that the inequality holds componentwise.

A representation theorem will be developed for all nonnegative semimartingales with the prescribed boundary values in terms of martingales of modified chains. A modified chain is one obtained by adding interior states to the boundary, and assigning value 0 to them. The representation is unique and leads to a simple geometric interpretation. U will be represented (except in certain de-

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generate cases) by a convex cubic s dimensional polyhedron. In degenerate cases the polyhedron reduces to smaller s dimensional polyhedra, including an s-simplex in the most degenerate case. U^* will be obtained from a reflection of U through the unique martingale.

These results will be applied to a treatment of certain sequential games, and to discrete subharmonic functions. In the latter application we will see that discrete subharmonic functions can be expressed as convex combinations of certain harmonic functions. And it is well known that the discrete harmonic function for given boundary values may be interpreted as the expected final value of a random walk. Hence we have a method of obtaining all discrete subharmonic functions in terms of certain random walks.

2. The basic semimartingales. Let T be a subset of I and denote by $P(T)$ the transition matrix obtained from P by changing the states in T into absorbing states. Let $Q(T) = \lim_{n\to\infty} [P(T)]^n$. Then the ijth entry of $Q(T)$, $q_{ij}(T)$ represents the probability, that starting at state i, the process will reach state j before reaching any element of T. Let $q_j(T)$ denote the jth column of $Q(T)$. Then since

$$Q(T) = P(T)\cdot Q(T),$$

$$q_{ij}(T) = \begin{cases} \sum_k p_{ik}q_{kj}(T), & i \not\in T, \\ 0, & i \in T, \end{cases}$$

we see that

$$Pq_j(T) \geqq q_j(T), \qquad j \in B.$$

Thus $q_j(T)$ is an upper semimartingale. It has the boundary value of 0 on all states of B except j, and has the value of 1 on this state. Thus the vector $z(T)$ given by

$$z(T) = \sum_{j=1}^{r} v_j\, q_j(T)$$

is a nonnegative upper semimartingale with the prescribed boundary values; *i.e.*, for each T, $z(T)$ is an element of U. We shall refer to $z(T)$ as a *basic upper semimartingale.*

The vector $z(T)$ may be interpreted in a game played as follows: The process starts in a given state, and continues until it reaches a state in T, or a state in B, and is then stopped. If it stops at a state j in B the player receives v_j; if it stops at a state in T, he receives 0. Then $\{z(T)\}_i$ represents the expected value of the game to the player starting at state i. We shall appeal to this interpretation for certain simple results, rather than give detailed proofs. For example:

LEMMA 1. *Assume that T_1 and T_2 are subsets of I such that $T_1 \subseteq T_2$. Then* $z(T_1) \geqq z(T_2)$.

From this interpretation we can easily determine $z(\phi)$ and $z(I)$. If $T = \phi$, then our game is always played till the boundary is reached, and hence $z(\phi)$ is the unique martingale with the prescribed boundary values. If $T = I$, then we can never reach the boundary from I. Hence

$$\{z(I)\}_i = \begin{Bmatrix} v_i, & i \in B \\ 0, & i \in I \end{Bmatrix}.$$

It can be seen from Lemma 1 that $z(\phi)$ and $z(I)$ are the largest and smallest $z(T)$, respectively. Since we will see later that all elements of U are convex combinations of $z(T)$'s, we see that $z(\phi)$ is the maximal and $z(I)$ the minimal element of U.

3. A special case. We shall first solve the problem of describing U for the case where the following hypothesis is satisfied.

HYPOTHESIS **A**: *The boundary values v_j are all positive, and for any state i in I there is at least one j in B such that $p_{ij} > 0$.*

In the case that hypothesis **A** is satisfied the game interpretation for $z(T)$ makes it clear that the following lemma holds.

LEMMA 2. *Under hypothesis* **A**, *the $z(T)$ have the property that*

$$\{Pz(T)\}_i = \{z(T)\}_i > 0 \qquad \textit{for } i \in I - T$$

and

$$\{Pz(T)\}_i > \{z(T)\}_i = 0 \qquad \textit{for } i \in T.$$

Thus for each component of $z(T)$ exactly one of the equalities in the defining conditions (a) and (b) of U holds.

LEMMA 3. *Let $x_1, x_2, \cdots, x_n$ be distinct nonnegative vectors. Let W_i be the set of components of x_i which are 0. Assume that if $W_i \subseteq W_k$ then $x_i \geqq x_k$. If so the vectors are convexly independent.*

PROOF. Assume that $x_i = \sum_k a_k x_k$ with $a_k > 0$ and $k \neq i$ and $\sum_k a_k = 1$. Then a component of x_i can be 0 only if all the x_k have this component 0. Hence $W_i \subseteq W_k$, and $x_i \geqq x_k$ for all k. But this can only be true if $x_i = x_k$ for all k, contrary to hypothesis.

DEFINITION. A convex n-dimensional polyhedron is *cubic* if in every j dimensional face for each $j - 1$ dimensional subface there is a unique nonintersecting $j - 1$ dimensional subface ($j = 1, 2, \cdots, n$).

THEOREM 1. *If hypothesis A is satisfied, then U is a convex cubic polyhedron with 2^s corner points. These corner points are the $z(T)$ for $T \subseteq I$.*

PROOF. We observe first that the 2^s $z(T)$ are distinct and convexly independent. This follows from Lemmas 1 and 3. We shall now prove that the convex set spanned by the $z(T)$ is a cubic polyhderon.

A j dimensional face of the convex set spanned by the ~~$z(T)$ is determined~~ by picking any $r - j$ interior states and requiring that one of the equalities

$$\text{(a) } \{Pz(T)\}_i = \{z(T)\}_i,$$

$$\text{(b) } \{z(T)\}_i = 0$$

hold for each i in the set chosen. To obtain a $j - 1$ dimensional subface of this face we impose an equality on one more component—say k. It follows from hypothesis **A** that $Pz > 0$. Hence it is not possible to have equality (a) and (b) for the same state. Hence this $j - 1$ dimensional face cannot intersect the face obtained by choosing the other equality for the kth component. By Lemma 2 we can find a $z(T)$ which has any prescribed set of equalities one for each of the pairs (a) and (b). Thus any $j - 1$ dimensional face obtained by choosing an equality for a component $i \neq k$ must intersect that obtained by choosing an equality for component i. Thus the set spanned by $z(T)$ satisfies the conditions for a cubic polyhedron.

To complete the proof of the theorem we must show that if z is in U then it must be in the cubic polyhedron spanned by the basic upper semimartingales $z(T)$. But if z is in U it must satisfy

$$\text{(a) } Pz \geqq z,$$

$$\text{(b) } z \geqq 0.$$

Thus for each interior state i it must lie between the hyperplane obtained by requiring $\{Pz\}_i = \{z\}_i$ and the hyperplane obtained by requiring $\{z\}_i = 0$ But this means that z must lie between each pair of opposite faces in the cubic polyhedron spanned by $z(T)$. Hence it must lie inside of this polyhedron.

Definition. A sequence $T_0 \subset T_1 \subset \cdots \subset T_k$ of subsets of I is called a *chain*. The corresponding sequence of corner points $z(T_0), z(T_1), \cdots, z(T_k)$ is called a z-chain. If $k = s$, the chain is called *maximal*.

It is clear that the elements of a z-chain are linearly independent and hence span a simplex. A simplex spanned by a z-chain will be called a *z-simplex*.

Lemma 4. *Every face (of every dimension) of the cube U has a maximal element.*

Proof. In the s-dimensional cube U, every j-face ($j = 0, 1, \cdots, s$) is a j dimensional cube. This is clear from the definition of the cubic polyhedron. The face of the cube is specified by imposing equalities of type (a) or (b) on $r - j$ components.

Since we have a polyhedral set, it suffices to show that there is a maximal corner. The corners are specified by imposing equalities of one of the two types on each of the j remaining components. It is a direct consequence of Lemma 1 that a corner $z(T)$ is maximal if its T is minimal. Hence we get a maximal corner by imposing equalities (a) on all j of the remaining components.

Lemma 5. *The intersection of two z-simplexes (if not empty) is a z-simplex which is a common face of the two original simplexes.*

Proof. Let $T_0 \subset T_1 \subset \cdots \subset T_k$ and $T'_0 \subset T'_1 \subset \cdots \subset T'_{k'}$. Let the two simplexes be determined by the corresponding z's. If there is a nonempty set of T's that the two chains have in common, then they span a common face. We will show that this is the intersection of the two simplexes.

It will suffice to show that all the remaining corners of the second simplex (if any) lie outside the first simplex. Let T' be one of the sets in the second chain that is not in the first chain. If $z(T')$ lies in the first simplex, then it is a convex combination of its corners. But this is impossible, since the $z(T)$'s are convexly independent. This completes the proof.

LEMMA 6. *Every point of U lies in at least one z-simplex.*

PROOF. Let z_0 be a point of U. Starting with ϕ we will construct a chain so that z_0 will lie in the simplex spanned by the corresponding z-chain.

First of all, draw a line from $z(\phi)$ through z_0 and continue it till it hits a face of U (of dimension less than s). Say it meets this face in the point z_1. Then z_0 is in the set spanned by $z(\phi)$ and z_1. In this face we pick the maximal point $z(T_1)$, which exists by Lemma 4, and draw a line from it through z_1 till we hit a face of lower dimension at a point z_2. Since z_1 lies in the set spanned by $z(T_1)$ and z_2, we know that z_0 is in the set spanned by $z(\phi)$ and $z(T_1)$ and z_2. We iterate this procedure until some z_n turns out to be a corner $z(T_n)$. This must happen, since the dimension of the face decreases at each step. Then we will have z_0 in the set spanned by $z(\phi), z(T_1), \cdots, z(T_n)$.

At each step we first introduced the minimal T in the face, hence the T's are monotone decreasing and hence form a chain. Thus the corners we found form a z-chain and the set they span is a z-simplex, which contains z_0.

THEOREM 2. *Any z_0 in U can be written uniquely as*

$$z_0 = \sum_{j=0}^{k} a_j z(T_j),$$

with $a_j > 0$ and $\sum a_j = 1$, where the $z(T)$'s used form a z-chain.

PROOF. Let z_0 be any point in U. By Lemma 6 it lies in at least one z-simplex Form the intersection of all z-simplexes that contain z_0. This intersection is not empty and hence by Lemma 5 it is a common face of all the z-simplexes. This smallest possible z-simplex serves the purpose of our representation. Its corners form a z-chain, and we can write z_0 as a convex combination of these. The weights a_j must all be positive, or else the point z_0 would lie in a smaller z-simplex.

To show the uniqueness of our representation we need only recall that the representation of a point in a simplex in our (barycentric) representation is unique. To get a representation of our form, the z-chain used must span a simplex containing z_0. Hence the minimal simplex is a face of it. Hence the a_j's can be all positive only if the simplex is the minimal one we found. This establishes the unique representation.

It is worth remarking that the theorem established only the uniqueness of the smallest z-simplex containing z_0. If this simplex is of a dimension smaller than s, then it is a common face of several z-simplexes. If hypothesis **A** is satisfied, then there are $s!$ maximal z-chains, and correspondingly $s!$ maximal z-simplexes. The cube is divided into these, and they overlap only in that they have common faces of lower dimension. If a point is in the interior of one of the maximal simplexes, then it is expressed by putting positive weights on all $s + 1$ corners. If it is on a face, we must apply the same consideration to the smaller simplexes in which it lies.

4. The general case. If we drop hypothesis **A**, most of the previous considerations still apply. However, the argument as to the distinctness of the $z(T)$'s breaks down. But by a continuity argument we can see that any case where the hypothesis is not fulfilled is a limiting case of ones where the hypothesis holds, and hence the worst that can happen is that some of the $z(T)$'s coincide, and hence we have fewer corners on U. While it is still a polyhedron of dimension s, it need not be cubic, and there will be fewer distinct z-simplexes. We will show how the distinct z-simplexes can be found in the general case.

Let B^* be the set of boundary points which have nonzero values assigned. We assume that B^* is not empty.

DEFINITION. A set T is *fundamental* if from any point in $I - T$ it is possible to reach B^* without going through T.

Let T be any set which is not fundamental. Add to T all states which are cut off from the set B^* by T. The new set T' is fundamental and $z(T)$ and $z(T')$ are the same. On the other hand the $z(T)$'s whose T is fundamental have 0 components exactly on T, and hence are distinct. Thus the extreme points of U are given by the $z(T)$'s with T fundamental.

LEMMA 7. *There exists at least one z-simplex of dimension s.*

PROOF. Let the index of an interior state be the minimum number of steps required to reach a state of B^* from it. Reorder the states in such a way that their indices are nonincreasing. Then from any state it must be possible to reach the boundary without going through a state appearing earlier in the sequence. Let T_j be the set of the first j states. Then $T_0, T_1, T_2, \cdots, T_s$ is a complete chain with all the T_j's fundamental. Hence $z(T_0), z(T_1), \cdots, z(T_s)$ form the corner points of a z-simplex of dimension s.

Lemma 7 is all that is needed to insure the construction used in Theorem 2. Hence the representation theorem applies equally well to the general case. The lemma also establishes that even in the degenerate cases U has dimension s.

5. The set of lower semimartingales U^*. The set U^* of all lower semimartingales having prescribed boundary conditions is determined by replacing the condition

$$\text{(a) } Pz \geqq z$$

by the condition

$$\text{(a}'\text{) } Pz \leqq z.$$

It is easy to determine the set U^* from what we know about U. Each face (of dimension $j - 1$) of U lies in a hyperplane determined by an equality $\{Pz\}_i = \{z\}_i$ or $\{z\}_i = 0$. The latter type faces lie in the coordinate planes. The former normally protrude, and they have the martingale $z(\phi)$ as maximal corner. U^* is obtained by taking the set that lies on the other side of the hyperplanes $\{Pz\}_i = \{z\}_i$. This is an s dimensional cone with $z(\phi)$ as minimal element. Thus U^* is the reflection of U through $z(\phi)$—and its linear extension. We also see that the martingale is the maximal upper semimartingale and the minimal lower semimartingale—the only point U and U^* have in common. It is possible to represent these lower semimartingales in terms of the $z(T)$'s. In fact let z^* be any point in U^*. Then a line from z^* through $z(\phi)$ will intersect a coordinate plane in a point z_1 in U. Then z may be uniquely written in the form

$$z^* = z(\phi) + A(z(\phi) - z_1),$$

where A is a nonnegative constant. On the other hand, by Theorem 2 $z_1 = \sum_j a_j z(T_j)$, $a_j > 0$, and $\sum a_j = 1$. Thus

$$z^* = z(\phi) + A(z(\phi) - \sum a_j\, z(T_j)),$$

and A and the a_k's are unique.

We can summarize this by saying that we have a unique representation for lower semimartingales:

$$z^* = \sum_{j=0}^{k} a_j\, z(T_j),$$

where $a_j < 0$ for $j \neq 0$, and $\sum a_j = 1$, with $\phi = T_0 \subset T_1 \subset \cdots \subset T_k$ forming a chain.

6. Arbitrary boundary values. We have assumed that specific boundary values were given. The particular convex polyhedron obtained for U depends on these boundary values. However, the extreme points $z(T)$ are easily obtained from $Q(T)$ for any choice of boundary values. In fact $z(I)$ is the vector with r components given by the boundary values and 0 for all other components. The vectors $z(T)$ are given by $z(T) = Q(T)z(I)$. The matrix $Q(T)$ does not depend upon the boundary values, thus when we find these $Q(T)$'s we have essentially solved the problem for all possible conditions.

7. Two examples. We shall give here two examples, one where hypothesis **A** is satisfied and one where it is not. For the first case let P be

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

The states $B = \{1, 2\}$ are the boundary states and $I = \{3, 4\}$ are the interior states. Assume that $v_1 = 2$ and $v_2 = 1$. The corner points are given by

$$\text{Martingale} = z(\phi) = \begin{pmatrix} 2 \\ 1 \\ \frac{7}{5} \\ \frac{9}{5} \end{pmatrix}; \quad z(\{3\}) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ \frac{4}{3} \end{pmatrix}; \quad z(\{4\}) = \begin{pmatrix} 2 \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}; \quad z(\{3,4\}) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The set of all upper semimartingales consists of the set of all vectors

$$z = \begin{pmatrix} 2 \\ 1 \\ x \\ y \end{pmatrix},$$

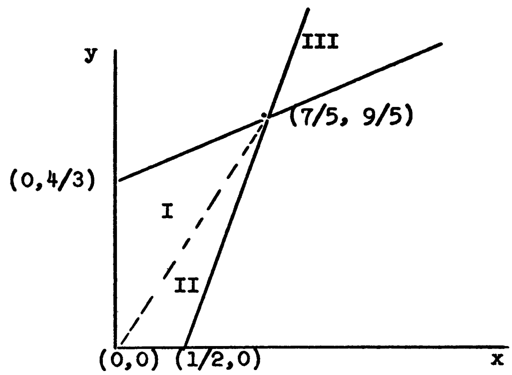

FIG. 1

where (x, y) is a point in the quadrilateral in Fig. 1. There are two maximal chains $\{0\}, \{3\}, \{3, 4\}$ and $\{0\}, \{4\}, \{3, 4\}$.

The regions above and below the dotted line, indicated by I and II, respectively, are the corresponding simplexes. The lower semimartingales are given by region III.

As an example of a case where we do not get a cubic polyhedron we consider the problem of random walk on the line with states 0 and $s + 1$ absorbing. Then the interior states are $I = (1, 2, \cdots, s)$. We require that $v(0) = 0$ and $v(s + 1) = 1$. It is clear that many subsets of I are not fundamental. In fact the only fundamental sets are the sets ϕ and $T_j = \{1, 2, \cdots, j\}$ for $1 \leqq j \leqq s$. Thus U is the s-dimensional simplex with corners $z(\phi), z(T_1), z(T_2), \cdots, z(T_s)$ These corner points are easily found from the ruin probabilities. They have coordinates for the interior states given by

$$\{z(T_j)\}_i = \begin{cases} 0, & i \leqq j, \\ \dfrac{i - j}{s + 1 - j} & j < i. \end{cases}$$

Thus any upper semimartingale vector

$$z = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \\ a_{s+1} \end{pmatrix},$$

with $a_0 = 0, a_{s+1} = 1$, is given by $z = \sum_{j=0}^{s} t_j z(T_j), \sum t_j = 1$. In this case it is easy to reverse the process and to find the t's from the z's. In fact for given z

$$t_0 = (s + 1)a_1,$$

$$t_j = (s + 1 - j)[a_{j+1} - 2a_j + a_{j-1}], \qquad 1 \leqq j \leqq s.$$

8. Application to sequential games. Consider an absorbing chain with r absorbing states and s interior states. Assume that we are given a vector

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{r+s} \end{pmatrix}$$

which determines the following game: The player starts in one of the states of the chain. If he is at an interior state i, he may either quit and collect v_i, or he may move on with the given transition probabilities. If he reaches a boundary state i, he collects v_i and the game ends. Let z_i be the value of the game to the player if he starts in state i. We wish to find the vector

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{r+s} \end{pmatrix}.$$

This is a special case of a problem considered in [2]. However, we can give a more precise description of the solution in the case considered here. It is clear that

$$z = \max [v, Pz], \tag{1}$$

since the player may by quitting or continuing have either of these. We shall now find a z having this property and then show that it is unique. Define

$$\bar{z} = \inf_z (z \geqq v, z \geqq Pz). \tag{2}$$

That is, $\bar{z}$ is the smallest lower semimartingale greater than v. If $\bar{z}$ did not have the property (1), then we could obtain a smaller semimartingale greater than v by replacing $\{\bar{z}\}_i$ by max $(v_i, \{P\bar{z}\}_i)$ in any component i for which $\{\bar{z}\}_i >$ max $(v_i, \{P\bar{z}\}_i$. Hence $\bar{z}$ must have the property (1).

Assume now that, for some other z, (1) is true. Let T be the set of interior states for which $\{z\}_i = v_i$, then

$$P(T)z = z$$

and thus

$$Q(T)z = \lim_{n\to\infty} [P(T)]^n z = z.$$

On the other hand,

$$P(T)\bar{z} \leqq \bar{z}$$

so that

$$Q(T)\bar{z} \leqq \bar{z}.$$

From the interpretation of $Q(T)$ (see Sec. 2), we know that $Q(T)z$ depends only on the components of z in $B \cup T$. And this is the set where $z = v$. Hence,

$$Q(T)z = Q(T)v.$$

Thus

$$z = Q(T)z = Q(T)v \leqq Q(T)\bar{z} \leqq \bar{z}.$$

But since $z \geqq v$ and $\geqq Pz$ we see from (2) that $z \geqq \bar{z}$. Therefore, $z = \bar{z}$. Hence $\bar{z}$ is the unique vector satisfying (1), and its components are the value of the game for various starting positions.

The optimal strategy is to continue on any component where $v_j < \{\bar{z}\}_j$.

A similar analysis shows that if the player wishes to minimize his fortune he should find the largest upper semimartingale z less than or equal to v and play only on states i such that $\{z\}_i < v_i$. This latter problem has application to statistical decision theory (see [2]).

For the first example given in Sec. 7, let the payoff vector be

$$v = \begin{pmatrix} 2 \\ 1 \\ v_3 \\ v_4 \end{pmatrix}.$$

Consider the case of the maximizing player. The various possibilities are indicated in Fig. 2. If $\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$ is in the interior of region IV then $z = z(\phi)$, $v_3 < z_3$, and $v_4 < z_4$. Hence the player should play on each interior state. If $\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$ is in the interior region II or its dotted boundary, then the smallest lower semimartingale greater than v is the point on the lower boundary of region I vertically above

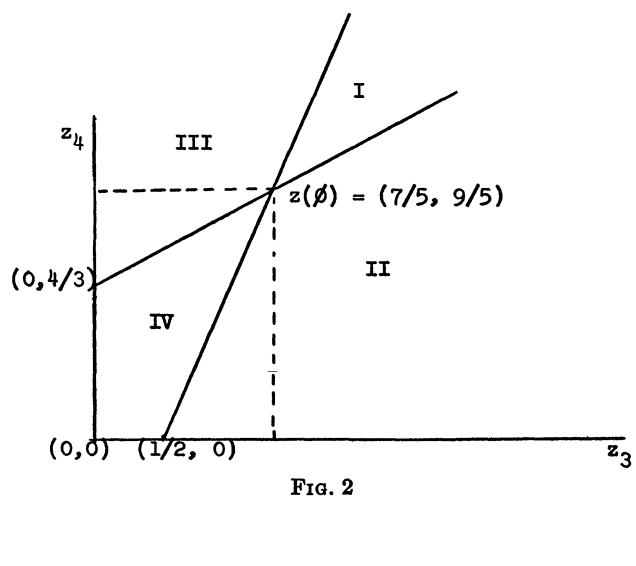

FIG. 2

v. Thus $z_3 = v_3$, $z_4 > v_4$. The player should stop on 3 and play on 4 in this region. Similarly, in region III he should stop on 4 and play on 3. If $\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$ is in region I, then $z = v$ and he should not play on any state.

9. Games with a fee for each play. The results of Sec. 8 can be extended to a game in which the player must pay a fee c_i if he wishes to continue playing in interior state i. Alternatively, we may think of c_i as the cost of carrying out an additional experiment. Let c be the column vector which is 0 in B and has components c_i in I. Then by an immediate extension of the previous argument, the vector z giving the values of the various states satisfies

$$(3) \qquad z = \max(v, Pz - c).$$

Let d be the column vector such that d_i is the expected cost to reach the boundary from state i. It can be shown that if we take the matrix $\mathcal{I} - P$, where $\mathcal{I}$ is the identity matrix, truncate it to the $s \times s$ matrix obtained by eliminating the boundary states, and take its inverse, then the ijth entry of the resulting matrix gives the expected number of times the process will be in state j if it starts in state i. (See [3], Chapter VII, Sec. 4.) This matrix multiplied into the truncated c-vector gives the truncated d-vector. Remembering that both vectors are 0 in B, we see that

$$(\mathcal{I} - P)\, d = c.$$

Hence

$$(4) \qquad Pd = d - c.$$

Since d is a fixed vector, we have from (3) that

$$z + d = \max(v + d, Pz - c + d)$$

and from (4) we see that

$$z + d = \max(v + d, P(z + d)).$$

But this is the problem we solved above. The vector $z + d$ is the least lower semimartingale greater than $v + d$. Thus the value of the game is given by the vector z that is found: First we find the least lower semimartingale greater than $v + d$, then we subtract d.

Thus the game with the cost vector c is strategically equivalent to a costless game in which the payoff vector v has added to it the expected cost of reaching the boundary.

10. Application to discrete subharmonic theory. Consider the lattice of points in the plane of the form (m, n) where m and n are integers. A random walk in the plane is a process which moves from (x, y) to $(x + 1, y)$, $(x - 1, y)$, $(x, y + 1)$, $(x, y - 1)$ with equal probabilities.

Let B and I be finite sets of lattice points such that from any point of I a random walk can reach a point of B but cannot reach any point not in $B \cup I$ without going through B. Then B is called a boundary set and I an interior set.

Consider a boundary set B and interior set I. Assume that boundary values $v(j, k)$ are given on B. Then there is a unique lattice function f defined on $B \cup I$ having the property that

$$f(j,k) = 1/4f(j+1,k) + 1/4f(j,k+1) + 1/4f(j-1,k) + 1/4f(j,k-1), \qquad (j,k) \varepsilon I,$$

and

$$f(j, k) = v(j, k), \qquad (j, k) \varepsilon B.$$

This function provides the discrete analogue for the solution of the Dirichlet problem; the function f is a discrete harmonic function. One should ask the corresponding problem for discrete subharmonic functions. That is, a function f is a discrete subharmonic function with prescribed boundary values if

$$f(j, k) \geqq 1/4f(j+1, k) + 1/4f(j, k+1) + 1/4f(j-1, k) + 1/4f(j, k-1), \qquad (j, k) \varepsilon I,$$

$$f(j, k) = v(j, k), \qquad (j, k) \varepsilon B.$$

In this case the solution would not be unique.

The random walk in $I \cup B$ forms an absorbing Markov chain. Assume that the boundary values are nonnegative. The harmonic function f is given by the vector $z(\phi)$. The subharmonic functions are the semimartingale vectors. Thus the set of all subharmonic functions forms a convex polyhedron and each such function may be represented in terms of a finite number of basic semimartingales. Each basic semimartingale $z(T)$ is simply the unique solution for the Dirichlet problem for boundary $B \cup T$ with the given values on B and 0 on T. Thus the set of all subharmonic functions for a given boundary B may be represented as convex combinations of the harmonic functions for the boundary sets $B \cup T$.

We have reason to believe that these results obtained for discrete subharmonic functions will lead to analogous results for ordinary subharmonic functions.

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