D	N	P	Q	L	O	H	A	A	B	B	C	C	F	F	K	K	J	J	G	G
E	R	U	T	M	S	I	N	R	H	I	H	I	D	E	D	E	D	E	D	E
A	B	F	C	J	K	G	O	S	L	M	M	L	L	M	I	H	H	I	M	K
B	F	C	J	K	G	A	P	T	P	Q	N	O	Q	O	N	Q	O	P	P	N
C	J	K	G	A	B	F	Q	U	T	U	S	R	S	T	T	R	U	S	R	U

The lower left hand group of blocks constitutes the design (b), and the lower right hand group of blocks is the GD design with parameters $v = b = 14$, $r = k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = 7$, $n = 2$. The groups are (D, E), (N, R), (P, U), (Q, T), (L, M), $O, S)$, and (H, I). Thus, for example, D occurs zero times with E in the GD design and once with N, R, P, U, Q, T, L, M, O, S and I.

A design with these parameters was obtained in [4] using the method of differences and is listed as number R24 in [5]. For $s = 3$ the resulting design has parameters,

$$v = b = 78, r = k = 9, \lambda_1 = 0, \lambda_2 = 1, m = 13, n = 6.$$

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ALTERNATIVE PROOF OF A THEOREM OF BIRNBAUM AND PYKE

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Let $U_1, U_2, \cdots, U_n$ be an ordered sample of a random variable (r.v.) X having a uniform distribution (0, 1). If i^* is the value of $i = 1, 2, \cdots, n$ at which $i/n - U_i$ is maximized and $U^* = U_{i^*}$, then U^* is a r.v. with values (0, 1). The probability that the sample cannot be ordered or that i^* is not uniquely defined is zero, and hence these possibilities are neglected. Theorem 3 [1] states that U^* has a uniform distribution (0, 1). Another proof of this fact was given in [2].

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In this note an alternative proof is given which entails little computation and is self-contained.

Replace the interval (0, 1) by the reals modulo 1, considered as a circle of circumference 1. Let c be an arbitrary point on the circle. Moving from c in the direction corresponding to increasing values (0, 1), one meets successively the points $U_{k+1}, U_{k+2}, \cdots, U_n, U_1, \cdots, U_k$ where k, so defined, is a r.v. depending on c. Rename these points $U_1^c, U_2^c, \cdots, U_n^c$ respectively. Define $i = i(j)$ by $U_j^c = U_i$. Let u_j^c denote the (arc) distance of U_j^c from c taken in the increasing direction. Therefore,

$$i = k + j; \qquad u_j^c = U_{k+j} - c \qquad \text{for } j = 1, \cdots, n - k$$

$$i = k + j - n; \qquad u_j^c = U_{k+j-n} + 1 - c \qquad \text{for } j = n - k + 1, \cdots, n$$

With the indicated relation between i and j observe that

$$j/n - u_j^c = (i - k)/n - U_i + c = i/n - U_i + c - k/n.$$

For a fixed c and a given sample, c and k are constants and hence $j/n - u_j^c$ attains its maximum at the same point $U^* = U_{i^*}$ as does $i/n - U_i$.

Given a sample $U_1, \cdots, U_n$, the point U^* on the circle of reals mod. 1 is therefore independent of the choice of the initial point c taken instead of 0 on this circle. Since the distribution of X mod. 1 is uniform, that is, is invariant under translations, the distribution of U^* mod. 1 is also invariant under translations. Thus U^* has a uniform distribution on (0, 1). q.e.d.

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QUASI-RANGES OF SAMPLES FROM AN EXPONENTIAL POPULATION

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In a study of the use of ranges and quasi-ranges in estimating the standard deviation of a population, Harter [4] has compared the results for samples from a normal population with those for samples from certain other populations, including the exponential. In this note are given the distributions of quasi-ranges from the exponential population and also formulas for the cumulants of these quasi-ranges.

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