$$\varphi_2(t) = \frac{1}{\cosh 2t} + i\sqrt{2}\,\frac{\sinh t}{\cosh 2t},$$

$$\varphi_3(t) = \frac{1}{(\cosh 2t)^{\frac{1}{2}}} \quad \text{and} \quad \varphi_4(t) = \frac{e^{it}}{(\cosh 2t)^{\frac{1}{2}}}.$$

If these functions φ are inverted (see [1], pp. 388–389, and [2], p. 30) and a change of variable made from $(4/\pi) \log |X|$ to X, assuming X symmetric, then the corresponding density functions are

$$p_1(x) = \frac{\sqrt{2}}{\pi}\,\frac{x^2}{1 + x^4}, \qquad -\infty < x < +\infty,$$

$$p_2(x) = \frac{2}{\pi}\,\frac{x^4}{(1 + x^2)(1 + x^4)}, \qquad -\infty < x < +\infty,$$

$$p_3(x) = \frac{1}{2\pi^2 \sqrt{2\pi}\,|x|}\left|\Gamma\left(\frac{1}{4} + \frac{i \log |x|}{\pi}\right)\right|^2, \qquad -\infty < x < +\infty,$$

and

$$p_4(x) = \frac{1}{2\pi^2 \sqrt{2\pi}\,|x|}\left|\Gamma\left(\frac{1 - i}{4} + \frac{i \log |x|}{\pi}\right)\right|^2, \qquad -\infty < x < +\infty.$$

Using $\theta(-t)$ instead of $\theta(t)$ provides additional densities $p^*(x) = p(1/x)/x^2$ (if p is the density function of X then p^* is the density function of $1/X$). For example,

$$p_1^*(x) = \frac{\sqrt{2}}{\pi}\,\frac{1}{1 + x^4}, \qquad -\infty < x < +\infty,$$

and

$$p_2^*(x) = \frac{2}{\pi}\,\frac{1}{(1 + x^2)(1 + x^4)}, \qquad -\infty < x < +\infty.$$

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ESTIMATION OF A REGRESSION LINE WITH BOTH VARIABLES SUBJECT TO ERROR UNDER AN UNUSUAL IDENTIFICATION CONDITION

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Suppose the random variables $w_j = (\xi_j, u_j, v_j)$ are independently and identically distributed with joint distribution F. Then if $\iiint e^{\alpha u + \beta v}\, dF(\xi, u, v)$ exists

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for all α, β in a neighborhood of 0 and $\int\int\int e^{t\xi}\, dF(\xi, u, v)$ does not exist for all t in any neighborhood of 0, Jeeves [1] has shown that the parameter θ in

$$\begin{aligned} x_j &= \xi_j \cos \theta + u_j , \\ y_j &= \xi_j \sin \theta + v_j , \end{aligned} \tag{1}$$

is identified. We shall construct a consistent estimate of θ (mod π) if these conditions are satisfied.

First, let us consider a univariate distribution G with moment generating function g. Then $g(t) = \sum (\mu_n / n!)\, t^n$, μ_n the nth moment, and if the radius of convergence is r, it is well known that

$$\rho = \frac{1}{r} = \overline{\lim} \left(\frac{|\mu_n|}{n!} \right)^{1/n} \tag{2}$$

As easy application of Liapounoff's inequality and Stirling's formula shows that

$$\rho = \overline{\lim} \frac{(\mu_{2n})^{1/2ne}}{2n} \tag{3}$$

Therefore a natural procedure would seem to be to consider the sample moments $m_{2n}(\phi)$ of $x_j \sin \phi - y_j \cos \phi$ and to define $\hat{\theta}$ as that value of ϕ minimizing $\max_n (m_{2n}(\phi))^{1/2n}/n$. For fixed sample size, this maximum exists. We shall show that this estimate is indeed consistent, and even converges with probability one to θ.

First let us show that $\max_n m_{2n}(\theta)^{1/2n}/n$ is bounded as a function of the sample size N with probability one. Let

$$\psi(t) = E(\cosh[t(u_j \sin \theta - v_j \cos \theta)]) = \sum \frac{\mu_{2n}\, t^{2n}}{(2n)!},$$

where μ_{2n} is the $2n$th moment of $u_j \sin \theta - v_j \cos \theta$. Then [2], for $|t| \leqq s < r$, $\psi_N(t) = \sum m_{2n} t^{2n}/(2n)!$ converges to $\psi(t)$ uniformly with probability one. Thus $\psi_N(t)$ is bounded with probability one, and since $m_{2n}{}^{1/2n}/n \leqq (K/t)\, [\psi_N(t)]^{1/2n}$, $\max_n m_{2n}(\theta)^{1/2n}/n$ is bounded with probability one. Hence with probability greater than $1 - \epsilon/3$, H_ϵ can be used for the bound. Similarly,

$$\max_{n,\phi} \frac{\left(\frac{1}{N} \sum (u_j \sin \phi - v_j \cos \phi)^{2n} \right)^{1/2n}}{n} < K_\epsilon$$

for all N with probability greater than $1 - \epsilon/3$. Let δ be given, $0 < \delta < \pi$, and let γ_n be the nth moment of ξ_j. Since $(\gamma_{2n})^{1/2n}/n$ can be made arbitrarily large by selecting n large enough, select n so that

$$\frac{(\gamma_{2n})^{1/2n}}{n} > \frac{H_\epsilon + K_\epsilon}{\sin \delta}.$$

Then with probability greater than $1 - \epsilon/3$,

$$\frac{(1/N \sum \xi_j^{2n})^{1/2n}}{n} > \frac{H_\epsilon + K_\epsilon}{\sin \delta}$$

for all N sufficiently large. By Minkowski's inequality

$$m_{2n}(\phi)^{1/2n} \geqq |\sin(\phi - \theta)| \left(\frac{1}{N}\sum \xi_j^{2n}\right)^{1/2n} - \left(\frac{1}{N}\sum (u_j \sin\phi - v_j \cos\phi)^{2n}\right)^{1/2n}$$

Therefore with probability greater than $1 - \epsilon$,

$$\frac{\max_n (m_{2n}(\phi))^{1/2n}}{n} > \frac{\max_n (m_{2n}(\theta))^{1/2n}}{n}$$

for all N sufficiently large for all ϕ not in the interval $(\theta - \delta, \theta + \delta)$ (mod π).

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ON THE DECOMPOSITION OF CERTAIN χ^2 VARIABLES

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It is well known that if the sum, say $Q = Q_1 + Q_2$, of two stochastically independent variables is χ^2 with r d.f., and if Q_1 is also χ^2 with r_1 d.f., then Q_2 is likewise χ^2 with $r_2 = r - r_1$ d.f. If the hypothesis of stochastic independence is removed, little can be said about Q_2. It seems to us quite interesting that if the variables under consideration are real symmetric quadratic forms in either central or non-central, stochastically independent or dependent normal variables, and if the hypothesis of stochastic independence of Q_1 and Q_2 is replaced by the weaker hypothesis $Q_2 \geqq 0$, then Q_1 and Q_2 are stochastically independent so that Q_2 is itself a χ^2 variable with $r_2 = r - r_1$ d.f.

Before we state our theorem, we recall [1] that the real symmetric quadratic form $Y'BY$ in n mutually stochastically independent normal variables $Y' = (y_1, y_2, \cdots, y_n)$ with unit variances and means $U' = (u_1, u_2, \cdots, u_n)$ has a non-central χ^2 distribution whose characteristic function is

$$\varphi(t) = \exp\left[\frac{it\theta}{1 - 2it}\right] \Big/ (1 - 2it)^{r/2}$$

if and only if $B^2 = B$. Here, $\theta = U'BU$ and r is the rank of B.

Theorem. *Let $Q = Q_1 + \cdots + Q_{k-1} + Q_k$, where $Q = X'AX$ and $Q_j = X'A_jX$, $j = 1, 2, \cdots, k$, are real symmetric quadratic forms in n normally distributed variables $X' = (x_1, x_2, \cdots, x_n)$ with means $M' = (m_1, m_2, \cdots, m_n)$ and real symmetric definite positive variance-covariance matrix V. Let $Q, Q_1, \cdots,$*

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