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NOTE ON THE FACTORIAL MOMENTS OF THE DISTRIBUTION OF LOCALLY MAXIMAL ELEMENTS IN A RANDOM SAMPLE

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0. Summary. The results reported by T. Austin, R. Fagen, T. Lehrer, and W. Penney [1] are extended to include a general recurrence relation for the factorial moments of the distribution. This recurrence relation is solved for the mean and second factorial moments, and it is shown that the method applied may also be used to obtain a general solution for any desired factorial moment of higher order.

1. Introduction. Austin, Fagen, Lehrer, and Penney [1] have discussed the distribution of locally maximal elements in a random sample. Among other results, the authors defined certain elements in an ordered random sample of n distinct real numbers to be locally k-maximal, provided such an element is the greatest of some set of k consecutive elements of the sample. Denoting by $f_k(n, t)$ the number of sequences of the first n positive integers which have exactly t elements which are locally k-maximal, and defining a generating function, $v_k(x, y)$,

$$v_k(x, y) \equiv \sum_{\alpha,\beta} f_k(\alpha, \beta)x^\alpha y^\beta/\alpha!, \tag{1.1}$$

a recurrence relation and partial differential equation were then derived:

$$f_k(n + 1, r + 1) = \sum_{m,t} \binom{n}{m} f_k(m, t)f_k(n - m, r - t), \qquad n \geqq k - 1 \tag{1.2}$$

$$\partial v_k/\partial x = yv_k^2 + (1 - y)\sum_{t=0}^{k-2} (t + 1)x^t. \tag{1.3}$$

Unless specified otherwise, the range of a summation variable may be taken as $(0, +\infty)$ in these and the subsequent sums.

The relations (1.1) and (1.3) may be employed to obtain a general recurrence relation for the factorial moments of the distribution. Information on such moments would be useful in any application of the distribution as a non-parametric test, and would generally be of value in characterizing the distribution.

2. Recurrence relation for the factorial moments of the distribution. Let the r-th factorial of β be defined as $\beta^{(r)} \equiv \beta(\beta - 1)\cdots(\beta - r + 1)$, with

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$\beta^{(0)} = 1$, $\beta^{(1)} = \beta$. The expected value of $\beta^{(r)}$ for n and k held fixed may be denoted as $E(\beta^{(r)}, n)$ such that

$$E(\beta^{(r)},n) = \sum_{\beta} \beta^{(r)} f_k(n, \beta)/n!. \tag{2.1}$$

With this notation, the defining relation (1.1) for v_k may be differentiated and evaluated at $y = 1$, with a result

$$\begin{aligned} \frac{\partial^r}{\partial y^r}\frac{\partial v_k}{\partial x}\bigg|_{y=1} &= \sum_{\alpha,\beta} (\alpha + 1)\beta^{(r)} f_k(\alpha + 1, \beta)x^\alpha/(\alpha + 1)! \\ &= \sum_{\alpha} (\alpha + 1)E(\beta^{(r)}, \alpha + 1)x^\alpha. \end{aligned} \tag{2.2}$$

A different representation of this same function may be obtained by differentiating (1.3), making use of (1.1), (2.1), and Leibnitz' expansion for the derivative of a product. There results

$$\begin{aligned} \frac{\partial^r}{\partial y^r}\frac{\partial v_k}{\partial x}\bigg|_{y=1} &= \frac{\partial^r}{\partial y^r}\left[y v_k^2 + (1 - y)\sum_{t=0}^{k-2} (t + 1)x^t \right]_{y=1} \\ &= \left[\left(\frac{\partial^r}{\partial y^r} + r\frac{\partial^{r-1}}{\partial y^{r-1}}\right) v_k^2 + \frac{\partial^r}{\partial y^r}(1 - y)\sum_{t=0}^{k-2}\left(t + 1)x^t\right.\right]_{y=1} \\ &= \left[\sum_{\lambda,\delta,\gamma} x^{\delta+\gamma}\left\{\binom{r}{\lambda} E(\beta^{(\lambda)}, \delta)E(\beta^{(r-\lambda)}, \gamma) + r\binom{r-1}{\lambda}\right.\right. \\ &\quad \left.\left. \cdot E(\beta^{(\lambda)}, \delta)E(\beta^{(r-1-\lambda)}, \gamma)\right\} + \frac{\partial^r}{\partial y^r}(1 - y)\sum_{t=0}^{k-2} (t + 1)x^t\right]_{y=1}. \end{aligned} \tag{2.3}$$

The boundary values of the $E(\beta^{(r)}, n)$ include the following:

$$\begin{aligned} E(\beta^{(0)}, n) &= 1, \; n \geqq 0 \\ E(\beta^{(0)}, n) &= 0, \; n < 0 \\ E(\beta^{(r)}, n) &= 0, \; r > 0, \; n < k \\ E(\beta^{(r)}, k) &= 1, \; r = 0, 1 \\ E(\beta^{(r)}, k) &= 0, \; r > 1 \end{aligned} \tag{2.4}$$

Upon equating the coefficients of x^{k+S}, $S \geqq -1$, in relations (2.2) and (2.3), after making some simple substitution changes on the dummy variables, there results

$$\begin{aligned} &(k + S + 1)E(\beta^{(r)}, k + s + 1) \\ &\quad = \sum_{\lambda,\gamma}\left[\binom{r}{\lambda} E(\beta^{(\lambda)}, k + S - \gamma)E(\beta^{(r-\lambda)}, \gamma)\right. \\ &\quad \left. + r\binom{r-1}{\lambda} E(\beta^{(\lambda)}, k + S - \gamma)E(\beta^{(r-1-\lambda)}, \gamma)\right], \qquad r \geqq 0, S \geqq -1 \end{aligned} \tag{2.5}$$

The relation (2.5) is a general recurrence relation for the factorial moments of the distribution. Computation of these moments may be expedited by use of the recurrence relation.

3. Solution of recurrence relation. For the special case, $r = 1$, the relation (2.5) reduces to the following:

$$(3.1)\quad (k+S+1)E(\beta^{(1)},k+S+1) = (k+S+1) + 2\sum_t E(\beta^{(1)},k+S-t).$$

This equation has the general solution $E(\beta^{(1)}, k+S) = (k+2S+1)/(k+1)$ in agreement with reference [1]. The solution may be proved by induction, or by the use of a method we shall employ for the case, $r = 2$.

For the special case, $r = 2$, relation (2.5) reduces to the form

$$(3.2)\quad \begin{aligned} &(k+S+1)E(\beta^{(2)}, k+S+1) \\ &\quad = \sum_\alpha [4E(\beta^{(1)}, k+S-\alpha) + 2E(\beta^{(2)}, k+S-\alpha) \\ &\qquad + 2E(\beta^{(1)}, k+S-\alpha)E(\beta^{(1)}, \alpha)]. \end{aligned}$$

The last term in the right member contributes only if $S \geqq k$. For this reason the cases $-1 \leqq S \leqq k-1$ and $S \geqq k$ will be considered separately.

If $-1 \leqq S \leqq k-1$, then (3.2) reduces to

$$(3.3)\quad \begin{aligned} &(k+S+1)E(\beta^{(2)}, k+S+1) \\ &\quad = \sum_\alpha [4E(\beta^{(1)}, k+S-\alpha) + 2E(\beta^{(2)}, k+S-\alpha)]. \end{aligned}$$

On subtracting from this equation the similar equation resulting when S is replaced by $S-1$, and making use of the general solution for the mean, a difference equation results,

$$(3.4)\quad \begin{aligned} &(k+S+1)E(\beta^{(2)}, k+S+1) - (k+S+2)E(\beta^{(2)}, k+S) \\ &\quad = 4(k+2S+1)/(k+1). \end{aligned}$$

This difference equation may be solved by making the substitution, $E(\beta^{(2)}, k+r) = (k+r+1)H(k+r)$, with boundary value $H(k) = 0$. The solution is

$$(3.5)\quad \begin{aligned} &E(\beta^{(2)}, k+S+1) \\ &\quad = \frac{4(k+S+2)}{(k+1)} \sum_{t=0}^{S} \frac{(k+2t+1)}{(k+t+1)(k+t+2)}, \qquad 0 \leqq S \leqq k-1. \end{aligned}$$

If $S \geqq k$, set $S = k+m$, $m \geqq 0$. The recurrence relation (3.2) takes the form

$$(3.6)\quad \begin{aligned} (2k+m+1)E(\beta^{(2)}, 2k+m+1) &= \sum_\alpha [4E(\beta^{(1)}, 2k+m-\alpha) \\ &+ 2E(\beta^{(2)}, 2k+m-\alpha) + 2E(\beta^{(1)}, 2k+m-\alpha)E(\beta^{(1)}, \alpha)]. \end{aligned}$$

On subtracting from this equation the similar equation with m replaced by $m-1$

and making use of the general solution for the mean, the following difference equation results:

$$(3.7)\quad \begin{aligned} &(2k+m+1)E(\beta^{(2)}, 2k+m+1) - (2k+m+2)E(\beta^{(2)}, 2k+m) \\ &= 4(3k+2m+1)/(k+1) + 2\sum_{\alpha} [E(\beta^{(1)}, \alpha)E(\beta^{(1)}, 2k+m-\alpha) \\ &\qquad - E(\beta^{(1)}, \alpha)E(\beta^{(1)}, 2k+m-1-\alpha)] \\ &= 4(3k+4m+1)/(k+1) + 4m(m-1)/(k+1)^2 + 2, \qquad m \geqq 0. \end{aligned}$$

This difference equation may be solved by placing $E(\beta^{(2)}, 2k+r) = (2k+r+1)H(2k+r)$, with the boundary value,

$$H(2k) = E(\beta^{(2)}, 2k)/(2k+1) = \frac{4}{(k+1)} \sum_{S=0}^{k-1} \frac{(k+2S+1)}{(k+S+1)(k+S+2)}.$$

The resulting solution to (3.7) is

$$(3.8)\quad \begin{aligned} &E(\beta^{(2)}, 2k+m+1) = (2k+m+2) \\ &\cdot \left\{ H(2k) + \sum_{S=0}^{m} \left[\frac{\dfrac{4(3k+4S+1)}{(k+1)} + \dfrac{4S(S-1)}{(k+1)^2} + 2}{(2k+S+1)(2k+S+2)} \right] \right\}, \end{aligned}$$

for $m \geqq 0$.

These results were verified by the use of the numerical examples included in reference [1]. For larger values of the arguments, computation of the values of the finite sums in the expressions for $E(\beta^{(2)}, r)$ may be expedited by expanding the summands into partial fraction form, resulting in expressions which could be evaluated by the use of tables of the logarithmic derivative of the Gamma function, according to methods discussed in references [2] and [3].

Similarly, a solution of (2.5) for $E(\beta^{(r)}, k+m)$ for general r may be obtained in terms of the factorial moments of all orders $\leqq r-1$. Such a solution is

$$(3.9)\quad \begin{aligned} &E(\beta^{(r)}, k+m+1) \\ &= (k+m+2)\left[H(k) + \sum_{S=0}^{m} P_{r-1}(k, S)/(k+S+1)(k+S+2) \right], \end{aligned}$$

where

$$(3.10)\quad \begin{aligned} P_{r-1}(k, S) = &\sum_{\lambda=1}^{r-1} \sum_{\gamma} \binom{r}{\lambda} E(\beta^{(r-\lambda)}, \gamma)\, [E(\beta^{(\lambda)}, k+S-\gamma) \\ &\qquad - E(\beta^{(\lambda)}, k+S-1-\gamma)] \\ &+ \sum_{\lambda,\gamma} \left\{ r\binom{r-1}{\lambda} E(\beta^{(r-1-\lambda)}, \gamma)[E(\beta^{(\lambda)}, k+S-\gamma) \right. \\ &\qquad \left. - E(\beta^{(\lambda)}, k+S-1-\gamma] \right\}, \end{aligned}$$

$$(3.11)\quad H(k) = 1/(k+1),\ r = 1;\ H(k) = 0,\ r > 1.$$

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THE LIMITING JOINT DISTRIBUTION OF THE LARGEST AND SMALLEST SAMPLE SPACINGS[1]

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1. Introduction and summary. $X_1, X_2, \cdots, X_n$ are independent chance variables, each with the same distribution. This common distribution assigns all the probability to the closed interval [0, 1], and has a density function $f(x)$ whose graph consists of any finite number of horizontal line segments. That is, there are H non-degenerate subintervals

$$I_1, I_2, \cdots, I_H, \qquad I_1 = [0, z_1), I_2 = [z_1, z_2), \cdots, I_H = [z_{H-1}, 1],$$

and for each x in I_j, $f(x) = a_j$. We assume that a_j is positive for all j. Let z_0 denote zero, and z_H denote unity. M will denote $\min_j a_j$, B will denote

$$\sum_{j:a_j=M} (z_j - z_{j-1}),$$

and S shall denote $\int_0^1 f^2(x)\,dx = \sum_{j=1}^H a_j^2(z_j - z_{j-1})$.

Let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ denote the ordered values of $X_1, \cdots, X_n$, and define $W_1 = Y_1$, $W_2 = Y_2 - Y_1, \cdots, W_n = Y_n - Y_{n-1}$, $W_{n+1} = 1 - Y_n$, $U_n = \min(W_1, \cdots, W_{n+1})$, $V_n = \max(W_1, \cdots, W_{n+1})$. In [1] it is shown that if $f(x)$ is the uniform density function over [0, 1], then

$$\lim_{n\to\infty} P\left[U_n > \frac{u}{(n+1)^2}, \quad V_n < \frac{\log(n+1) - \log v}{n+1}\right] = \exp\{-(u+v)\},$$

for any positive numbers u, v. It is easy to see that the convergence must be uniform over any bounded rectangle in the space of u and v. In this paper it is shown that if $f(x)$ is of the type described above, then

$$\lim_{n\to\infty} P\left[U_n > \frac{u}{(n+1)^2}, V_n < \frac{\log(n+1) + \log M - \log v}{M(n+1)}\right] = \exp\{-(Su + Bv)\},$$

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