Using this expression for a,

$$E\{a[x(r+i)-x(r-i)]\} = \sigma\{x(r)\} + O(n^{-5/2+2\alpha}) + O(n^{-3/2})$$
$$\sigma\{a[x(r+i)-x(r-i)]\} = O(n^{-1/2-\alpha/2}).$$

Thus increasing α decreases the order of magnitude of

$$\sigma\{a[x(r+i)-x(r-i)]\},$$

but increases the order of

$$E\{a[x(r+i)-x(r-i)]\} - \sigma\{x(r)\}.$$

Hence the order of the error is minimized when

$$-1/2 - \alpha/2 = -5/2 + 2\alpha.$$

Thus $\alpha = 4/5$ appears to be the most desirable choice for α.

In $\sigma\{a[x(r+i)-x(r-i)]\}$, the parameter ϵ appears predominantly as the factor $1/\sqrt{\epsilon}$. In $E\{a[x(r+i)-x(r-i)]\} - \sigma\{x(r)\}$ the predominant factor is ϵ^2. Solution of the equation

$$\epsilon^2 = 1/\sqrt{\epsilon}$$

suggests that $\epsilon = 1$ is an appropriate compromise choice for ϵ.

Use of $\alpha = 4/5$, $\epsilon = 1$, and the expression for a yields the results

$$i = (n+1)^{4/5}, \qquad a = \tfrac{1}{2}(n+1)^{-3/10}\sqrt{p_r q_r},$$

and verifies the properties stated for $s[x(r)]$.

REFERENCE

[1] F. N. David and N. L. Johnson, "Statistical treatment of censored data. Part I. Fundamental formulae," *Biometrika*, Vol. 41 (1954), p. 228–240.

A UNIQUENESS PROPERTY NOT ENJOYED BY THE NORMAL DISTRIBUTION

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1. Summary. It is well known that if X and Y (or $1/X$ and $1/Y$) are independently normally distributed with mean zero and variance σ^2, then X/Y has a Cauchy distribution. It is the purpose of this note to show that the converse statement is not true. That is, the fact that the ratio of two independent, identically distributed, random variables X and Y follows a Cauchy distribution is not sufficient to imply that X and Y (or $1/X$ and $1/Y$) are normally distributed. This will be shown by exhibiting several counterexamples.

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2. Construction of counterexamples. Let X and Y be independent, identically distributed, random variables with common symmetric density function p. Let φ denote the characteristic function of $(4/\pi)\log|X|$, let $Z = X/Y$, and let ω denote the characteristic function of $(4/\pi)\log|Z|$. The fact that Z has a Cauchy distribution implies that

$$p_{4/\pi \log|Z|}(u) = \frac{1}{4\cosh \pi u/4}, \qquad -\infty < u < +\infty,$$

and, hence,

$$\omega(t) = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{itu}\,du}{\cosh \pi u/4} = \frac{1}{\cosh 2t}.$$

Since

$$\frac{4}{\pi}\log|Z| = \frac{4}{\pi}\log|X| - \frac{4}{\pi}\log|Y|,$$

it follows that

$$\varphi(t)\cdot\varphi(-t) = \frac{1}{\cosh 2t},$$

and, therefore,

$$\varphi(t) = \frac{e^{i\theta(t)}}{(\cosh 2t)^{\frac{1}{2}}} = \frac{\cos\theta(t)}{(\cosh 2t)^{\frac{1}{2}}} + i\,\frac{\sin\theta(t)}{(\cosh 2t)^{\frac{1}{2}}}, \tag{1}$$

where θ is continuous, real, odd, and of such a form that φ is a characteristic function.

Since $\varphi(t)$ must be inverted by contour integration to find corresponding density functions, equation (1) suggests that θ be chosen so as to eliminate the square root. The relations

$$\cosh 2t = \cosh^2 t + \sinh^2 t = 1 + 2\sinh^2 t$$

provide two functions θ which accomplish this, namely,

$$\theta_1(t) = \arctan\tanh t$$

and

$$\theta_2(t) = \arctan\sqrt{2}\sinh t.$$

Other functions θ which immediately suggest themselves, even though they do not eliminate the square root, are

$$\theta_3(t) \equiv 0, \quad \text{and} \quad \theta_4(t) = t.$$

The corresponding functions φ are

$$\varphi_1(t) = \frac{\cosh t}{\cosh 2t} + i\,\frac{\sinh t}{\cosh 2t},$$

$$\varphi_2(t) = \frac{1}{\cosh 2t} + i\sqrt{2}\,\frac{\sinh t}{\cosh 2t},$$

$$\varphi_3(t) = \frac{1}{(\cosh 2t)^{\frac{1}{2}}} \quad \text{and} \quad \varphi_4(t) = \frac{e^{it}}{(\cosh 2t)^{\frac{1}{2}}}.$$

If these functions φ are inverted (see [1], pp. 388–389, and [2], p. 30) and a change of variable made from $(4/\pi) \log | X |$ to X, assuming X symmetric, then the corresponding density functions are

$$p_1(x) = \frac{\sqrt{2}}{\pi}\frac{x^2}{1 + x^4}, \qquad -\infty < x < +\infty,$$

$$p_2(x) = \frac{2}{\pi}\frac{x^4}{(1 + x^2)(1 + x^4)}, \qquad -\infty < x < +\infty,$$

$$p_3(x) = \frac{1}{2\pi^2\sqrt{2\pi}\,|x|}\left|\Gamma\left(\frac{1}{4} + \frac{i \log |x|}{\pi}\right)\right|^2, \qquad -\infty < x < +\infty,$$

and

$$p_4(x) = \frac{1}{2\pi^2\sqrt{2\pi}\,|x|}\left|\Gamma\left(\frac{1 - i}{4} + \frac{i \log |x|}{\pi}\right)\right|^2, \qquad -\infty < x < +\infty.$$

Using $\theta(-t)$ instead of $\theta(t)$ provides additional densities $p^*(x) = p(1/x)/x^2$ (if p is the density function of X then p^* is the density function of $1/X$). For example,

$$p_1^*(x) = \frac{\sqrt{2}}{\pi}\frac{1}{1 + x^4}, \qquad -\infty < x < +\infty,$$

and

$$p_2^*(x) = \frac{2}{\pi}\frac{1}{(1 + x^2)(1 + x^4)}, \qquad -\infty < x < +\infty.$$

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[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of integral transforms*, Vol. 1, McGraw-Hill, 1954.

ESTIMATION OF A REGRESSION LINE WITH BOTH VARIABLES SUBJECT TO ERROR UNDER AN UNUSUAL IDENTIFICATION CONDITION

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Suppose the random variables $w_j = (\xi_j, u_j, v_j)$ are independently and identically distributed with joint distribution F. Then if $\int\int\int e^{\alpha u + \beta v}\, dF(\xi, u, v)$ exists

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